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KO-GROUP OF $PSp(2^{4n})$

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Let $Sp(n)$ be the symplectic group of degree n and $PSp(n)$ be the projective group associated with $Sp(n)$, that is, $PSp(n) = Sp(n)/C$ where C denotes the center of $Sp(n)$ which is generated by the scalar matrix with all diagonal entries -1.

Our purpose here is to compute the real K-group $KO^*(PSp(2^{4n}))$. As for the complex K-group, $K^*(PSp(\ell))$ has been determined in [7,9] for any $\ell \geq 1$. But we begin with the calculation of $K^*(PSp(2^{4n}))$ by our method for convenience of calculation. The way getting these groups is quite parallel to that of [12]. As it turns out that there is a $\boldsymbol{Z}/2$-map from S^{8n+3} to $Sp(2^{4n})$ where the generator of $\boldsymbol{Z}/2$ acts on S^{8n+3} as antipodal involution and on $Sp(2^{4n})$ as the generator of C respectively, the multiplicative structures of the K-groups of $PSp(2^{4n})$ can be reduced to those of the K-groups of P^{8n+3} and $Sp(2^{4n})$ just as in the case of $SO(8\ell)$ [12] by making use of this $\boldsymbol{Z}/2$-map and applying a device to the equivariant K-theories associated with $\boldsymbol{Z}/2$.

This paper is arranged as follows. Section 1 consists of preparations for the subsequent sections. Sections 2 and 3 deal with the computation of $K^*(PSp(2^{4n}))$ and $KO^*(PSp(2^{4n}))$ respectively.

1. Let Γ denote the multiplicative group generated by -1 and H denote the canonical non-trivial 1-dimensional real representation of Γ.

We write nH for the direct sum of n copies of H. And by $B(pH \oplus \boldsymbol{R}^q)$ and $S(pH \oplus \boldsymbol{R}^q)$ we denote the unit ball and the unit sphere in $pH \oplus \boldsymbol{R}^q$ centered at the origin o, and let $\Sigma^{p,q} = B(pH \oplus \boldsymbol{R}^q)/S(pH \oplus \boldsymbol{R}^q)$ with the collapsed $S(pH \oplus \boldsymbol{R}^q)$ as base point. Here $\boldsymbol{R}$ denotes the field of real numbers. Also, for later use we fix the notations $\boldsymbol{C}$ and $\boldsymbol{H}$ for the fields of complex numbers and quaternions as usual.

Let $\Delta^+ \colon Spin(8n+4) \to U(2^{4n+1})$ be one of the half-spin representations of $Spin(8n+4)$. It is known [10], §13 that Δ^+ is the restriction of a quaternionic representation of $Spin(8n+4)$, denoted by

$$\bar{\Delta}^+ : Spin(8n+4) \longrightarrow Sp(2^{4n})$$

below. Assume that the generator of Γ acts on $Spin(8n+4)$ and $Sp(2^{4n})$ as the elements -1 and $-I$ of these groups respectively where I is the unit matrix, and thus consider these two groups as Γ-spaces. Then $\bar{\Delta}^+$ becomes a Γ-map obviously. Moreover we know [6] that $Spin(8n+4)$ contains $S^{8n+4,0}$ as an invariant subspace. This follows from the fact that $Spin(8n+4)$ is a subgroup of the Clifford algebra C_{8n+3} multiplicatively generated by the elements of the unit sphere S^{8n+3} ([10], §11). Therefore we have the following result similar to [6],(1.14).

(1.1) There exists a Γ-map ι: $S^{8n+4,0} \to Sp(2^{4n})$, so that we have a homeomorphism

$$(S^{8n+4,0} \times Sp(2^{4n}))/\Gamma \approx P^{8n+3} \times Sp(2^{4n}).$$

In fact, this homeomorphism is induced by the assignment $(x,g) \mapsto (\pi(x), \iota(x)^{-1}g)$ for $x \in S^{8n+4,0}$ and $g \in Sp(2^{4n})$, where $P^{8n+3} = S^{8n+4,0}/\Gamma$, the real projective space of dimension $8n+3$, and π is the canonical projection from $S^{8n+4,0}$ to P^{8n+3}.

A Real (Γ-)vector bundle is a complex (Γ-)vector bundle together with a conjugate (equivariant) involutive automorphism and a quaternionic (Γ-)vector bundle is a complex (Γ-)vector bundle together with a conjugate (equivariant) anti-involutive automorphism. It is clear by definition that the external tensor product $E \hat{\otimes}_C F$ of two quaternionic (Γ-)vector bundles E and F admits an obvious Real structure.

Let KR and KSp denote the Real and quaternionic K-theories and let KR_Γ and KSp_Γ denote the equivariant ones associated with Γ. But $KR(X) \cong KO(X)$ and $KR_\Gamma(X) \cong KO_\Gamma(X)$ canonically if X has a trivial Real structure. Since all spaces of this note are such ones, we identify these isomorphisms throughout this paper. Then the above external tensor product $x \hat{\otimes}_C y$ defines uniquely an element $x \wedge_C y$ of either $\widetilde{KO}(X \wedge Y)$ or $\widetilde{KO}_\Gamma(X \wedge Y)$ according as $x \in \widetilde{KSp}(X)$, $y \in \widetilde{KSp}(Y)$ or $x \in \widetilde{KSp}_\Gamma(X)$, $y \in \widetilde{KSp}_\Gamma(Y)$.

Considering $S^{0,3}$ to be the unit quaternions $Sp(1)$ yields a generator of $\widetilde{KSp}(\Sigma^{0,4})$ in a canonical way. We write α for this element. Then

$$\widetilde{KSp}(\Sigma^{0,4}) = \boldsymbol{Z} \cdot \alpha$$

and also α satisfies

$$(1.2) \qquad \alpha \otimes_C \boldsymbol{H} = \eta_4, \quad \alpha \wedge_C \alpha = \eta_8 \quad \text{and} \quad s(\alpha) = \mu^2$$

where η_4, η_8 and μ denote the canonical generators of $\widetilde{KO}(\Sigma^{0,4})$, $\widetilde{KO}(\Sigma^{0,8})$ and $\widetilde{K}(\Sigma^{0,2})$, (the last two generators are called the Bott class), and s denotes the natural complexification $KSp \to K$.

From [3,11,14] we now recall the equivariant Thom isomorphism theorems. Consider the isomorphism $S^{8n+4,0} \times \boldsymbol{H}^{2^{4n}} \cong S^{8n+4,0} \times \boldsymbol{H}^{2^{4n}} \otimes_{\boldsymbol{R}} H$ of Γ-quaternionic vector bundles over $S^{8n+4,0}$ given by the assignment $(x, v) \mapsto (x, \iota(x)v)$ for $x \in S^{8n+4,0}$, $v \in \boldsymbol{H}^{2^{4n}}$ where ι is as in (1.1). Then, in a canonical manner, this isomorphism yields a generator $\tau_{\boldsymbol{H}}$ of $\widetilde{KSp}_\Gamma(\Sigma^{8n+4,0})$ such that its restriction to $o \in B((8n+4)H)$ is $2^{4n}(\boldsymbol{H} - \boldsymbol{H} \otimes_{\boldsymbol{R}} H) \in KSp_\Gamma(o)$ $(= RSp(\Gamma)$, the quaternionic representation ring of Γ).

Set

$$(1.3) \qquad \begin{aligned} \tau &= s(\tau_{\boldsymbol{H}}) \in \widetilde{K}_\Gamma(\Sigma^{8n+4,0}) \quad \text{and} \\ \omega &= \tau_{\boldsymbol{H}} \wedge_{\boldsymbol{C}} \alpha \in \widetilde{KO}_\Gamma(\Sigma^{8n+4,4}). \end{aligned}$$

Then their restrictions to o and $\Sigma^{0,4}$ are $2^{4n+1}(1-L) \in K_\Gamma(o) = R(\Gamma)$ and $2^{4n}(1-H)\eta_4 \in \widetilde{KO}_\Gamma(\Sigma^{0,4}) = RO(\Gamma)\cdot\eta_4$ respectively where $L = \boldsymbol{C} \otimes_{\boldsymbol{R}} H$, and multiplications by τ and ω give isomorphisms $\widetilde{K}^*_\Gamma(X) \cong \widetilde{K}^*_\Gamma(\Sigma^{8n+4,0} \wedge X)$ and $\widetilde{KO}^*_\Gamma(X) \cong \widetilde{KO}^*_\Gamma(\Sigma^{8n+4,4} \wedge X)$ for any Γ-space X with base-point respectively. Here $R(\Gamma)$ and $RO(\Gamma)$ are the complex and real representation rings of Γ and $R\cdot g$ denotes an R-module generated by a single element g for a ring R.

By h we denote the K- or KO-functor. For $X = +$ (a point), $Sp(2^{4n})$ we consider the exact sequence of the pair $(B((8n+4)H) \times X, S((8n+4)H) \times X)$ in h_Γ-theory. In general if Γ acts on X freely then there is a natural isomorphism $h^*_\Gamma(X) \cong h^*(X/\Gamma)$. Combining this with (1.1) and (1.3) gives rise to the following exact sequences.

$$(1.4a) \qquad \cdots \xrightarrow{\delta} h^*_\Gamma(+) \xrightarrow{J} h^*_\Gamma(+) \xrightarrow{I} h^*(P^{8n+3}) \xrightarrow{\delta} \cdots,$$

$$(1.4b) \qquad \cdots \xrightarrow{\delta} h^*(PG) \xrightarrow{J} h^*(PG) \xrightarrow{I} h^*(P^{8n+3} \times G) \xrightarrow{\delta} \cdots$$

where $G = Sp(2^{4n})$ and there holds the equality $\delta(xI(y)) = \delta(x)y$ in either case.

We write G for $Sp(2^{4n})$ for simplicity in the subsequent sections.

2. By the same symbol $\bar{\sigma}$ we denote the reduced bundles of the canonical line bundles $(S^{8n+4,0} \times H)/\Gamma \to P^{8n+3}$ and $(G \times H)/\Gamma \to PG$. And we write $\sigma = c(\bar{\sigma})$ where c denotes the complexification $KO \to K$. Since

$H^2 = 1$ in $RO(\Gamma)$ there hold obviously

$$\bar{\sigma}^2 + 2\bar{\sigma} = 0 \quad \text{and} \quad \sigma^2 + 2\sigma = 0.$$

Let $\bar{\nu} = p^*(\eta_8^{n+1}) \in \widetilde{KO}^{-5}(P^{8n+3})$ and $\nu = p^*(\mu^{4n+2}) \in \widetilde{K}^{-1}(S^{8n+3})$ where p is the map $P^{8n+3} \to S^{8n+3}$ obtained by collapsing the outside of a top dimensional cell in P^{8n+3} to a point. Then the equalities

$$c(\bar{\nu}) = \mu^2\nu \quad \text{and} \quad r(\nu) = \eta_4\bar{\nu}$$

follow from the relations $c(\eta_4) = 2\mu^2$ and $\eta_4^2 = 4$.

We consider the complex and real K-theories the $Z/2$-and $Z/8$-graded cohomology theories with the coefficient rings $K^*(+) = Z[\mu]/(\mu^2 - 1)$ and $KO^*(+) = Z[\eta_1, \eta_4, \eta_8]/(2\eta_1, \eta_1^3, \eta_1\eta_4, \eta_4^2 - 4, \eta_8 - 1)$ respectively where $\eta_1 \in KO^{-1}(+)$ and the others are as in Section 1. But the complex K-theory is viewed as $Z/8$-graded, so that $K^*(+) = Z[\mu]/(\mu^4 - 1)$, when we discuss the relation between these two kinds of K-theories.

Here we calculate $K^*(P^{8n+3})$ and $KO^*(P^{8n+3})$ whose additive structures are given in [2,5]. Consider the exact sequence of (1.4a). First note that $h_\Gamma^*(+) \cong h^*(+)[t]/(t^2 - 1)$ because of $\Gamma \cong Z/2$ where $t = L$ or H according as $h = K$ or KO. From inspecting the definitions of the maps it follows that

$$(2.1)\quad \begin{aligned} &\delta(\nu) = 1 + L,\ J(1) = 2^{4n+1}(1 - L) \text{ and } I(L) = \sigma + 1 \text{ for } h = K, \\ &\delta(\bar{\nu}) = 1 + H,\ J(1) = 2^{4n}\eta_4(1 - H) \text{ and } I(H) = \bar{\sigma} + 1 \text{ for } h = KO. \end{aligned}$$

Moreover we have a unique element ζ of $KO^{-6}(P^{8n+3})$ satisfying $\delta(\zeta) = \eta_1$.

Using this and the equality $\delta(xI(y)) = \delta(x)y$ we obtain by the exactness of (1.4a) the following.

With the notation as above

$$(2.2a)\qquad \widetilde{K}(P^{8n+3}) = Z/2^{4n+1} \cdot \sigma, \quad \widetilde{K}^{-1}(P^{8n+3}) = Z \cdot \nu$$

where the ring structure is given by

$$\sigma^2 + 2\sigma = 0, \quad \nu^2 = 0,$$

(2.2b)
$$
\begin{aligned}
\widetilde{KO}(P^{8n+3}) &= Z/2^{4n+2}\cdot\bar{\sigma},\\
\widetilde{KO}^{-1}(P^{8n+3}) &= Z/2\cdot\eta_1\bar{\sigma}\oplus Z\cdot\eta_4\bar{\nu},\\
\widetilde{KO}^{-2}(P^{8n+3}) &= Z/2\cdot\eta_1^2\bar{\sigma},\\
\widetilde{KO}^{-3}(P^{8n+3}) &= 0,\\
\widetilde{KO}^{-4}(P^{8n+3}) &= Z/2^{4n}\cdot\eta_4\bar{\sigma},\\
\widetilde{KO}^{-5}(P^{8n+3}) &= Z\cdot\bar{\nu},\\
\widetilde{KO}^{-6}(P^{8n+3}) &= Z/2\cdot\eta_1\bar{\nu}\oplus Z/2\cdot\zeta,\\
\widetilde{KO}^{-7}(P^{8n+3}) &= Z/2\cdot\eta_1^2\bar{\nu}\oplus Z/2\cdot\eta_1\zeta
\end{aligned}
$$

where the ring structure is given by

$$
\begin{aligned}
&\bar{\sigma}^2+2\bar{\sigma}=0, \qquad \bar{\nu}^2=0, \qquad \zeta^2=0, \qquad \eta_4\zeta=0,\\
&\bar{\sigma}\zeta=\eta_1\bar{\nu}, \qquad \eta_1^2\zeta=2^{4n+1}\bar{\sigma}.
\end{aligned}
$$

Now we are ready for computing the K-groups of PG.

Let ρ be the canonical, non-trivial, 2^{4n}-dimensional complex representation of G and $\lambda^i\rho$ be the i-th exterior power of ρ. Since the restriction of $\lambda^{2i}\rho$ to the center of G is trivial clearly, it factors through the canonical projection $\pi\colon\to PG$. So we view $\lambda^{2i}\rho$ also as a representation of PG below. Moreover, as is well known, an element of $\widetilde{K}^{-1}(PG)$ is represented as the homotopy class of a map from PG to the infinite dimensional unitary group U. Hence we see that $\lambda^{2i}\rho$ yields naturally an element $\beta(\lambda^{2i}\rho)$ of $K^{-1}(PG)$, which is called the β-construction of $\lambda^{2i}\rho$ [8]. Because $\dim_C\lambda^{2i+1}\rho = \binom{2^{4n+1}}{2i+1}$ and $2^{4n+1}\|\binom{2^{4n+1}}{2i+1}$, $d_{2i+1} = \binom{2^{4n+1}}{2i+1}/2^{4n+1}$ is odd. Let $\ell\rho$ denote the direct sum of ℓ copies of ρ. The map $PG\to U\left(\binom{2^{4n+1}}{2i+1}\right)$ given by the assignment $\pi(g)\mapsto(d_{2i+1}\rho)(g)\lambda^{2i+1}\rho(g)$ defines a similar element $\beta(d_{2i+1}\rho+\lambda^{2i+1}\rho)$ of $K^{-1}(PG)$.

We describe explicitly the image of $\beta(\rho)\in K^{-1}(G)$ by the transfer map $\pi_*\colon K^{-1}(G)\to K_\Gamma^{-1}(G)=K^{-1}(PG)$. Let us view $E=G\times(C^{2^{4n+1}}\oplus C^{2^{4n+1}})$ as a product Γ-vector bundle over G provided with the Γ-action given by $(g,u,v)\mapsto(-g,v,u)$ for $g\in G$, $u,v\in C^{2^{4n+1}}$. Then the assignment $(g,u,v)\mapsto(g,\rho(g)u,-\rho(g)v)$ gives an equivariant bundle automorphism of E. In a canonical way this gives rise to an element of $K_\Gamma^{-1}(G)$ which is just $\pi_*(\beta(\rho))$ and is written $\beta(\rho,\Gamma)$ below.

Then we have

Theorem 2.3 ([7,9]). *With the notation as above*

$$K^*(PSp(2^{4n})) = Z[\sigma]/(2^{4n+1}\sigma, \sigma^2+2\sigma)$$
$$\otimes \Lambda(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho), \beta(\lambda^{2j}\rho), \beta(\rho, \Gamma)$$
$$(2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1}))/I$$

as a ring where I is the ideal generated by

$$\sigma\beta(\rho, \Gamma).$$

Proof. We observe the exact sequence of (1.4b). According to [8]

$$K^*(G) = \Lambda(\beta(\rho), \beta(\lambda^2\rho), \cdots, \beta(\lambda^{2^{4n}}\rho)).$$

Since $K^*(G)$ is torsion-free we have the Künneth isomorphism

$$K^*(P^{8n+3} \times G) \cong K^*(P^{8n+3}) \otimes K^*(G).$$

Then we get similarly to (2.1) the following.

(2.4) $\delta(\nu \times 1) = \sigma + 2, \ \ J(1) = -2^{4n+1}\sigma$ and $I(\sigma) = \sigma + 1.$

Now $2^{4n+1}\sigma = 0$ follows because of $\rho(-1) = -I$. Hence (1.4b) becomes a short exact sequence

$$0 \longrightarrow K^*(PG) \xrightarrow{I} K^*(P^{8n+3} \times G) \longrightarrow_{\delta} K^*(PG) \longrightarrow 0$$

provided with $\delta(xI(y)) = \delta(x)y$. Further by inspecting definition we have

(2.5)
$$\begin{aligned} &I(\beta(\lambda^{2i}\rho)) = 1 \times \beta(\lambda^{2i}\rho), \\ &I(\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho)) \\ &\qquad = (\sigma+1) \times d_{2i-1}\beta(\rho) + 1 \times \beta(\lambda^{2i-1}\rho) + d_{2i-1}\nu \times 1, \\ &I(\beta(\rho, \Gamma)) = (\sigma+2) \times \beta(\rho) + \nu \times 1, \\ &\delta(1 \times \beta(\rho)) = -1. \end{aligned}$$

Let R denote the ring on the right-hand side of the equality of the theorem. Using the last formula of (2.4) and the first three formulas of (2.5), the injectivity of I shows that R is a subring of $K^*(PG)$.

To prove the theorem it therefore suffices to verify that $\mathrm{Im}\,\delta = R$ since δ is surjective. The images of generators of $K^*(P^{8n+3} \times G)$ as a module by δ can be calculated by using (2.5) together with the equality $\delta(xI(y)) = \delta(x)y$. For example, we have

$$\begin{aligned}
&\delta(1 \times \beta(\lambda^{2i-1}\rho)) = -d_{2i-1}(\sigma + 1),\\
&\delta(\nu \times 1) = -(\sigma + 2),\\
&\delta(\nu \times \beta(\rho)) = \beta(\rho, \Gamma),\\
&\delta(1 \times \beta(\rho)\beta(\lambda^{2i-1}\rho)) = -\beta(d_{2i-1}\rho + \lambda^{2i-1}\rho) - d_{2i-1}\beta(\rho, \Gamma).
\end{aligned}$$

Thus by repeating such a computation inductively we get $\operatorname{Im}\delta = R$, which completes our proof.

3. In this section we compute $KO^*(PG)$. First we consider the exact sequence (1.4b) for KO-theory. The complex representation ρ of G is, of course, the complexification of the 2^{4n}-dimensional quaternionic representation, for which we write $\bar{\rho}$. Clearly $\bar{\rho}$ yields an isomorphism $G \times \boldsymbol{H}^{2^{4n}} \otimes_{\boldsymbol{R}} H \cong G \times \boldsymbol{H}^{2^{4n}}$ of Γ-quaternionic vector bundles over G. Now we have $J(1) = 2^{4n}\eta_4\bar{\sigma}$ similarly to the 2nd formula of (2.1) and also $\alpha \otimes_{\boldsymbol{C}} \boldsymbol{H} = \eta_4$ by (1.2). Hence we see that $J(1) = 0$, so that (1.4b) becomes a short exact sequence

$$(3.1)\quad 0 \longrightarrow KO^*(PG) \xrightarrow{I} KO^*(P^{8n+3} \times G) \xrightarrow{\delta} KO^*(PG) \longrightarrow 0$$

provided with $\delta(xI(y)) = \delta(x)y$.

Using this exact sequence we proceed as the same way as for $K^*(PG)$.

Let $\lambda^k_{\boldsymbol{C}}\bar{\rho}$ be the exterior power $\bar{\rho} \wedge_{\boldsymbol{C}} \cdots \wedge_{\boldsymbol{C}} \bar{\rho}$ of $\bar{\rho}$ over $\boldsymbol{C}$. Then in general $\lambda^k_{\boldsymbol{C}}\bar{\rho}$ is quaternionic. But if k is even then it has a natural Real structure. So we consider $\lambda^{2i}_{\boldsymbol{C}}\bar{\rho}$ to be real. By the β-construction we have

$$\beta(\lambda^{2i-1}_{\boldsymbol{C}}\bar{\rho}) \in \widetilde{KSp}^{-1}(G) \quad \text{and} \quad \beta(\lambda^{2i}_{\boldsymbol{C}}\bar{\rho}) \in \widetilde{KO}^{-1}(G)$$

and we set

$$\bar{\beta}(\lambda^{2i-1}_{\boldsymbol{C}}\bar{\rho}) = \alpha \wedge_{\boldsymbol{C}} \beta(\lambda^{2i-1}_{\boldsymbol{C}}\bar{\rho}) \in \widetilde{KO}^{-1}(\Sigma^{0,4} \wedge G) = \widetilde{KO}^{-5}(G).$$

Then, according to [15], Theorem 5.6,

$$(3.2)\qquad KO^*(G) = \Lambda_{KO^*(+)}(\bar{\beta}(\lambda^{2i-1}_{\boldsymbol{C}}\bar{\rho}), \beta(\lambda^{2i}_{\boldsymbol{C}}\bar{\rho}) \quad (1 \le i \le 2^{4n-1}))$$

as a $KO^*(+)$-module. Further by [4], §6 and [13], Corollary 2.3 we see that its generators satisfy the relations

$$(3.3)\qquad \bar{\beta}(\lambda^{2i-1}_{\boldsymbol{C}}\bar{\rho})^2 = \eta_1\beta(\lambda^{4i-2}_{\boldsymbol{C}}\bar{\rho}), \quad \beta(\lambda^{2i}_{\boldsymbol{C}}\bar{\rho})^2 = \eta_1\beta(\lambda^{4i}_{\boldsymbol{C}}\bar{\rho}).$$

Here we note that $\lambda_C^k\bar{\rho} = \lambda_C^{2^{4n+1}-k}\bar{\rho}$ for $1 \leq k \leq 2^{4n}$. Of course this equality holds for $\lambda_C^{2k}\bar{\rho}$ viewed as a representation of PG.

Because $KO^*(G)$ is torsion-free, there holds the Künneth isomorphism $KO^*(P^{8n+3} \times G) \cong KO^*(P^{8n+3}) \otimes_{KO^*(+)} KO^*(G)$. Therefore by using (2.2b), (3.2) and (3.3), the multiplicative structure of $KO^*(P^{8n+3} \times G)$ centered in the sequence (3.1) can be described explicitly.

In order to state our theorem we provide generators of $KO^*(PG)$. Similarly to the complex case we have

$$\begin{aligned}&\beta(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}\bar{\rho}), \quad \beta(\bar{\rho}, \Gamma) \in \widetilde{KSp}^{-1}(PG) \quad \text{and}\\ &\beta(\lambda_C^{2i}\bar{\rho}) \in \widetilde{KO}^{-1}(PG)\end{aligned}$$

and so we set

$$\begin{aligned}&\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}\bar{\rho}) = \alpha \wedge_C \beta(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}\bar{\rho}),\\ &\bar{\beta}(\bar{\rho}, \Gamma) = \alpha \wedge_C \beta(\bar{\rho}, \Gamma) \in \widetilde{KO}^{-5}(PG).\end{aligned}$$

Moreover we see that

(3.4) There exists an element $\bar{\zeta} \in KO^{-6}(PG)$ such that

$$I(\bar{\zeta}) = \eta_1 \times \bar{\beta}(\bar{\rho}) + \zeta \times 1.$$

This is shown below.

Then we obtain the following.

Theorem 3.5. *With the notation as above*

$$KO^*(PSp(2^{4n})) = \boldsymbol{Z}[\bar{\sigma}]/(\bar{\sigma}^2 + 2\bar{\sigma}) \otimes E \otimes \Lambda_{\boldsymbol{Z}/2}(\bar{\zeta})/I$$

as a ring where E is a $KO^*(+)$-module

$$\begin{aligned}\Lambda_{KO^*(+)}(\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}), \beta(\lambda_C^{2j}\bar{\rho}), \bar{\beta}(\bar{\rho}, \Gamma)\\ (2 \leq i \leq 2^{4n-1}, 1 \leq j \leq 2^{4n-1}))\end{aligned}$$

with the relations

$$\begin{aligned}&\bar{\beta}(d_{2i-1}\bar{\rho} + \lambda_C^{2i-1}\bar{\rho})^2 = \eta_1(\beta(\lambda_C^2\bar{\rho}) + \beta(\lambda_C^{4i-2}\bar{\rho})),\\ &\beta(\lambda_C^{2j}\bar{\rho})^2 = \eta_1\beta(\lambda_C^{4j}\bar{\rho}),\\ &\bar{\beta}(\bar{\rho}, \Gamma)^2 = 0\end{aligned}$$

and I is the ideal generated by

$$2^{4n}\bar{\sigma}\eta_4,\quad \bar{\sigma}\bar{\beta}(\bar{\rho},\Gamma),\quad \eta_4\bar{\zeta},\quad \bar{\sigma}\bar{\zeta}-\eta_1\bar{\beta}(\bar{\rho},\Gamma),\quad \eta_1^2\bar{\zeta}-2^{4n+1}\bar{\sigma}$$

(the $\otimes$'s are omitted).

Proof. Observe (3.1). By looking at the definitions of the maps and elements we have

(i) $I(\bar{\sigma})=\bar{\sigma}\times 1$,

(ii) $I(\beta(\lambda_C^{2i}\bar{\rho}))=1\times\beta(\lambda_C^{2i}\bar{\rho})$,

(iii) $I(\bar{\beta}(d_{2i-1}\bar{\rho}+\lambda_C^{2i-1}\bar{\rho}))=(\bar{\sigma}+1)\times d_{2i-1}\bar{\beta}(\bar{\rho})+1\times\bar{\beta}(\lambda_C^{2i-1}\bar{\rho})$
$+\,d_{2i-1}\bar{\nu}\times 1$,

(iv) $I(\bar{\beta}(\bar{\rho},\Gamma))=(\bar{\sigma}+2)\times\bar{\beta}(\bar{\rho})+\bar{\nu}\times 1$,

(v) $I(1\times\bar{\beta}(\bar{\rho}))=-1$,

(vi) $\delta(\bar{\nu}\times 1)=(\bar{\sigma}+2)\times 1$,

(vii) $\delta(\zeta\times 1)=\eta_1$.

(3.4) is immediate from (v) and (vii). Let $\bar{R}$ denote the ring on the right-hand side of the equality of Theorem 3.5. Then using (i)–(iv) and (3.4) we see that $\bar{R}\subset KO^*(PG)$ because of the injectivity of I, and by using (v)–(vii) and the equality $\delta(xI(y))=\delta(x)y$ in addition we can verify that $\bar{R}$ fills $KO^*(PG)$ because of the surjectivity of δ. This completes the proof of the theorem.

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