

ON THE ASYMPTOTIC EXPANSION FOR THE TRACE OF THE HEAT KERNEL ON LOCALLY SYMMETRIC EINSTEIN SPACES AND ITS APPLICATION

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0. Introduction. Let (M, g) be an n -dimensional closed and connected Riemannian manifold and Δ be the Laplacian for functions defined by

$$(0.1) \quad \Delta f = -g^{ij} \nabla_i \nabla_j f.$$

Let $\text{Spec}(M, g) = \{\lambda_i\}_{i=0}^{\infty}$ be the spectrum of the Laplacian, that is, the set of eigenvalues of Δ counting with multiplicities. It is well-known that the coefficients a_i of Minakshisundaram-Pleijel's asymptotic expansion

$$(0.2) \quad \sum_{i=0}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i t^i, \quad t \rightarrow +0,$$

are determined by the spectrum. a_0 , a_1 and a_2 are easily calculated by Taylor asymptotic expansion of the metric tensor g_{ij} . a_3 was calculated by Sakai [12] and Gilkey [6]. Similarly we can treat the spectrum $\text{Spec}^1(M, g)$ of the Laplacian for 1-forms. In this paper we calculate a_4 for a locally symmetric Einstein space and give some geometric applications. The main result is the following:

Proposition. *For two oriented closed Riemannian manifolds (M, g) and (M', g') assume that one of them is an 8-dimensional locally symmetric Einstein space. If (M, g) and (M', g') have the same spectra for functions and for 1-forms, respectively, i.e.,*

$$\text{Spec}(M, g) = \text{Spec}(M', g') \quad \text{and} \quad \text{Spec}^1(M, g) = \text{Spec}^1(M', g'),$$

then (1) $\chi(M) = \chi(M')$ and (2) $|\sigma(M)| = |\sigma(M')|$ are equivalent, where $\chi(M)$, $\sigma(M)$ denote the Euler characteristic and signature of M , respectively.

1. Preliminaries. We assume that (M, g) is an n -dimensional closed locally symmetric Einstein space. We define the curvature tensor as

$$(1.1) \quad R_{ijk}{}^l \partial_l = R(\partial_i, \partial_j) \partial_k,$$

$$(1.2) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Then the contracted values of curvature tensors given in the following (1.3) are constant on (M, g) .

$$\begin{aligned} \tau &:= \sum_{ij=1}^n R_{ij}{}^{ji} : \text{ scalar curvature,} \\ R^2 &:= \sum R_{ijkl} R^{ijkl} : \\ &\quad \text{the square of the norm of the curvature tensor,} \\ (1.3) \quad R^3 &:= \sum R_{ijkl} R^{klmn} R_{mn}{}^{ij}, \\ \tilde{R}^3 &:= \sum R_{ikjl} R^{kmln} R_m{}^i{}_n{}^j, \\ R^4 &:= \sum R_{ijkl} R^{kl}{}_{mn} R^{mn}{}_{ab} R^{abij}, \\ \tilde{R}^4 &:= \sum R_{ikjl} R^k{}_{m}{}^l{}_n R^m{}_a{}^n{}_b R^{abij}, \end{aligned}$$

and so on. We take the local expression for tensors in a normal coordinate at the center. In the following we follow the Einstein's convention on summation.

The relations characterizing a locally symmetric Einstein space are

$$(1.4) \quad \nabla_i R_{jklm} = 0,$$

$$(1.5) \quad \rho_{ij} = \frac{\tau}{n} \delta_{ij},$$

where ρ_{ij} is the Ricci curvature tensor. We use notations a_i and u_i in the sense of [2],[12];

$$(1.6) \quad a_i = \int_M u_i dv.$$

In the following we compute a_4 for a locally symmetric Einstein space (M, g) . Since (M, g) is locally symmetric, all we have to do is to compute u_4 . In computing u_4 we shall find what kind of terms appear in u_4 . Taylor asymptotic expansion of g_{ij} tells that (see e.g. [2],[12])

(1) each term consists of a function which is obtained by contracting four curvature tensors,

(2) the coefficient of each term is independent of the shape of the Riemannian manifold (however depends on the dimension n because of (1.5)).

We use a notation $(abcd)$ for the curvature tensor instead of R_{abcd} , that is, discribe only indices. We try to classify the contraction of four curvature tensors. We begin with the classification in the case of two curvature tensors. We obtain

$$\begin{aligned}
 (1.7) \quad & (abcd)(abcd) = R^2, \\
 & -R^2 = (abcd)(abdc) = (abcd)(bacd) = (abcd)(dcab) \\
 & \quad = (abcd)(cdba), \\
 & R^2 = (abcd)(badc) = (abcd)(cdab) = (abcd)(dcba).
 \end{aligned}$$

By the Bianchi identity

$$(1.8) \quad (abcd)(abcd) = 2(abcd)(acbd) = -2(abcd)(adbc),$$

we obtain

$$\begin{aligned}
 (1.9) \quad & -\frac{1}{2}R^2 = (abcd)(acdb) = (abcd)(cabd) = (abcd)(bdca) \\
 & \quad = (abcd)(dbac) = (abcd)(dacb) = (abcd)(bcad) \\
 & \quad = (abcd)(cbda), \\
 & \frac{1}{2}R^2 = (abcd)(cadb) = (abcd)(bdac) = (abcd)(dbca) \\
 & \quad = (abcd)(dabc) = (abcd)(adcb) = (abcd)(bcd a) \\
 & \quad = (abcd)(cbad).
 \end{aligned}$$

So in this case the independent factor is only R^2 . As for the case containing τ , we obtain

$$(1.10) \quad (abca)(dbcd) = \frac{\tau^2}{n}, \quad (abba)(dccc) = \tau^2.$$

In the case of three curvature tensors, we obtain the following relations by the Bianchi identity,

$$\begin{aligned}
 (1.11) \quad & R^3 = (abcd)(cdef)(efab) = 2(abcd)(cdef)(eafb) \\
 & \quad = 4(abcd)(cedf)(eafb), \\
 & \tilde{R}^3 = (acbd)(cedf)(eafb) = (adbc)(cedf)(eafb) + \frac{1}{4}R^3.
 \end{aligned}$$

By the Ricci identity we obtain

$$\begin{aligned}
 (1.12) \quad & 0 = \nabla_u \nabla_v R_{uabc} - \nabla_v \nabla_u R_{uabc} \\
 & \quad = R_{uvlu} R_{labc} + R_{uvla} R_{ulbc} + R_{uvlb} R_{ualc} + R_{uvlc} R_{uabl},
 \end{aligned}$$

and by multiplying R_{vabc} to the both side of (1.12) and contracting, we obtain

$$(1.13) \quad \tilde{R}^3 = -\frac{1}{2} \frac{\tau}{n} R^2 - \frac{1}{4} R^3.$$

In the case of four curvature tensors, we classify them into three types except the cases containing τ and R^2 . In each type we neglect the difference of the arrangement of the indices in R_{abcd} . They are

$$(1.14) \quad \text{3-type:} \quad (abc)(def)(vabc)(vdef),$$

$$(1.15) \quad (2, 2)\text{-type:} \quad (abcd)(cdef)(efgh)(ghab),$$

$$(1.16) \quad (1, 1, 2)\text{-type:} \quad (abcd)(abkl)(uvck)(uvdl).$$

Remark. The cases containing τ and R^2 are

$$(1.17) \quad \tau^4, \quad \tau^2 R^2, \quad \tau R^3 \quad \text{and} \quad R^2 R^2.$$

2. The classification in the case of four curvature tensors. In the following we obtain the relations among the changes of the arrangement of indices.

Step 1: The relations obtained by using the Bianchi identity. As for 3-type:

$$(2.1) \quad \begin{aligned} (abc)(def)(vabc)(vdef) &= 2(abc)(uedf)(vabc)(vdef) \\ &= 4(ubac)(uedf)(vabc)(vdef) \\ &= 4(ubac)(uedf)(vabc)(vedf). \end{aligned}$$

As for (2, 2)-type: We denote

$$(2.2) \quad R^4 = (abcd)(cdef)(efgh)(ghab), \quad \tilde{R}^4 = (acbd)(cedf)(egfh)(gahb),$$

and get

$$(2.3) \quad \begin{aligned} R^4 &= 2(abcd)(cdef)(efgh)(gahb) = 4(abcd)(cdef)(egfh)(gahb) \\ &= 4(abcd)(cedf)(efgh)(gahb) = 8(abcd)(cedf)(egfh)(gahb), \end{aligned}$$

$$(2.4) \quad \begin{aligned} \tilde{R}^4 &= (adbc)(cfde)(egfh)(gahb) = (adbc)(cedf)(ehfg)(gahb) \\ &= (adbc)(cedf)(egfh)(gahb) + \frac{1}{8} R^4. \end{aligned}$$

As for (1, 1, 2)-type:

$$\begin{aligned}
(2.5) \quad & (abcd)(abkl)(uvck)(uvdl) \\
& = 2(abcd)(abkl)(uvck)(udvl) = 4(acbd)(abkl)(ucvk)(uvdl) \\
& = 4(abcd)(akbl)(ucvk)(uvdl), \\
(2.6) \quad & (acbd)(akbl)(uvck)(uvdl) \\
& = \frac{1}{2}(abcd)(abkl)(uvck)(uvdl) + (adbk)(akbl)(uvck)(uvdl), \\
(2.7) \quad & (acbd)(akbl)(ucvk)(uvdl) \\
& = (acbd)(akbl)(uvck)(uvdl) + (acbd)(akbl)(ukvc)(uvdl) \\
& = -(acbd)(akbl)(ukvc)(uvdl), \\
& 2(acbd)(akbl)(ucvk)(uvdl) \\
& = (acbd)(akbl)(uvck)(uvdl), \\
(2.8) \quad & (adbk)(akbl)(ucvk)(uvdl) \\
& = -\frac{1}{4}(abcd)(abkl)(uvck)(uvdl) + \frac{1}{2}(acbd)(akbl)(uvck)(uvdl), \\
(2.9) \quad & (acbd)(akbl)(ucvk)(udvl) \\
& = \frac{1}{2}(acbd)(akbl)(uvck)(uvdl) + (acbd)(akbl)(ucvk)(ulvd), \\
& (adbk)(akbl)(ukvc)(udvl) \\
& = (acbd)(akbl)(ucvk)(udvl) - (acbd)(akbl)(uvck)(uvdl) \\
& \quad + \frac{1}{4}(abcd)(abkl)(uvck)(uvdl), \\
& (acbd)(akbl)(ucvk)(ulvd) \\
& = (acbd)(albk)(ukvc)(ulvd), \\
& (acbd)(akbl)(ucvk)(udvl) \\
& = (acbd)(akbl)(ukvc)(ulvd) = (adbk)(albk)(ukvc)(udvl).
\end{aligned}$$

Then we may choose independent factors in each type, and use the following notations:

$$\begin{aligned}
3\text{-type} \quad & (t) := (uabc)(udef)(vabc)(vdef), \\
(2, 2)\text{-type} \quad & R^4 := (abcd)(cdef)(efgh)(ghab), \\
& \tilde{R}^4 := (acbd)(cedf)(egfh)(gahb), \\
(1, 1, 2)\text{-type} \quad & (a) := (abcd)(abkl)(uvck)(uvdl), \\
& (b) := (acbd)(akbl)(uvck)(uvdl), \\
& (c) := (acbd)(akbl)(ucvk)(udvl).
\end{aligned}$$

Step 2: The relations obtained by using the Ricci identity.

Proposition 1. *We obtain the following relations:*

$$(2.10) \quad (a) + 2(b) = (t),$$

$$(2.11) \quad \frac{\tau}{n}R^3 + \frac{1}{2}R^4 = -2(b),$$

$$(2.12) \quad \frac{1}{2}\frac{\tau^2}{n^2}R^2 + \frac{1}{4}\frac{\tau}{n}R^3 = \tilde{R}^4 + \frac{3}{2}(b) - (c) - \frac{1}{4}(a).$$

Proof. By the Ricci identity and $\nabla_i R_{abcd} = 0$, we obtain

$$(2.13) \quad \begin{aligned} & (uv al)(l b c d)(k b c d)(u v k a) + (u v b l)(a l c d)(k b c d)(u v k a) \\ & + (u v c l)(a b l d)(k b c d)(u v k a) + (u v d l)(a b c l)(k b c d)(u v k a) = 0, \\ & (u v u l)(l b c d)(c d g h)(g h v b) + (u v b l)(u l c d)(c d g h)(g h v b) \\ & + (u v c l)(u b l d)(c d g h)(g h v b) + (u v d l)(u b c l)(c d g h)(g h v b) = 0, \\ & (u v u l)(l b c d)(v g c h)(g b h d) + (u v b l)(u l c d)(v g c h)(g b h d) \\ & + (u v c l)(u b l d)(v g c h)(g b h d) + (u v d l)(u b c l)(v g c h)(g b h d) = 0. \end{aligned}$$

Then we apply (2.1), ..., (2.9) to the above.

Therefore we can choose the following sets of independent factors to describe the terms of u_4 as

$$(2.14a) \quad (a), (b), (c), \quad \text{or} \quad (2.14b) \quad (t), R^4, (c),$$

and can set u_4 as

$$(2.15) \quad u_4 = \left(c_1 + \frac{c_2}{n} + \frac{c_3}{n^2} + \frac{c_4}{n^3}\right)\tau^4 + \left(c_5 + \frac{c_6}{n} + \frac{c_7}{n^2}\right)\tau^2 R^2 \\ + \left(c_8 + \frac{c_9}{n}\right)\tau \tilde{R}^3 + c_{10} R^2 R^2 + c_{11}(a) + c_{12}(b) + c_{13}(c),$$

where c_1, c_2, \dots, c_{13} are constants.

3. The calculation of u_4 . We use the following data;

(1) Sphere $S^n(1)$ of radius 1:

$$(3.1) \quad \begin{aligned} \tau &= n(n-1), & R^2 &= 2n(n-1), \\ R^3 &= -4n(n-1), & R^4 &= 8n(n-1), \\ \tilde{R}^3 &= -n(n-1)(n-2), & \tilde{R}^4 &= n(n-1)(n^2-3n+4), \\ (a) &= 4n(n-1), & (b) &= 2n(n-1)(n-2), \\ (c) &= n(n-1)(3n-5), & (t) &= 4n(n-1)^2, \end{aligned}$$

(2) Complex projective space $CP^n(4)$ ($m = 2n$) of constant holomorphic sectional curvature 4:

$$\begin{aligned}
 (3.2) \quad & \tau = m(m+2), & R^2 &= 8m(m+2), \\
 & R^3 = -8m(m+2)(m+6), & R^4 &= 16m(m+2)(m^2+6m+16), \\
 & \tilde{R}^3 = -2m(m+2)(m-2), & \tilde{R}^4 &= 2m(m+2)(m^2+6m+48), \\
 & (a) = 16m(m+2)(3m+10), & (b) &= 8m(m+2)(m-2), \\
 & (c) = 4m(m+2)(3m+10), & (t) &= 64m(m+2)^2.
 \end{aligned}$$

Computation of u_4 is divided into the following steps.

Step 1: We express u_4 for $S^n(1)$ as a polynomial of n .

Step 2: We express u_4 for $S^n(1) \times S^n(1)$, $S^n(1) \times S^n(1) \times S^n(1)$ and $\times^4 S^n(1)$.

Step 3: We express the curvature data as polynomials of n .

Step 4: We make up the system of equations for c_1, c_2, \dots, c_{13} .

Step 5: We carry out the same procedure for $S^2(1) \times S^6(\sqrt{5})$, etc.

Step 6: We carry out the same procedure for $CP^4(4)$, etc.

Step 1: Take a normal coordinate system and r denotes the distance to the center y of the normal coordinate neighbourhood. Then on $S^n(1)$ or $CP^n(4)$

$$(3.3) \quad v(r) = (\det(g_{ij}(y)))^{-\frac{1}{4}},$$

depends only on r . In fact, on $S^n(1)$ we obtain $v(r) = (\sin r/r)^{(1-n)/2}$ and on $CP^n(4)$ we have $v(r) = (\cos r)^{-1/2}(\sin r/r)^{(1-2n)/2}$. By Taylor asymptotic expansion of $v(r)$ (see e.g. [1]), on $S^n(1)$ we obtain

$$\begin{aligned}
 (3.4) \quad & u_0 = 1, \\
 & u_1 = \frac{1}{6}n(n-1), \\
 & u_2 = \frac{1}{360}n(n-1)(5n^2-7n+6), \\
 & u_3 = \frac{1}{45360}n(n-1)(35n^4-112n^3+187n^2-110n+96), \\
 & u_4 = \frac{1}{5443200}n(n-1)(175n^6-945n^5+2389n^4 \\
 & \quad - 3111n^3+3304n^2-516n+2160).
 \end{aligned}$$

Step 2: We use the following formula (see [2])

$$u_i(M \times N) = \sum_{i_1+i_2=i} u_{i_1}(M)u_{i_2}(N).$$

Then on $S^n(1) \times S^n(1)$, $\overset{3}{\times} S^n(1)$ and $\overset{4}{\times} S^n(1)$, we obtain the following, respectively;

$$(3.5) \quad \begin{aligned} u_4 &= \frac{1}{340200} n(n-1)(175n^6 - 735n^5 + 1516n^4 - 1638n^3 \\ &\quad + 1243n^2 - 399n + 270), \\ u_4 &= \frac{1}{604800} n(n-1)(1575n^6 - 5985n^5 + 10789n^4 \\ &\quad - 10359n^3 + 6368n^2 - 1956n + 720), \\ u_4 &= \frac{1}{85050} n(n-1)(700n^6 - 2502n^5 + 4141n^4 - 3630n^3 \\ &\quad + 1924n^2 - 534n + 135). \end{aligned}$$

Step 3 and 4: By putting (3.1) into (2.15), we obtain the following formulas for u_4 on $S^n(1)$, $S^n(1) \times S^n(1)$, $\overset{3}{\times} S^n(1)$ and $\overset{4}{\times} S^n(1)$, respectively;

$$(3.6) \quad \begin{aligned} u_4 &= n(n-1)\{c_1n^6 + (c_2 - 3c_1)n^5 + (3c_1 - 3c_2 + c_3 + 2c_5)n^4 \\ &\quad - (c_1 - 3c_2 + 3c_3 - c_4 + 4c_5 - 2c_6 + c_8)n^3 \\ &\quad - (c_2 - 3c_3 + 3c_4 - 2c_5 + 4c_6 - 2c_7 - 3c_8 + c_9 - 4c_{10})n^2 \\ &\quad - (c_3 - 3c_4 - 2c_6 + 4c_7 + 2c_8 - 3c_9 + 4c_{10} - 2c_{12} - 3c_{13})n \\ &\quad - c_4 + 2c_7 - 2c_9 + 4c_{11} - 4c_{12} - 5c_{13}\}, \\ u_4 &= n(n-1)\{16c_1n^6 + (8c_2 - 48c_1)n^5 + (48c_1 - 24c_2 + 4c_3 + 16c_5)n^4 \\ &\quad - (16c_1 - 24c_2 + 12c_3 - 2c_4 + 32c_5 - 8c_6 + 4c_8)n^3 \\ &\quad - (8c_2 - 12c_3 + 6c_4 - 16c_5 + 16c_6 - 4c_7 - 12c_8 + 2c_9 - 16c_{10})n^2 \\ &\quad - (4c_3 - 6c_4 - 8c_6 + 8c_7 + 8c_8 - 6c_9 + 16c_{10} - 4c_{12} - 6c_{13})n \\ &\quad + (\text{lower order terms})\}, \\ u_4 &= n(n-1)\{(\text{higher order terms}) \\ &\quad - (27c_2 - 27c_3 + 9c_4 - 54c_5 + 36c_6 - 6c_7 - 27c_8 + 3c_9 - 36c_{10})n^2 \\ &\quad - (9c_3 - 9c_4 - 18c_6 + 12c_7 + 18c_8 - 9c_9 + 36c_{10} - 6c_{12} - 9c_{13})n \\ &\quad + (\text{lower order terms})\}, \\ u_4 &= n(n-1)\{(\text{higher order terms}) \\ &\quad - (64c_2 - 48c_3 + 12c_4 - 128c_5 + 64c_6 - 8c_7 - 48c_8 + 4c_9 - 64c_{10})n^2 \\ &\quad - (16c_3 - 12c_4 - 32c_6 + 16c_7 + 32c_8 - 12c_9 + 64c_{10} - 8c_{12} - 12c_{13})n \\ &\quad + (\text{lower order terms})\}. \end{aligned}$$

We can compare the coefficients of the polynomials of n for u_4 in (3.4),

(3.5) and (3.6), then we obtain

$$\begin{aligned}
 c_1 &= \frac{1}{31104}, & c_2 &= -\frac{1}{12960}, & c_3 &= -\frac{59}{1360800}, \\
 c_4 &= -\frac{1}{113400}, & c_5 &= \frac{1}{12960}, & c_6 &= (\text{parameter}), \\
 (3.7) \quad c_7 &= (\text{parameter}), & c_8 &= \frac{41}{340200} + 2c_6, & c_9 &= \frac{1}{5670} + 2c_7, \\
 c_{10} &= \frac{1}{13608} - \frac{c_6}{2}, & c_{11} &= (\text{parameter}), & c_{12} &= 2c_7 + 6c_{11} - \frac{4}{4725}, \\
 c_{13} &= -2c_7 - 4c_{11} + \frac{1}{1890},
 \end{aligned}$$

where c_6 , c_7 and c_{11} are undetermined. So we express them as parameters.

Step 5: We have to treat the product of spheres of distinct dimensions.

Proposition 2. $S^k(\sqrt{k-1}) \times S^l(\sqrt{l-1})$ is a locally symmetric Einstein space, where $S^n(r)$ denotes the sphere of radius r .

We obtain the following table.

Table 1

	$S^2(1) \times S^6(\sqrt{5})$	$S^3(\sqrt{2}) \times S^5(2)$	$\tilde{\chi}^2 S^3(\sqrt{2}) \times S^2(1)$	$\tilde{\chi}^2 S^2(1) \times S^4(\sqrt{3})$
τ^4	4096	4096	4096	4096
$\tau^2 R^2$	$\frac{2048}{5}$	352	640	$\frac{2048}{3}$
$\tau \tilde{R}^3$	$-\frac{192}{25}$	$-\frac{27}{2}$	-12	$-\frac{64}{9}$
$R^2 R^2$	$\frac{1024}{25}$	$\frac{121}{4}$	100	$\frac{1024}{9}$
(a)	$\frac{1024}{125}$	$\frac{29}{16}$	11	$\frac{448}{27}$
(b)	$\frac{48}{125}$	$\frac{39}{32}$	$\frac{3}{2}$	$\frac{16}{27}$
(c)	$\frac{328}{125}$	$\frac{73}{32}$	5	$\frac{136}{27}$
u_4	$\frac{74243}{590625}$	$\frac{15}{128}$	$\frac{41}{280}$	$\frac{19541}{127575}$

Now we restrict the dimension to $n = 8$. Then we obtain

$$(3.8) \quad u_4 = d_1 \tau^4 + d_2 \tau^2 R^2 + d_3 \tau \tilde{R}^3 + d_4 R^2 R^2 + d_5(a) + d_6(b) + d_7(c),$$

where

$$\begin{aligned}
d_1 &= c_1 + \frac{c_2}{8} + \frac{c_3}{64} + \frac{c_4}{512} = \frac{3799}{174182400}, \\
d_2 &= c_5 + \frac{c_6}{8} + \frac{c_7}{64} = \frac{1}{12960} + \frac{c_6}{8} + \frac{c_7}{64}, \\
d_3 &= c_8 + \frac{c_9}{8} = \frac{97}{680400} + 2c_6 + \frac{c_7}{4}, \\
d_4 &= c_{10} = \frac{1}{13608} - \frac{c_6}{2}, & d_5 &= c_{11}, \\
d_6 &= c_{12} = 2c_7 + 6c_{11} - \frac{4}{4725}, \\
d_7 &= c_{13} = -2c_7 - 4c_{11} + \frac{1}{1890}.
\end{aligned}$$

By putting the data in Table 1 into (3.8), we obtain

$$\begin{aligned}
(3.9) \quad d_1 &= \frac{3799}{174182400}, & d_2 &= (\text{parameter}), \\
d_3 &= 16d_2 - \frac{743}{680400}, & d_4 &= \frac{1}{64800}, \\
d_5 &= (\text{parameter}), & d_6 &= 128d_2 + 6d_5 - \frac{107}{8505}, \\
d_7 &= -128d_2 - 4d_5 + \frac{149}{12150},
\end{aligned}$$

where d_2 and d_5 are undetermined. So we express them as parameters.

Step 6: We carry out the following calculation. On $CP^n(4)$ we obtain $v(r) = (\cos r)^{-1/2}(\sin r/r)^{(1-2n)/2}$. By Taylor asymptotic expansion we obtain on $CP^2(4)$ and $CP^4(4)$, respectively;

$$\begin{aligned}
(3.10) \quad u_0 &= 1, \quad u_1 = 4, \quad u_2 = \frac{124}{15}, \quad u_3 = \frac{3856}{315}, \quad u_4 = \frac{5008}{315}, \\
u_0 &= 1, \quad u_1 = \frac{40}{3}, \quad u_2 = 88, \quad u_3 = 384, \quad u_4 = \frac{1184368}{945}.
\end{aligned}$$

Then we obtain $u_4 = 103984/525$ on $CP^2(4) \times CP^2(4)$. By putting (3.10) and (3.2) into (3.9), we obtain (as for the curvature data refer to Table 2

below)

$$(3.11) \quad d_2 = \frac{101}{1088640}, \quad d_5 = \frac{11}{113400},$$

$$\left(\begin{array}{l} c_6 = \frac{79}{680400}, \quad c_7 = \frac{1}{14175}, \quad c_8 = \frac{1}{2835}, \quad c_9 = \frac{1}{3150} \\ c_{10} = \frac{1}{64800}, \quad c_{11} = \frac{11}{113400}, \quad c_{12} = -\frac{1}{8100}, \quad c_{13} = 0 \end{array} \right).$$

By the Ricci identity (1.13), (2.10) and (2.11), we have

$$(3.12) \quad \begin{aligned} \tau \tilde{R}^3 &= -\frac{1}{2} \frac{\tau^2}{n} R^2 - \frac{1}{4} \tau R^3, \\ (a) &= (t) + \frac{\tau}{n} R^3 + \frac{1}{2} R^4, \\ (b) &= -\frac{1}{4} R^4 - \frac{1}{2} \frac{\tau}{n} R^3. \end{aligned}$$

Then we obtain

$$(3.13) \quad \begin{aligned} u_4 &= \left(\frac{1}{31104} - \frac{1}{12960} \frac{1}{n} - \frac{59}{1360800} \frac{1}{n^2} - \frac{1}{113400} \frac{1}{n^3} \right) \tau^4 \\ &\quad + \left(\frac{1}{12960} - \frac{41}{680400} \frac{1}{n} - \frac{1}{11340} \frac{1}{n^2} \right) \tau^2 R^2 \\ &\quad + \left(-\frac{1}{11340} + \frac{1}{12600} \frac{1}{n} \right) \tau R^3 + \frac{1}{64800} R^2 R^2 \\ &\quad + \frac{11}{113400} (t) + \frac{1}{12600} R^4 + 0(c), \end{aligned}$$

for an n -dimensional locally symmetric Einstein space. Especially in the case of $n = 8$, we obtain

$$(3.14) \quad \begin{aligned} u_4 &= \frac{3799}{174182400} \tau^4 + \frac{743}{10886400} \tau^2 R^2 - \frac{71}{907200} \tau R^3 + \frac{1}{64800} R^2 R^2 \\ &\quad + \frac{11}{113400} (t) + \frac{1}{12600} R^4 + 0(c). \end{aligned}$$

Remark. Avramidi [1] also gave the explicit expression of u_4 . But it is too complicated to apply for geometry.

Remark. For an n -dimensional locally symmetric Einstein space

$$(3.15) \quad \begin{aligned} u_3 &= \left(\frac{1}{1296} - \frac{1}{1080} \frac{1}{n} - \frac{1}{2835} \frac{1}{n^2} \right) \tau^3 \\ &\quad + \left(\frac{1}{1080} - \frac{1}{5670} \frac{1}{n} \right) \tau R^2 - \frac{1}{1890} R^3, \end{aligned}$$

was given by Sakai [12].

4. The calculation of $\chi(M)$. We calculate the Euler characteristic $\chi(M)$ of an 8-dimensional locally symmetric Einstein space. We know that $\chi(M^8)$ is given by the following

$$(4.1) \quad \begin{aligned} \chi(M^8) &= \frac{1}{2^{12}\pi^4 4!} \int_M \varepsilon_{i_1 \dots i_8} \varepsilon_{j_1 \dots j_8} R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8} dv \\ &= \frac{1}{\pi^4} \int_M (e_1 \tau^4 + e_2 \tau^2 R^2 + e_3 \tau \tilde{R}^3 + e_4 R^2 R^2 \\ &\quad + e_5(a) + e_6(b) + e_7(c)) dv, \end{aligned}$$

where e_1, \dots, e_7 are constants. We take the following models

- (1) $S^8(1)$, (2) $S^4(1) \times S^4(1)$, (3) $\overset{4}{\times} S^2(1)$,
 (4) $S^2(1) \times S^6(\sqrt{5})$, (5) $S^3(\sqrt{2}) \times S^5(2)$, (6) $\overset{2}{\times} S^3(\sqrt{2}) \times S^2(1)$,
 (7) $\overset{2}{\times} S^2(1) \times S^4(\sqrt{3})$, (8) $CP^4(4)$, (9) $CP^2(4) \times CP^2(4)$.

By (3.1) and (3.2) we obtain the following table.

Table 2

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
τ^4	9834496	331776	4096	4096	4096	4096	4096	40960000	5308416
$\tau^2 R^2$	351232	27648	1024	$\frac{2048}{5}$	352	640	$\frac{2048}{3}$	4096000	884736
$\tau \tilde{R}^3$	-18816	-1152	0	$-\frac{192}{25}$	$-\frac{27}{2}$	-12	$-\frac{64}{9}$	-76800	-9216
$R^2 R^2$	12544	2304	256	$\frac{1024}{25}$	$\frac{121}{4}$	100	$\frac{1024}{9}$	409600	147456
(a)	224	96	32	$\frac{1024}{125}$	$\frac{29}{16}$	11	$\frac{448}{27}$	43520	16896
(b)	672	96	0	$\frac{48}{125}$	$\frac{39}{32}$	$\frac{3}{2}$	$\frac{16}{27}$	3840	768
(c)	1064	168	8	$\frac{328}{125}$	$\frac{73}{32}$	5	$\frac{136}{27}$	10880	4224
$\frac{\chi(M)}{\text{Vol}}$	$\frac{105}{16\pi^4}$	$\frac{9}{16\pi^4}$	$\frac{1}{16\pi^4}$	$\frac{3}{400\pi^4}$	0	0	$\frac{1}{48\pi^4}$	$\frac{120}{\pi^4}$	$\frac{36}{\pi^4}$

We put the data in Table 2 into (4.1). For the spherical data of (1), ..., (7) we have

$$\begin{aligned}
e_1 &= -\frac{1}{98304}, \quad e_2 = (\text{parameter}), \quad e_3 = 16e_2 - \frac{23}{3072}, \\
(4.2) \quad e_4 &= \frac{1}{2048}, \quad e_5 = -32e_2 - \frac{e_7}{4} - \frac{1}{1536}, \quad e_6 = -62e_2 - \frac{3}{2}e_7 - \frac{23}{384}, \\
e_7 &= (\text{parameter}),
\end{aligned}$$

where e_2 and e_7 are undetermined. So we express them as parameters. By adding (8) and (9) we obtain

$$(4.3) \quad e_2 = -\frac{1}{49152}, \quad e_3 = -\frac{1}{128}, \quad e_5 = \frac{1}{512}, \quad e_6 = -\frac{3}{64}, \quad e_7 = -\frac{1}{128}.$$

By (3.12) we express $\chi(M)$ in another form

$$\begin{aligned}
(4.4) \quad \chi(M^8) &= \frac{1}{\pi^4} \int_M \left(-\frac{1}{98304} \tau^4 + \frac{23}{49152} \tau^2 R^2 + \frac{21}{4096} \tau R^3 \right. \\
&\quad \left. + \frac{1}{2048} R^2 R^2 + \frac{1}{512} (t) + \frac{13}{1024} R^4 - \frac{1}{128} (c) \right) dv.
\end{aligned}$$

Remark. For a 6-dimensional locally symmetric Einstein space Sakai [12] obtained

$$(4.5) \quad \chi(M^6) = \frac{1}{\pi^3} \int_M \left(\frac{1}{3456} \tau^3 - \frac{5}{1152} \tau R^2 - \frac{1}{64} R^3 \right) dv.$$

Similarly we can calculate the signature $\sigma(M^8)$. By the following fact

$$(4.6) \quad \sigma(S^{2n}) = 0, \quad \sigma(CP^{2n}) = 1, \quad \sigma(M \times N) = \sigma(M)\sigma(N),$$

we obtain for the signature

$$(4.7) \quad \sigma(M^8) = \frac{1}{\pi^4} \int_M \left(-\frac{17}{92160} \tau R^3 - \frac{7}{11520} R^4 + \frac{1}{2880} (t) - \frac{7}{2880} (c) \right) dv,$$

up to sign for an oriented 8-dimensional locally symmetric Einstein space.

Remark. For a 4-dimensional Kähler Einstein space we had better refer to Donnelly's paper [4].

$$(4.8) \quad \chi(M^4) = \frac{1}{32\pi^2} \int_M R^2 dv, \quad \sigma(M^4) = \frac{1}{96\pi^2} \int_M (\tau^2 - 2R^2) dv.$$

Remark. Lovelock [9] also gave the explicit expression of $\chi(M^8)$ by direct tensor calculation.

5. a_4 for 1-form. We consider the spectrum of the Laplacian for 1-forms. Similarly we can treat the asymptotic expansion for the trace of the heat kernel [5]. The coefficients a_i^1 contain geometric informations and are spectral invariants. a_2^1 was calculated by Patodi [11]. a_3^1 was calculated by Li [7] (for an Einstein space). Their approach is different from each other. The one is a combinatorial method and the other is a method using Taylor asymptotic expansion. In this section we calculate a_4^1 on an n -dimensional locally symmetric Einstein space by a combinatorial method.

Step 1: We can set

$$(5.1) \quad u_4^1 = c_1(n)\tau^4 + c_2(n)\tau^2 R^2 + c_3(n)\tau R^3 + c_4(n)R^2 R^2 \\ + c_5(n)(t) + c_6(n)R^4 + c_7(n)(c).$$

If we express u_4^1 as a polynomial of the independent contracted values of the product of for curvature tensors in a locally symmetric space, then its coefficients are polynomials of degree 1. However, in our Einstein case, by (1.5) the coefficients $c_i(n)$ corresponding to the term containing τ in (5.1) are polynomials containing powers of the factor $1/n$ (see [7]).

Step 2: On the product space $M \times M$, we have ([11])

$$(5.2) \quad u_4^1(M \times M) = 2 \sum_{i+j=4} u_i^1(M) u_j^0(M) \\ = 2(u_4^1 u_0 + u_3^1 u_1 + u_2^1 u_2 + u_1^1 u_3 + u_0^1 u_4).$$

In the above we denote $u_i^p = u_i^p(M)$ for the sake of simplicity.

$$(5.3) \quad \begin{aligned} \tau(M \times M) &= 2\tau(M), & R^2(M \times M) &= 2R^2(M), \\ R^3(M \times M) &= 2R^3(M), & R^4(M \times M) &= 2R^4(M), \\ (t)(M \times M) &= 2(t)(M), & (c)(M \times M) &= 2(c)(M). \end{aligned}$$

In the following we denote simply $\tau = \tau(M)$, $R^2 = R^2(M)$, etc.

Step 3: On the space M^n , u_i and u_i^1 take the following forms;

$$\begin{aligned}
 (5.4) \quad & u_0 = 1, \quad u_1 = \frac{1}{6}\tau, \\
 & u_2 = \left(\frac{1}{72} - \frac{1}{180}\frac{1}{n}\right)\tau^2 + \frac{1}{180}R^2, \\
 & u_3 = \left(\frac{1}{1296} - \frac{1}{1080}\frac{1}{n} - \frac{1}{2835}\frac{1}{n^2}\right)\tau^3 + \left(\frac{1}{1080} - \frac{1}{5670}\frac{1}{n}\right)\tau R^2 \\
 & \quad - \frac{1}{1890}R^3, \\
 & u_4 = \left(\frac{1}{31104} - \frac{1}{12960}\frac{1}{n} - \frac{59}{1360800}\frac{1}{n^2} - \frac{1}{113400}\frac{1}{n^3}\right)\tau^4 \\
 & \quad + \left(\frac{1}{12960} - \frac{41}{680400}\frac{1}{n} - \frac{1}{11340}\frac{1}{n^2}\right)\tau^2 R^2 \\
 & \quad + \left(-\frac{1}{11340} + \frac{1}{12600}\frac{1}{n}\right)\tau R^3 + \frac{1}{64800}R^2 R^2 \\
 & \quad + \frac{11}{113400}(t) + \frac{1}{12600}R^4,
 \end{aligned}$$

$$\begin{aligned}
 (5.5) \quad & u_0^1 = n, \quad u_1^1 = \left(\frac{n}{6} - 1\right)\tau, \\
 & u_2^1 = \left(\frac{n}{72} - \frac{31}{180} + \frac{1}{2}\frac{1}{n}\right)\tau^2 + \left(\frac{n}{180} - \frac{1}{12}\right)R^2, \\
 & u_3^1 = \left(\frac{n}{1296} - \frac{2}{135} + \frac{251}{2835}\frac{1}{n} - \frac{1}{6}\frac{1}{n^2}\right)\tau^3 \\
 & \quad + \left(\frac{n}{1080} - \frac{89}{4536} + \frac{7}{90}\frac{1}{n}\right)\tau R^2 + \left(-\frac{n}{1890} + \frac{1}{120}\right)R^3, \\
 & \quad (u_3^1 \text{ was obtained by Li [7]}).
 \end{aligned}$$

Then we put (5.3), ..., (5.5) into (5.2) and obtain the following

$$\begin{aligned}
 (5.6) \quad & 16c_1(2n)\tau^4 + 8c_2(2n)\tau^2 R^2 + 4c_3(2n)\tau R^3 + 4c_4(2n)R^2 R^2 \\
 & + 2c_5(2n)(t) + 2c_6(2n)R^4 + 2c_7(2n)(c) \\
 & = \left(2c_1(n) + \frac{5n}{5184} - \frac{77}{6480} + \frac{10651}{226800}\frac{1}{n} - \frac{571}{9450}\frac{1}{n^2}\right)\tau^4 \\
 & + \left(2c_2(n) + \frac{7n}{6480} - \frac{2917}{226800} + \frac{739}{22680}\frac{1}{n}\right)\tau^2 R^2 \\
 & + \left(2c_3(n) - \frac{n}{1890} + \frac{151}{37800}\right)\tau R^3 + \left(2c_4(n) + \frac{n}{10800} - \frac{1}{1080}\right)R^2 R^2 \\
 & + \left(2c_5(n) + \frac{11n}{56700}\right)(t) + \left(2c_6(n) + \frac{n}{6300}\right)R^4 + 2c_7(n)(c).
 \end{aligned}$$

Step 4: Since the curvature data $\tau^4, \dots, (c)$ are independent, we can compare the coefficients. For example

$$(5.7) \quad 16c_1(2n) = 2c_1(n) + \frac{5n}{5184} - \frac{77}{6480} + \frac{10651}{226800} \frac{1}{n} - \frac{571}{9450} \frac{1}{n^2}.$$

Then we can set $c_1(n) = p_1 n + p_2 + p_3/n + p_4/n^2 + c_1/n^3$, and we obtain $p_1 = 1/31104$, $p_2 = -11/12960$, $p_3 = 10651/1360800$, $p_4 = -571/18900$, $c_1 = (\text{parameter})$, that is,

$$(5.8) \quad c_1(n) = \frac{n}{31104} - \frac{11}{12960} + \frac{10651}{1360800} \frac{1}{n} - \frac{571}{18900} \frac{1}{n^2} + \frac{c_1}{n^3}.$$

Similarly

$$(5.9) \quad \begin{aligned} c_2(n) &= \frac{n}{12960} - \frac{2917}{1360800} + \frac{739}{45360} \frac{1}{n} + \frac{c_2}{n^2}, \\ c_3(n) &= -\frac{n}{11340} + \frac{151}{75600} + \frac{c_3}{n}, \\ c_4(n) &= \frac{n}{64800} - \frac{1}{2160} + \frac{c_4}{n}, \end{aligned}$$

(since $R^2 R^2$ does not contain τ , the factor of $1/n$ does not appear, i.e. $c_4 = 0$.)

$$c_5(n) = \frac{11n}{113400} + c_5, \quad c_6(n) = \frac{n}{12600} + c_6, \quad c_7(n) = c_7.$$

Step 5: To determine c_1, c_2, \dots, c_7 we need the explicit data of u_4^1 for some model spaces. We calculate u_4^1 for model spaces by the following formula (5.10)

$$(5.10) \quad Z^1(t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \sim \frac{\text{Vol}(M^n, g)}{(4\pi t)^{\frac{n}{2}}} \sum_{i=0}^{\infty} u_i^1 t^i, \quad t \rightarrow +0,$$

where λ_i 's are the eigenvalues counted with multiplicities of the Laplacian for 1-forms. For the perpose we need the explicit data of the spectrum for $S^n(1)$ and $CP^n(4)$ which are determined by Ikeda [8] (see Table 3).

We use the following formulae (see [3],[10])

$$(5.11) \quad \begin{aligned} \sum_{n=0}^{\infty} (2n+1) e^{-(n+\frac{1}{2})^2 t} &\sim \frac{1}{t} + \sum_{n=0}^{\infty} p_n t^n, \\ \sum_{n=0}^{\infty} 2n e^{-n^2 t} &\sim \frac{1}{t} + \sum_{n=0}^{\infty} q_n t^n, \end{aligned} \quad \sum_{n=-\infty}^{\infty} e^{-n^2 t} \sim \sqrt{\pi t}^{-\frac{1}{2}},$$

Table 3

	Eigenvalue	Multiplicity	Range
$S^n(1)$	$(k+1)(n+k)$	$\frac{(n+2k+1)(n+k-1)!}{(n-1)!(k+1)!}$	$k \geq 0$
	$(k+1)(n+k-2)$	$\frac{(n+2k-1)(n+k-1)!}{(n-2)!(k-1)!(k+1)(n+k-2)}$	$k \geq 1$
$CP^n(4)$	$4(k+1)(k+n+1)$	$\frac{n(2k+n+2)(k+n)!^2}{(k+1)!^2 n!^2}$	$k \geq 0$
	$4(k+2)(k+n+1)$	$\frac{(2k+n+3)(k+n+1)!(k+n+2)!}{(k+2)^2(k+n+1)^2(k+1)!k!(n-2)!n!}$	$k \geq 0$
	$4k(k+n)$	$\frac{n(2k+n)(k+n-1)!^2}{k!^2 n!^2}$	$k \geq 1$
	$4k(k+n-1)$	$\frac{(2k+n-1)(k+n)!(k+n-1)!}{k^2(k+n-1)^2(k-1)!(k-2)!(n-2)!n!}$	$k \geq 2$

where

$$p_n = \frac{(-1)^n}{(n+1)!} B_{2n+2} \left(1 - \frac{1}{2^{2n+1}}\right), \quad q_n = -\frac{(-1)^n}{(n+1)!} B_{2n+2}.$$

B_i is the Bernolli number, so that $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$. Then we obtain the following table.

Table 4

	u_0^{-1}	u_1^{-1}	u_2^{-1}	u_3^{-1}	u_4^{-1}
$S^2(1)$	2	$-\frac{4}{3}$	$\frac{2}{15}$	$\frac{8}{315}$	$\frac{2}{315}$
$S^3(1)$	3	-3	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$
$S^4(1)$	4	-4	$-\frac{4}{15}$	$\frac{44}{63}$	$\frac{116}{315}$
$S^5(1)$	5	$-\frac{10}{3}$	$-\frac{10}{3}$	$\frac{2}{3}$	$\frac{35}{18}$
$S^6(1)$	6	0	-8	$-\frac{88}{21}$	$\frac{22}{5}$
$CP^2(4)$	4	-8	$-\frac{104}{15}$	$\frac{1312}{315}$	$\frac{4384}{315}$
$CP^3(4)$	6	0	-32	$-\frac{5312}{105}$	$\frac{64}{35}$

By putting (3.1), (3.2) and the data in table 4 into (5.8) and (5.9), we obtain the following system of equations

$$\begin{aligned}
& 2c_1 + 4c_2 - 8c_3 + 8c_5 + 16c_6 + 2c_7 - \frac{4}{945} = \frac{2}{315}, \\
& 48c_1 + 48c_2 - 48c_3 + 48c_5 + 48c_6 + 24c_7 - \frac{17}{24} = \frac{1}{24}, \\
& 324c_1 + 216c_2 - 144c_3 + 144c_5 + 96c_6 + 84c_7 - \frac{2164}{315} = \frac{116}{315}, \\
(5.12) \quad & 1280c_1 + 640c_2 - 320c_3 + 320c_5 + 160c_6 + 200c_7 - \frac{860}{27} = \frac{35}{18}, \\
& 3750c_1 + 1500c_2 - 600c_3 + 600c_5 + 240c_6 + 390c_7 - \frac{10988}{105} = \frac{22}{5}, \\
& 5184c_1 + 6912c_2 - 11520c_3 + 9216c_5 + 21504c_6 + 2112c_7 - \frac{15392}{315} = \frac{4384}{315}, \\
& 24576c_1 + 24576c_2 - 36864c_3 + 24576c_5 + 67584c_6 + 5376c_7 - \frac{45056}{105} = \frac{64}{35}.
\end{aligned}$$

Then we have

$$\begin{aligned}
(5.13) \quad & c_1 = \frac{1}{24}, \quad c_2 = -\frac{107}{3024}, \quad c_3 = -\frac{1}{180}, \quad c_4 = 0, \\
& c_5 = \frac{37}{15120}, \quad c_6 = -\frac{1}{20160}, \quad c_7 = \frac{1}{360},
\end{aligned}$$

$$\begin{aligned}
(5.14) \quad u_4^1 = & \left(\frac{n}{31104} - \frac{11}{12960} + \frac{10651}{1360800} \frac{1}{n} - \frac{571}{18900} \frac{1}{n^2} + \frac{1}{24} \frac{1}{n^3} \right) \tau^4 \\
& + \left(\frac{n}{12960} - \frac{2917}{1360800} + \frac{739}{45360} \frac{1}{n} - \frac{107}{3024} \frac{1}{n^2} \right) \tau^2 R^2 \\
& + \left(-\frac{n}{11340} + \frac{151}{75600} - \frac{1}{180} \frac{1}{n} \right) \tau R^3 + \left(\frac{n}{64800} - \frac{1}{2160} \right) R^2 R^2 \\
& + \left(\frac{11n}{113400} + \frac{37}{15120} \right) (t) + \left(\frac{n}{12600} - \frac{1}{20160} \right) R^4 + \frac{1}{360} (c).
\end{aligned}$$

Remark. c_1, c_2, \dots, c_7 are determined except $S^6(1)$.

6. Applications. Summing up we obtain the following formulae for an oriented 8-dimensional locally symmetric Einstein space (M, g) ;

$$\begin{aligned}
(6.1) \quad a_4 = & \int_M \left(\frac{3799}{174182400} \tau^4 + \frac{743}{10886400} \tau^2 R^2 - \frac{71}{907200} \tau R^3 \right. \\
& \left. + \frac{1}{64800} R^2 R^2 + \frac{11}{113400} (t) + \frac{1}{12600} R^4 + 0(c) \right) dv,
\end{aligned}$$

$$(6.2) \quad a_4^1 = \int_M \left(-\frac{673}{174182400} \tau^4 - \frac{1859}{43545600} \tau^2 R^2 + \frac{271}{453600} \tau R^3 \right. \\ \left. - \frac{11}{32400} R^2 R^2 + \frac{731}{226800} (t) + \frac{59}{100800} R^4 + \frac{1}{360} (c) \right) dv,$$

$$(6.3) \quad \chi(M^8) = \frac{1}{\pi^4} \int_M \left(-\frac{1}{98304} \tau^4 + \frac{23}{49152} \tau^2 R^2 + \frac{21}{4096} \tau R^3 \right. \\ \left. + \frac{1}{2048} R^2 R^2 + \frac{1}{512} (t) + \frac{13}{1024} R^4 - \frac{1}{128} (c) \right) dv,$$

$$(6.4) \quad \sigma(M^8) = \frac{1}{\pi^4} \int_M \left(-\frac{17}{92160} \tau R^3 - \frac{7}{11520} R^4 + \frac{1}{2880} (t) - \frac{7}{2880} (c) \right) dv.$$

Remark. As for the signature the ambiguity of the sign occurs by the orientation of M .

Proposition 3. *Let (M, g) and (M', g') be oriented 8-dimensional locally symmetric Einstein spaces. Assume that $\text{Spec}(M, g) = \text{Spec}(M', g')$ holds. Then if (1) $R^4 = R'^4$ and (2) $(c) = (c)'$ hold, we have $\chi(M) = \chi(M')$ and $|\sigma(M)| = |\sigma(M')|$.*

Proof. By $\text{Spec}(M, g) = \text{Spec}(M', g')$, $a_i = a'_i$ hold for each i . From $a_0 = a'_0$, $a_1 = a'_1$, $a_2 = a'_2$, $a_3 = a'_3$ and the local symmetricity we have

$$(6.5) \quad \begin{aligned} \text{Vol}(M) &= \text{Vol}(M'), & \tau^4 &= \tau'^4, & \tau^2 R^2 &= \tau'^2 R'^2, \\ \tau R^3 &= \tau' R'^3, & R^2 R^2 &= R'^2 R'^2, \end{aligned}$$

and they are constant on M, M' . If (1) and (2) hold, by putting (6.5), (1) and (2) into $a_4 = a'_4$ of (6.1) we obtain $(t) = (t)'$. Then we can conclude $\chi(M) = \chi(M')$ and $|\sigma(M)| = |\sigma(M')|$.

Remark. For 6-dimensional locally symmetric Einstein spaces this proposition holds without conditions (1) and (2) (see [12]).

Proposition 4 (Patodi [11]). *Let (M, g) , (M', g') be closed Riemannian manifolds. Assume that (M, g) is a locally symmetric Einstein space. If $\text{Spec}(M, g) = \text{Spec}(M', g')$ and $\text{Spec}^1(M, g) = \text{Spec}^1(M', g')$ hold, then the other (M', g') is also a locally symmetric Einstein space with the same dimension.*

Proposition 5. *For two oriented closed Riemannian manifolds (M, g) and (M', g') assume that one of them is an 8-dimensional locally symmetric Einstein space. If (M, g) and (M', g') have the same*

spectra for functions and for 1-forms, respectively, i.e., $\text{Spec}(M, g) = \text{Spec}(M', g')$ and $\text{Spec}^1(M, g) = \text{Spec}^1(M', g')$, then (1) $\chi(M) = \chi(M')$ and (2) $|\sigma(M)| = |\sigma(M')|$ are equivalent.

Proof. By Proposition 4 M' is also an 8-dimensional locally symmetric Einstein space. Then we can apply (6.1), ..., (6.4) for M and M' . From the assumptions $a_0 = a'_0$, $a_1 = a'_1$, $a_2 = a'_2$, $a_3 = a'_3$ and the local symmetry, we obtain (6.5) and they are constant on M, M' . If (1) (resp. (2)) holds, we obtain a system of equations (6.1), (6.2) and (6.3) (resp. (6.4)) of (t) , R^4 , and (c) . Then it suffices to solve them.

Acknowledgement. The author wishes to express his hearty thanks to Professor T. Sakai for his valuable suggestions and patient check of the contents. Especially, he pointed out some defects of this paper on the sign of the signature.

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(Received June 1, 1995)
(Revised October 31, 1995)