Math. J. Okayama Univ. 37(1995), 201–204

MORSE INDICES OF GENERALIZED HENNEBERG'S SURFACES

KAORI SEGAWA

Introduction. A minimal surface in $\mathbf{R}^3$ is called *stable* if the second variation of area is nonnegative for every compactly supported variation of the surface. A beautiful theorem due to do Carmo-Peng [1] and Fischer-Colbrie-Schoen [4] states that a complete orientable stable minimal surface must be a plane. Actually, this result continues to hold if the surface is allowed to have finitely many branch points. In the recent past, the stability of a *non-orientable* minimal surface has been investigated by several authors. Among others, Ross [9] has proved that any complete nonorientable minimal surface of *finite total curvature* cannot be stable. He also discusses the instability of finitely branched surfaces. It should be mentioned here that, unlike the orientable case, the above result of Ross does not extend to finitely branched surfaces. In fact, Henneberg's surface, which has finite total curvature -2π and possesses two branch points, is stable as was observed by Choe [2].

On a related front, Fischer-Colbrie [3] initiated the study of the Morse index of a minimal surface, which measures how far the surface is from being stable. Since then, quantitative study of this invariant has been done by a number of authors.

In this note, we present a series of examples of complete nonorientable minimal surfaces of finite total curvature, generalizing Henneberg's surface, and compute their Morse indices.

1. Preliminaries. Let (M, x) be a nonorientable minimal surface in $\mathbf{R}^3$, that is, $x\colon M \to \mathbf{R}^3$ is a branched minimal immersion of nonorientable two-dimensional manifold M. Let N be the orientable double cover of M, and let $pr\colon N \to M$ denote the covering projection. We have an orientation-reversing involution $\sigma\colon N \to N$ without fixed points satisfying $pr \circ \sigma = pr$. Setting $y = x \circ pr$, we obtain an orientable minimal surface (N, y), which may be expressed in terms of the Weierstrass representation (see [8]):

$$y(p) = \int^p \left(1 - g^2, i(1 + g^2), 2g\right) \omega, \quad p \in N, \tag{1}$$

where g is a meromorphic function and ω is a holomorphic one-form on N. The fact that (N, y) is a double cover of a nonorientable surface is echoed in the two equations (see [7])

$$g \circ \sigma = -\frac{1}{\bar{g}}, \tag{2}$$

$$\sigma^* \omega = -\overline{g^2 \omega}. \tag{3}$$

If M (and hence N) is complete, of finite total curvature and at most finitely branched, then N is conformally a compact Riemann surface with finitely many punctures, $N = \overline{N} \setminus \{p_1, \ldots, p_k\}$, and g and ω extend meromorphically to $\overline{N}$ (see [8]).

Let ν be a unit normal vector field on N, which then satisfies $\nu \circ \sigma = -\nu$. For $\phi \in C_0^\infty(N)$, let y_t be a compactly supported variation of y whose variation vector field $d/dt|_{t=0} y_t$ has normal component $\phi\nu$. Then the second variation of area with respect to y_t is given by

$$Q(\phi) = \int_N \left(|d\phi|^2 + 2K\phi^2 \right) dA,$$

where K and dA denote the Gauss curvature and the area element of N respectively. We note that $\phi\nu$ projects to a normal vector field on M if and only if $\phi \in C_0^\infty(N)_- = \{\phi \in C_0^\infty(N) | \phi \circ \sigma = -\phi\}$. This motivates us to define the *Morse index* $\mathrm{Ind}(M, x)$ of (M, x) as the dimension of a maximal subspace of $C_0^\infty(N)_-$ on which the quadratic form Q is negative definite. Notice that (M, x) is stable if and only if $\mathrm{Ind}(M, x) = 0$. We also mention that there is an obvious way to define $\mathrm{Ind}(M, x)$ by working extensively on M, and both of the definitions give the same number.

2. Generalized Henneberg's surfaces. In this section, we present examples of complete nonorientable minimal surfaces in $\mathbf{R}^3$, and compute their Morse indices.

For each positive integer k, we choose Weierstrass data (g, ω) on $N = \mathbf{C} \setminus 0$ as

$$g(\zeta) = \zeta^{2k-1},$$
$$\omega = f(\zeta) d\zeta = (1 - \zeta^{-4k}) d\zeta.$$

It is easy to verify that the formula (1) then gives a well-defined map $y_k \colon N \to \mathbf{R}^3$, producing an orientable minimal surface (N, y_k). Let $\sigma \colon N \to$

N be the involution defined by $\sigma(\zeta) = -1/\bar{\zeta}$. Factoring N by σ, we obtain a nonorientable surface M homeomorphic to the once-punctured real projective plane, or the Möbius band. Since g and ω satisfy the equations (2) and (3) respectively, y_k induces a map x_k: $M \to \mathbf{R}^3$. Thus we obtain a nonorientable minimal surface (M, x_k). It is easy to verify that (M, x_k) is complete, has total curvature $-2\pi(2k-1)$ and possesses two branch points. Notice that (M, x_1) is nothing but Henneberg's surface (see [6]).

We now prove

Theorem. *For each positive integer k, we have*

$$\mathrm{Ind}(M, x_k) = 2(k-1).$$

Proof. Let ν be a unit normal vector field on N, which defines a map ν: $N \to S^2$, the so-called Gauss map of N. As is well-known, ν and g are related by $g = \pi \circ \nu$, where π: $S^2 \to \mathbf{C} \cup \infty$ is the stereographic projection from the north pole. In particular, ν is holomorphic. Since N has finite total curvature, ν extends to a holomorphic map from $\bar{N} = \mathbf{C} \cup \infty$, which we denote by the same symbol.

We now pull back the metric on the unit sphere S^2 to $\bar{N}$ by ν, and denote its negative Laplacian by Δ. Then the Morse index of (M, x_k) coincides with the number of negative eigenvalues of the operator $L = -\Delta - 2$ which correspond to eigenfunctions ϕ satisfying

$$\phi \circ \sigma = -\phi, \tag{4}$$

where σ is extended to $\bar{N}$ as well [3, Corollary 2, p.131]. We note that this assertion holds for any complete nonorientable minimal surface with finite total curvature, and follows by modifying a proof of the corresponding fact in the orientable case.

For g as above, eigenfunctions as well as eigenvalues of L are explicitly computed in [5]. Indeed, the eigenvalues of L are exhausted by

$$\lambda_i = \frac{i}{2n}\left(\frac{i}{n} + 1\right), \quad i = 0, 1, 2, \ldots,$$

where $n = 2k - 1$, and the eigenspace of λ_i is spanned by

$$\{v_{p,q}(r) \cos q\theta, v_{p,q}(r) \sin q\theta\}_{pn+q=i}.$$

Here p, q are nonnegative integers, (r,θ) are the polar coordinates on $\mathbf{C}$, and $v_{p,q}(r)$ are functions explicitly representable in terms of the Gauss hypergeometric function. Notice that, in terms of (r,θ), (4) is rewritten as $\phi(1/r,\theta+\pi)=-\phi(r,\theta)$.

If $\lambda_i<1$, then $i\leq n-1=2k-2$, and hence $p=0$ and $q\leq 2k-2$. Since $v_{0,q}(r)=r^q/(r^{2n}+1)^{q/n}$ up to a constant multiple (see [5]) and so $v_{0,q}(1/r)=v_{0,q}(r)$, the eigenfunctions $v_{0,q}(r)\cos q\theta$, $v_{0,q}(r)\sin q\theta$ satisfy (4) if and only if q is odd. We thus obtain $\mathrm{Ind}(M,x_k)=2(k-1)$ as desired.

Remark. By the Theorem, (M,x_k) is unstable if $k\geq 2$. A result of Ross [9, Corollary 6] also implies this fact.

Acknowledgements. The author would like to express her sincere gratitude to Prof. K. Kenmotsu for his encouragement and sage advice. She would also like to thank Prof. S. Nayatani for useful discussions.

REFERENCES

[1] M. P. DO CARMO and C. K. PENG: Stable complete minimal surfaces in $\mathbf{R}^3$ are planes, Bull. Amer. Math. Soc. **1** (1979), 903–906.
[2] J. CHOE: Index, vision number, and stability of complete minimal surfaces, Arch. Rat. Mech. Anal. **109** (1990), 195–212.
[3] D. FISCHER-COLBRIE: On complete minimal surfaces with finite Morse index in three manifolds, Invent. Math. **82** (1985), 121–132.
[4] D. FISCHER-COLBRIE and R. SCHOEN: The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, Comm. Pure Appl. Math. **33** (1980), 199–211.
[5] S. NAYATANI: Lower bounds for the Morse index of complete minimal surfaces in Euclidean 3-space, Osaka J. Math. **27** (1990), 453–464.
[6] J. C. C. NITSCHE: Lectures on Minimal Surfaces, Vol. 1, Cambridge University Press, Cambridge, 1989.
[7] M. E. G. G. DE OLIVEIRA: Some new examples of nonorientable minimal surfaces, Proc. Amer. Math. Soc. **98** (1986), 629–636.
[8] R. OSSERMAN: A Survey of Minimal Surfaces, Dover, New York, 1986.
[9] M. ROSS: Complete nonorientable minimal surfaces in $\mathbf{R}^3$, Comment. Math. Helv. **67** (1992), 64–76.

SENDAI SHIRAYURI GAKUEN SENIOR HIGH SCHOOL
1-7-1 HONCHO, AOBA-KU, SENDAI 980, JAPAN

(Received December 9, 1994)
(Revised March 20, 1995)