Math. J. Okayama Univ. 37(1995), 205–217

STRATIFIED POISSON SPACES AND REDUCTION

SEIKI YAMANISHI

1. Introduction. In this paper we show a generalized local reduction theorem for a Poisson manifold on which a Lie group G acts with moment map, following Sjamaar and Lerman's work in the symplectic context.

Marsden and Weinstein showed the following reduction theorem. Let G be a Lie group acting symplectically on a symplectic manifold (M,ω). We denote by $\mathfrak{g}^*$ the dual space of the Lie algebra $\mathfrak{g}$ of G.

Theorem (Marsden-Weinstein [9]). *Let $J\colon M \to \mathfrak{g}^*$ be an* Ad*-*equivariant moment map for the G-action. Let $\mu \in \mathfrak{g}^*$ be a regular value of J. Suppose that the isotropy group G_μ of μ acts freely and properly on $J^{-1}(\mu)$. Then there is the unique symplectic structure ω_μ on $M_\mu := J^{-1}(\mu)/G_\mu$ such that $i_\mu^*\omega=\pi_\mu^*\omega_\mu$, where $i_\mu\colon J^{-1}(\mu) \to M$ is the inclusion and $\pi_\mu\colon J^{-1}(\mu) \to M_\mu$ is the projection.*

Furthermore Marsden-Weinstein showed the following "shifting trick" for the reduction of the action via G. If μ is a regular value of J different from zero, consider the symplectic manifold $M \times O_{-\mu}$, where $O_{-\mu}$ is the co-adjoint orbit through $-\mu$. The diagonal action of G on $M \times O_{-\mu}$ is Hamiltonian with a moment map J_μ sending $(m, f) \in M\times O_{-\mu}$ to $J(m)+f$. Then zero is a regular value of J_μ, and the Marsden-Weinstein reduced space M_μ at μ can be identified with $J_\mu^{-1}(0)/G$. So we can reduce the discussion about reduction to the 0-value.

Recently Sjamaar and Lerman showed a generalized reduction theorem in the symplectic context where the regularity assumptions are dropped.

Theorem (Sjamaar-Lerman [10]). *Let (M,ω) be a Hamiltonian G-space with a moment map $J\colon M \to \mathfrak{g}^*$. The intersection of the stratum $M_{(H)}$ of orbit type (H) with the zero level set Z of the moment map is a manifold, and the orbit space*

$$(M_0)_{(H)} = (M_{(H)} \cap Z)/G$$

has a natural symplectic structure $(\omega_0)_{(H)}$ whose pullback to $Z_{(H)} := M_{(H)} \cap Z$ coincides with the restriction to $Z_{(H)}$ of the symplectic form *on M. Consequently the stratification of M by orbit types induces a decomposition of the reduced space $M_0 = Z/G$ into a disjoint union of symplectic manifolds*

$$M_0 = \bigcup_{H<G} (M_0)_{(H)}.$$

Sjamaar and Lerman proved the above theorem modeling a neighborhood of the orbit through zero by a symplectic vector bundle, using the local normal form for the moment map. (The local normal form for the moment map is discovered independently by Marle [7] and Guillemin and Sternberg [5]).

The purpose of this paper is to show a generalized reduction theorem in the Poisson context in the case where the transversal Poisson structure $\{,\}$ can be approximated by the Lie-Poisson structure.

Recall that a Poisson structure on a differentiable manifold P is defined as a Lie algebra structure on $C^\infty(P)$ satisfying the Libnitz identity on the derivation. The bracket operation $\{\ ,\ \}$ is a derivation and for each $H \in C^\infty(P)$ there is a vector field ξ_H such that $\xi \cdot F = \{H, F\}$ for all $F \in C^\infty(P)$. ξ_H is called the Hamiltonian vector field generated by H. There is a bundle map B: $T^*P \to TP$ such that $\xi_H = B \circ dH$ for all $H \in C^\infty(P)$. One of the basic properties of Poisson manifolds is the following.

Splitting theorem (Weinstein [14]). *Let p be any point of a Poisson manifold P. Then there is a neighborhood U of p in P and a Poisson isomorphism $\phi = \phi_S \times \phi_N$ from U onto the product $S \times N$ such that S is symplectic and the rank of Poisson structure of N at $\phi_N(p)$ is zero. The factors S and N are unique up to local isomorphisms.*

In the Poisson context Marsden and Ratiu proved the following reduction theorem that is similar to a symplectic one:

Theorem (Marsden-Ratiu [8]). *Let G be a compact Lie group acting as a Poisson automorphism on the Poisson manifold $(P, \{\ \})$. Let J: $P \to \mathfrak{g}^*$ is an Ad^*-equivariant moment map for the G-action. Let $\mu \in \mathfrak{g}^*$ is a regular value of J. Suppose that the isotropy group G_μ of μ acts freely and properly on $J^{-1}(\mu)$. Then there is a unique Poisson structure on $P_\mu := J^{-1}(\mu)/G_\mu$ such that $\{\ ,\ \}_P \circ i = \{\ ,\ \}_{P_\mu} \circ \pi$, where i: $P_\mu \to P$ is the inclusion and π: $P \to P_\mu$ is the projection.*

The action of a Poisson automorphism is called canonical when the action preserves Poisson/symplectic structure with an equivariant moment map. The shifting trick holds also in the Poisson context. Now the main result of this paper is the following local reduction theorem for Poisson G-manifolds:

Theorem. *Suppose that a compact Lie group G acts canonically on a Poisson manifold P with an Ad^*-equivariant moment map $J\colon P \to \mathfrak{g}^*$. Let $p \in J^{-1}(0)$. Then for the G-orbit $G{\cdot}p$ there exists a G-invariant neighborhood U of $G{\cdot}p$ in P and a Poisson isomorphism $\phi = \phi_S \times \phi_N\colon U \to S \times N$ stated in Proposition 2.1 where the rank of N at $\phi_N(p)$ is zero. Now assume that N is linearlizable near $\phi_N(p)$. Then, the intersection of the stratum $U_{(H)}$ of orbit type (H) with the zero level set of the moment map is a manifold, and the orbit space*

$$(U_0)_{(H)} := (U_{(H)} \cap J^{-1}(0))/G$$

has a natural Poisson structure whose pull back to $U_{(H)} \cap J^{-1}(0)$ coincides with the restriction of the Poisson structure on P to $U_{(H)} \cap J^{-1}(0)$. Consequently the stratification of U by orbit types induces a decomposition of the reduced space $U_0 = (U \cap J^{-1}(0))/G$ into a disjoint union of Poisson manifolds

$$U_0 = (U \cap J^{-1}(0))/G = \bigcap_{H<G} (U_0)_{(H)}.$$

We will apply the method due to Sjamaar and Lerman to prove the above result.

2. A decomposition of phase space in the Poisson context.

Proposition 2.1. *Suppose that a compact Lie Group G acts canonically on a Poisson manifold P. Namely, $g \in G$ acts as an automorphism of the Poisson manifold P. Let $G{\cdot}p$ be a G-orbit through $p \in P$. Then there exist a neighborhood U of $G{\cdot}p$ in P and an isomorphism $\phi = \phi_S \times \phi_N$ from U onto the product $S \times N$, where S is a symplectic manifold and the rank of N at every point of $\phi_N(G{\cdot}p)$ is zero. The factors S and N are unique up to local isomorphisms.*

Proof. Because of the assumption, $G{\cdot}p$ is on a symplectic leaf S' of P containing p. From Weinstein's splitting theorem [14] there exist a neighborhood U_p of p in P and an isomorphism $\phi_p = \phi_{p_S} \times \phi_{p_N}$ from U_p to $S_p \times N_p$. Since $G{\cdot}p$ is compact, there exist finite points $g_1p, \dots, g_np$ in $G{\cdot}p$ such that $G{\cdot}p \subset \bigcup g_iU_p$. Put $U_i = g_iU_p$ and $U = \bigcup U_i$. (U_i may be written as $S_i \times N_i$). We will construct an isomorphism ϕ: $U \to S \times N$. Since the above splittings are unique up to isomorphisms, we can identify all N_i. We denote this Poisson factor by N. We may assume that $U_i \cap U_{i+1} \neq \emptyset$ by changing indices if necessary. Uniqueness of the above splitting implies that $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are isomorphic. We define a map

$$\psi_1 : U_1 \cup U_2 \longrightarrow (S_1 \times N) \cup (S_2 \times N) \cong (S_1 \cup S_2) \times N$$
$$\text{by } \psi_1(u_i) = \phi_i(u_i),\ u_i \in U_i,\ i = 1, 2.$$

Since $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are isomorphic, we can identify $\phi_1(u)$ and $\phi_2(u)$ for any $u \in U_1 \cap U_2$. So ψ_1 is an isomorphism from $U_1 \cup U_2$ to $(S_1 \cup S_2) \times N$. Repeating the above procedure we obtain an isomorphism

$$\psi_{n-1} : U_1 \cup U_2 \cup \cdots \cup U_n \longrightarrow (S_1 \cup S_2 \cup \cdots S_n) \times N.$$

Thus we have a desired isomorphism $\phi = \psi_{n-1}$, and symplectic factor $S = S_1 \cup S_2 \cup \cdots \cup S_n$.

Remark 2.2. For $S \times N$ constructed in the above proposition, there exists a symplectic structure ω on S which is induced from the Poisson structure on P. Suppose that for a submanifold $Y \subset S$ there exists another symplectic structure ω_0 on S such that $\omega_{|Y} = \omega_{0|Y}$. Then there exist a neighborhood U of Y in S and a diffeomorphism f: $U \times N \to S \times N$ such that

(i) $f(y) = y, \quad \forall y \in Y$,

(ii) $f^*\omega_0 = \omega$.

In fact this is Darboux-Weinstein's theorem [12] when we restrict above claim to S. We can see that f may be got by integrating Hamiltonian vector field on S because Hamiltonian vector fields preserve the Poisson structure. This gives another proof of the theorem above.

Let P be a Poisson manifold. Put $O = \{p \in P \,|\, \text{rank of Poisson structure at } p \text{ is maximal}\}$. O is open and dense in P. Furthermore, O is a Poisson submanifold of P.

We will show that the same result holds in O as Sjamaar and Lerman's theorem [10].

Theorem 2.3. *Suppose that a compact Lie group G acts canonically on a Poisson manifold P with a moment map J: $P \to \mathfrak{g}^*$. Let O be the subset of P defined above. Then the intersection of the stratum $O_{(H)}$ of* *orbit type (H) with the zero level set of the moment map is a manifold, and the orbit space*

$$(O_0)_{(H)} = (O_{(H)} \cap J^{-1}(0))/G$$

has a Poisson structure whose pull back to $O_{(H)} \cap J^{-1}(0)$ coincides with the restriction of the Poisson structure on P to $O_{(H)} \cap J^{-1}(0)$. Consequently the stratification of O by orbit type induces a decomposition of the reduced space $O_0 = (O_{(H)} \cap J^{-1}(0))/G$ into a disjoint union of Poisson manifold

$$O_0 = \bigcup (O_0)_{(H)}.$$

Proof. We restrict the action of G to O. Let $q \in O$. Then from Proposition 2.1, $G{\cdot}q$ has a neighborhood in O which is isomorphic to $S \times N$. Since rank of Poisson structure at q is maximal, rank of Poisson structure on N is identically zero. Let $p \in O \cap J^{-1}(0)$. Then $G{\cdot}p$ is isotropic in S. According to the local normal form of moment map and Remark 2.2, a neighborhood of $G{\cdot}p$ in S is symplectic isomorphic to the model manifold $Y = G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V)$, where H is the stabilizer of G at p, V is symplectic slice of $G{\cdot}p$ in S and $\mathfrak{g}$ (resp. $\mathfrak{h}$) is the Lie algebra of G (resp. H). In this case, H acts trivially on N because rank of the Poisson structure on N is identically zero. So the model coincides with $G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V) \times N$. $G{\cdot}p$ is embedded as a zero section of Y, and the neighborhood of $G{\cdot}p$ in O is equivariantly Poisson diffeomorphic to a neighborhood of the zero section of Y with the G-moment map given by the formula

$$J([\mathfrak{g}, \mu, v], n) = \mathrm{Ad}_{\mathfrak{g}^*}(\mu + \Phi_v(v))$$

where Φ_v is a moment map corresponding to the linear symplectic action of G on the symplectic vector space V (see Remark 2.5). Thus reduction of this case depends only on symplectic part. Applying the Sjamaar and Lerman's theorem to the symplectic part, assertion of the theorem follows. Further we have

$$\begin{aligned}(O_0)_{(H)} &= (O_{(H)} \cap J^{-1}(0))/G \\ &\cong (S_{(H)} \cap J^{-1}(0))/G \times N \\ &= (S_0)_{(H)} \times N.\end{aligned}$$

Now we consider the case where the Poisson structure on N may be approximated by the Lie-Poisson structure. We begin by recalling the linear approximation of a Poisson structure near a point of rank zero (for more details see Weinstein [14]). Let P be a Poisson manifold and x be any point in P. In general the cotangent space T_x^*P can be identified with the quotient m_x/m_x^2, where m_x^k is the ideal in $C^\infty(P)$ generated by functions which vanish at x together with all derivatives of order up to $k-1$. As P is a Poisson manifold, if the rank is zero at x then m_x is a Lie subalgebra of $C^\infty(P)$. Furthermore, m_x^2 is a Lie ideal of m_x.

Thus $T_x^*P \cong m_x/m_x^2$ has te structure of a Lie algebra. We will denote this by $\mathfrak{g}_x$ and $T_xP \cong \mathfrak{g}_x^*$ carries a Lie-Poisson structure. This is called the linear approximation to the Poisson structure at x. The linearlization problem raised by Weinstein asks whether a Poisson structure is locally isomorphic to the Lie-Poisson structure on the dual of Lie algebra.

Conn ([3],[4]) answered this question. According to his result

(i) If the Poisson structure is analytic, then the answer is yes if its Lie algebra is semisimple.

(ii) If the Poisson structure is smooth, then the answer is yes if its Lie algegra is semisimple of compact type.

Let P be a Poisson manifold and G a compact Lie group acting on P canonically with an Ad*-equivariant moment map $J\colon P \to \mathfrak{g}^*$. Let $p \in J^{-1}(0)$. Then by Proposition 2.1, for the G-orbit through p there exist a neighborhood U in P isomorphic to $S\times N$, where rank of N at $\phi_N(p)$ is zero. In the following discussion we assume that the Poisson structure on N is linearlizable near the point $\phi_N(p)$ in the sense of Conn's condition (ii), namely we have $T_{\phi_N(p)}N \cong \mathfrak{k}^*$, where $\mathfrak{k}^*$ is the dual space of a semisimple Lie algebra $\mathfrak{k}$ of compact type, and the Poisson structure on N is isomorphic to the Lie-Poisson structure of $\mathfrak{k}^*$ near the point $\phi_N(p)$.

Similar by to the symplectic case, we will construct a model space that is equivariantly Poisson diffeomorphic to a neighborhood of $G\cdot p$. First we will show the following lemma.

Lemma 2.4. *Let $\mathfrak{k}^*$ be the dual space of a semisimple Lie algebra $\mathfrak{k}$ of compact type. We endow $\mathfrak{k}^*$ with the Lie-Poisson structure. Assume that a compact Lie group H acts on $\mathfrak{k}^*$ as a linear Poisson automorphism group. Then, the action of H is canonical and the moment map of the action is given by the dual map of a homomorphism from $\mathfrak{h}$ to $\mathfrak{k}$, where $\mathfrak{h}$ is the Lie algebra of H.*

Proof. Let K be a compact Lie group with the Lie algebra $\mathfrak{k}$. By assumption the Killing form B on $\mathfrak{k}$ is nondegenerate. Then the linear map from $\mathfrak{k}^*$ to $\mathfrak{k}$ given by the formula

$$\xi^*(\eta) = B(\xi, \eta), \quad \xi, \eta \in \mathfrak{k}$$

is nondegenerate. Further we have for $a \in K$

$$B(\mathrm{Ad}_{a^{-1}}\xi, \eta) = B(\xi, \mathrm{Ad}_a \eta) = \xi^*(\mathrm{Ad}_a \eta) = (\mathrm{Ad}^*_{a^{-1}}\xi^*)\eta.$$

So we may identify the adjoint action on $\mathfrak{k}$ with the coadjoint action on $\mathfrak{k}^*$. By definition of the Lie-Poisson structure, we may identify a linear Poisson action of H on $\mathfrak{k}^*$ with an automorphism group of $\mathfrak{k}$. Namely we have a representation of H over $\mathfrak{k}$. Since $\mathfrak{k}$ is semisimple, we may identify the Lie algebra $\mathrm{ad}(\mathfrak{k})$ with the Lie algebra $\partial(k)$, where $\mathrm{ad}(k)$ denotes the Lie algebra of the group consisting of all adjoint actions on $\mathfrak{k}$ and $\partial(\mathfrak{k})$ denotes the Lie algebra of the group of automorphisms of $\mathfrak{k}$. Because $\mathfrak{k}$ is semisimple, $\mathrm{ad}(\mathfrak{k})$ is isomorphic to $\mathfrak{k}$. Thus the differential of the action of H may be expressed by $\mathrm{ad}_{\psi(\zeta)}$ for $\zeta \in \mathfrak{h}$, where ψ is a homomorphism from $\mathfrak{h}$ to $\mathfrak{k}$. Therefore the action of H on $\mathfrak{k}^*$ is canonical and the moment map is given by the dual map $\psi^*\colon \mathfrak{k}^* \to \mathfrak{h}^*$ of ψ where ψ is the above homomorphism. It is obvious that the moment map is H-equivariant.

Now let ω be the symplectic form on S induced by the Poisson structure on P. We denote by V the symplectic vector space $(T_{\phi_S(p)}(G\cdot p))^\omega / T_{\phi_S(p)}(G\cdot p)$, which is a fiber of the symplectic normal bundle of the orbit in S. Let H be the stabilizer of G at p. And by our assumption $T_{\phi_N(p)}N$ is isomorphic to $\mathfrak{k}^*$. We will regard $V \times \mathfrak{k}^*$ as a Poisson slice of the action. The Poisson normal bundle of the orbit is given by $G \times_H (V \times \mathfrak{k}^*)$, which is a vector bundle associated to the principle fibration

$$H \longrightarrow G \longrightarrow G \cdot p.$$

Now in the following we will give a Poisson structure on the total space Y of the associated bundle $G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V \times \mathfrak{k}^*)$ such that the embedding $G/H \to Y$ defined by the zero section is isotropic in $G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V)$ and the corresponding normal bundle is given by $G \times_H (V \times \mathfrak{k}^*)$.

Assume that $\mathfrak{g}^*$ is splitted in the form $\mathfrak{g}^* = \mathfrak{m}^* + \mathfrak{h}^*$, $\mathfrak{m}^* \cong (\mathfrak{g}/\mathfrak{h})^*$ by fixing an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

Now we consider the triple product action of H on $T^*G \times V \times \mathfrak{g}^*$, where H acts on T^*G as a lift of the right multiplication, on V as a linear symplectic action and on $\mathfrak{k}^*$ as a linear Poisson action, respectively.

Remark 2.5. The action of the Lie group G on T^*G given by

$$R^*(a): (g,\eta) \longrightarrow (ga^{-1}, \mathrm{Ad}_a^*\eta), \quad a \in G, \ g \in G, \ \eta \in \mathfrak{g}^*$$

is Hamiltonian and the corresponding moment map is given by the formula

$$\Phi_R(g,\eta) = -\eta.$$

Also the action given by

$$L^*(a): (g,\eta) \longrightarrow (ag,\eta)$$

is Hamiltonian too, and the corresponding moment map is given by the formula

$$\Phi_L(g,\eta) = \mathrm{Ad}_g^*\eta.$$

Linear symplectic action of G on the symplectic vector space (V,ω_V) is also Hamiltonian. The corresponding moment map Φ_V is given by the formula

$$\langle \eta, \Phi_V(v)\rangle = \frac{1}{2}\omega_V(\xi_v \cdot v, v).$$

For more details, see for example Sjamaar and Lerman [10], Abraham and Marsden [1].

Above action of H on $T^*G \times V \times \mathfrak{k}^*$ is canonical, and the corresponding H-equivariant moment map Φ: $G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \times \mathfrak{k}^* \to \mathfrak{h}^*$ is given by $\Phi_R + \Phi_V + \psi^*$, i.e. we get

$$\Phi(g,\mu,\eta,v,\xi) = \Phi_V(v) - \eta + \psi^*(\xi).$$

Then zero is a regular value of Φ and $\Phi^{-1}(0)$ consists of all points of the form $(g,\mu,\Phi_V(v)+\psi^*(\xi),v,\xi)$. The map given by

$$\begin{aligned} G \times \mathfrak{m}^* \times V \times \mathfrak{k}^* &\longrightarrow \Phi^{-1}(0) \subset G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \times \mathfrak{k}^* \\ (g,\mu,v,\xi) &\longrightarrow (g,\mu,\Phi_V(v)+\psi^*(\xi),v,\xi) \end{aligned}$$

is an H-equivariant diffeomorphism. Since H acts freely on $\Phi^{-1}(0)$, from the reduction theorem of Poisson case the reduced space $\Phi^{-1}(0)/H$ is a Poisson manifold. We may identify $\Phi^{-1}(0)/H$ with $Y = G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V \times \mathfrak{k}^*)$ and a Poisson structure is induced on Y. Note that the symplectic factor of Y is given by $G \times_H ((\mathfrak{g}/\mathfrak{h})^* \times V)$.

Now $\Phi_R^{-1}(0)$ may be identified with $\{(g,\mu,0,0,0) \in G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \times \mathfrak{k}^*\}$ $(\subset \Phi^{-1}(0))$. Thus $\Phi_R^{-1}(0)/H$ may be canonically identified with $T^*(G{\cdot}p) \cong T^*(G/H)$, and under this identification $T^*(G{\cdot}p)$ is a symplectic submanifold of the symplectic factor of Y with the symplectic structure induced by the Poisson structure on Y. $G{\cdot}p$ is embedded in $T^*(G{\cdot}p)$ as a zero section. Clearly this embedding is isotropic and its normal bundle in Y is given by $G \times_H (V \times \mathfrak{k}^*)$.

Therefore from the equivariant version of isotropic embedding theorem [6], Remark 2.2 and the assumption that N is linearlizable by $\mathfrak{k}^*$, there exist a neighborhood U_0 of the zero section of Y, a neighborhood U in P of the orbit $G{\cdot}p$ and a G-equivariant Poisson diffeomorphism

$$\psi : U_0 \longrightarrow U.$$

Next we describe the Hamiltonian action of G on the model space Y. Actions of L^* and R^* of G on T^*G commute each other. We regard L^* as an action of G on the triple product $T^*G \times V \times \mathfrak{k}^*$ where G acts trivially on V and $\mathfrak{k}^*$. Then L^* commutes with the product action of H, and the moment map J_L: $G \times \mathfrak{g}^* \times V \times \mathfrak{k}^* \to \mathfrak{g}^*$ given by

$$J_L(g,\eta,v,\xi) = \mathrm{Ad}_g^*\eta$$

is H-invariant.

Therefore the action L^* of G descends to an action on the H-reduced space $\Phi^{-1}(0)/H$, and the corresponding moment map

$$J : G \times_H (\mathfrak{m}^* \times V \times \mathfrak{k}^*) \longrightarrow \mathfrak{g}^*$$

sends a point $[g,\mu,v,\xi]$ to $\mathrm{Ad}_g^*(\mu+\Phi_V(v)+\psi^*(\xi))$. Here $[g,\mu,v,\xi]$ denotes the conjugate class of $(g,\mu,v,\xi) \in g \times \mathfrak{m}^* \times V \times \mathfrak{k}^*$ under the H-action.

Thus we get a result analogous to that of the theorem of the local normal form theorem for the moment map.

Proposition 2.6. *Let H be the stabilizer of G at $p \in J^{-1}(0)$ and $V \times \mathfrak{k}^*$ $(\mathfrak{k}^* \cong T_{\phi_N(p)}N)$ the Poisson slice of the orbit $G{\cdot}p$. Suppose that N is linearlizable on a neighborhood of $\phi_N(p)$. Then a neighborhood of the orbit is equivariantly Poisson diffeomorphic to a neighborhood of the zero section of $Y = G \times_H (\mathfrak{m}^* \times V \times \mathfrak{k}^*)$ with respect to the G-moment map J given by the formula*

$$J([g,\mu,v,\xi]) = \mathrm{Ad}_g^*(\mu + \Phi_V(v) + \psi^*(\xi)).$$

Now we will show a reduction theorem of Poisson manifold. We begin with the following lemma.

Lemma 2.7. *Under the situation of Lemma 2.4, let*

$$(\psi^{*-1}(0))^H := \{p \in \psi^{*-1}(0) |\ h(p) = p \text{ for any } h \in H\}$$

denote the set consisting of H-fixed points in the zero level set of the moment map ψ^*. *Then* $(\psi^{*-1}(0))^H$ *is a Poisson manifold.*

Proof. Let K be a compact Lie group with the Lie algebra $\mathfrak{k}$. For $p \in \psi^{*-1}(0)$, the symplectic leaf in $\mathfrak{k}^*$ through p is given by the K-orbit $K \cdot p$ where the action of K on $\mathfrak{k}^*$ is given by the coadjoint action. Since H acts canonically on $\mathfrak{k}^*$, the action of H leaves invariant symplectic leaves. Restricting the action of H to $K \cdot p$, we have a Hamiltonian H-space $K \cdot p$ with the moment map $\psi^*_{|K \cdot p}$. Then from the theorem of Sjamaar and Lerman [10], we have

$$(\psi^{*-1}(0) \cap (K \cdot p))/H = \bigcup_{L<H} (\psi^{*-1}(0) \cap (K \cdot p)(L))/H,$$

where $K \cdot p(L)$ is a stratum of orbit type (L) and $(\psi^{*-1}(0)(L))/H$ is a symplectic submanifold of $K \cdot p$. Especially all H-fixed points in $\psi^{*-1}(0) \cap (K \cdot p)$ form a symplectic submanifold $(\psi^{*-1}(0) \cap (K \cdot p(H))/H$ of $K \cdot p$. Since zero is a regular value of ψ^*, $\psi^{*-1}(0)$ is a submanifold of $\mathfrak{k}^*$ and so is $(\psi^{*-1}(0))^H$. We will show that $(\psi^{*-1}(0))^H$ is a Poisson manifold. It is obvious from above discussion that $T_x(\psi^{*-1}(0))^H + \operatorname{Im} B_x = T_x\mathfrak{k}^*$ holds for all $x \in (\psi^{*-1}(0))^H$, $x \neq 0$ where B is a bundle map $B: T^*P \to TP$ defined by $\xi_H = B \circ dH$ for all $H \in C^\infty(P)$. Furthermore, the induced symplectic form on $\operatorname{Im} B_x \cap T_x(\psi^{*-1}(0))^H$ is nondegenerate because $(\psi^{*-1}(0) \cap (K \cdot p(H))/H$ is a symplectic submanifold of $K \cdot p$. Thus from the Weinstein's result (Proposition 1.4 in [14]) $(\psi^{*-1}(0))^H$ is a Poisson manifold.

Theorem 2.8. *Suppose that a compact Lie group G acts canonically on a Poisson manifold P with an* Ad^**-equivariant moment map* $J: P \to \mathfrak{g}^*$. *Let* $p \in J^{-1}(0)$. *Then for the G-orbit* $G \cdot p$ *there exist a G-invariant neighborhood U of* $G \cdot p$ *in P and a Poisson isomorphism* $\phi = \phi_S \times \phi_N: U \to S \times N$ *stated in Proposition 2.1. Where the rank of N at* $\phi_N(p)$ *is zero.*

Now assume that the Poisson structure is smooth, and the cotangent Lie algebra is semisimple of compact type. Then, the intersection of the *stratum $U_{(H)}$ of orbit type (H) with the zero level set of the moment map is a manifold, and the orbit space*

$$(U_0)_{(H)} := (U_{(H)} \cap J^{-1}(0))/G$$

has a natural Poisson structure whose pull back to $U_{(H)} \cap J^{-1}(0)$ coincides with the restriction of the Poisson structure on P to $U_{(H)} \cap J^{-1}(0)$. Consequently the stratification of U by orbit types induces a decomposition of the reduced space $U_0 = (U \cap J^{-1}(0))/G$ into a disjoint union of Poisson manifolds

$$U_0 = (U \cap J^{-1}(0))/G = \bigcap_{H<G} (U_0)_{(H)}$$

Proof. The existence of U is proved in Proposition 2.1 and Remark 2.2. From Proposition 2.6 computing in the model space Y we see that the intersection of the zero level set of J with the fiber of the fiber bundle $\mathfrak{m}^* \times V \times \mathfrak{k}^* \to Y \to G/H$ is given by $\{0\} \times \Phi_V^{-1}(0) \times \psi^{*-1}(0)$. On the other hand, the intersection of the stratum $Y_{(H)}$ of orbit type (H) with the fiber consists of points in $\mathfrak{m}^* \times V \times \mathfrak{k}^*$ at which stabilizers of G are conjugate to H. Therefore we have

$$\begin{aligned}(\mathfrak{m}^* \times V \times \mathfrak{k}^*) \cap J^{-1}(0) \cap Y_{(H)} &\cong (V_{(H)} \cap \Phi_V^{-1}(0)) \times (\mathfrak{k}^*_{(H)} \cap \psi^{*-1}(0)) \\ &= V^H \times (\mathfrak{k}^*_{(H)} \cap \psi^{*-1}(0)).\end{aligned}$$

Since $(\mathfrak{k}^*_{(H)} \cap \psi^{*-1}(0))$ consists of points of $\mathfrak{k}^*$ fixed by H and included in $\psi^{*-1}(0)$, we have $(\mathfrak{k}^*_{(H)} \cap \psi^{*-1}(0)) = (\psi^{*-1}(0))^H$, because $\psi^{*-1}(0)$ is a submanifold of $\mathfrak{k}^*$. Since the set $J^{-1}(0) \cap Y_{(H)}$ is G-invariant and the action of G on G/H is transitive, it follows that

$$\begin{aligned}J^{-1}(0) \cap Y_{(H)} &= G \cdot ((\mathfrak{m}^* \times V \times \mathfrak{k}^*) \cap J^{-1}(0) \cap Y_{(H)}) \\ &\cong G \times_H (V^H \times (\psi^{*-1}(0))^H) \\ &\cong G/H \times V^H \times (\psi^{*-1}(0))^H.\end{aligned}$$

Therefore the orbit space $(Y_{(H)} \cap J^{-1}(0))/G$ is the Poisson manifold $V^H \times (\psi^{*-1}(0))^H$. The first assertion of the theorem follows because the reduced space $(U \cap J^{-1}(0))/G$ is decomposed into a union of Poisson manifolds $(U_0)_{(H)}$, $H < G$.

We will show that the Poisson pieces satisfy the frontier condition, namely if $(U_0)_{(H)}$ intersects nontrivially the closure of a piece $(U_0)_{(L)}$ then the closure of $(U_0)_{(L)}$ contains every connected component of $(U_0)_{(H)}$ that the closure of $(U_0)_{(L)}$ intersects nontrivially. For the symplectic part, our assertion follows by the same discussion as that of Sjamaar and Lerman's Theorem 2.1 in [10]. As for the Poisson part $\psi^{*-1}(0)$, for each symplectic leaf $K\cdot q$, where $K\cdot q$ is a coadjoint orbit of K through $q \in \mathfrak{k}^*$, applying the above Sjamaar and Lerman's theorem to $\psi^{*-1}(0) \cap K\cdot q$ the frontier condition is satisfied for the $(K\cdot q)_0 = (\psi^{*-1}(0) \cap K\cdot q)/G$. Thus the frontier condition is satisfied also for Poisson part $\psi^{*-1}(0)$. The proof of the theorem is complete.

Acknowledgement. The results of this paper are based on the author's master dissertation in 1993 at Okayama University. I would like to thank Prof. T. Sakai and Prof. A. Katsuda for their generous help during the preparation of this paper and encouragements. I would also like to thank Prof. K. Mikami for his helpful discussions.

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KAWASAKI STEEL SYSTEMS R&D CORPORATION
MIZUSHIMA OFFICE
1-CHOME, KAWASAKI-DORI, MIZUSHIMA
KURASHIKI 712, JAPAN

(*Received November 30, 1994*)
(*Revised March 15, 1995*)