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A NON-IMMERSION RESULT FOR LENS SPACES $L^n(2^m)$

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1. Introduction. The lens space $L^n(2^m)$ is the quotient of the sphere S^{2n+1} by the free action of the cyclic group $Z/2^m$ given by:

$$\zeta^k z = (\zeta^k z_0, \zeta^k z_1, \ldots, \zeta^k z_n),$$

where $\zeta = \exp(i\pi/2^{m-1})$ is the generator of $Z/2^m$, and $z = (z_0, z_1, \ldots, z_n) \in C^{n+1}$ is such that $\sum_{i=0}^n |z_i|^2 = 1$. A classical question is to determine the smallest integer k such that $L^n(2^m)$ immerses into R^{2n+1+k}. In [3], we have seen that for m sufficiently large, k is greater or equal than $2n - 2\alpha(n)$, where $\alpha(n)$ denotes the number of 1 in the dyadic expansion of n. More precisely, we have proved the following theorem

Theorem 1.1. *For $m \geq [\log_2 n] + [n/2]$, $L^n(2^m)$ does not immerse into $R^{4n-2\alpha(n)}$.*

Here $[x]$ denotes the integer part of x. Some other results have been published in the same direction, (see [1], [5], [6] and [7]). In this note, we are completing theorem 1.1 for the case $m \leq [\log_2 n] + [n/2] - 1$. Let $l(n)$ be the integer

$$l(n) = \max\left\{1 \leq i \leq n-1 \ \text{ such that } \binom{n+i+1}{n} \not\equiv 0 \pmod 4\right\}.$$

We prove:

Theorem 1.2. *Let $m \geq 2$.*

a) *If $n \neq 2^s + 1$ and $n \geq 2$, $L^n(2^m)$ does not immerse in $R^{2n+1+2l(n)}$.*

b) *If $n = 2^s + 1$, with $s \geq 1$, $L^n(2^m)$ does not immerse into $R^{2n+2l(n)} = R^{4n-4}$.*

We apply theorem 1.2 to some particular values of n, and we obtain

Corollary 1.1. *Let $m \geq 2$.*

a) *If $n = 2^s$ with $s \geq 1$, $L^n(2^m)$ does not immerse in R^{4n-1}.*

b) *If $n = 2^s + 2^t$, with $s > t \geq 1$, $L^n(2^m)$ does not immerse in R^{4n-3}.*

This improves for these two cases the results obtained by theorem 1.1. Recalling that for $n = 2^s$, the space $L^n(2^m)$ immerses in $\mathbf{R}^{4n}$, we note that our result is the best possible for this case.

2. Preliminaries. In this section we establish some cohomology properties of the spaces $B(n,k)$ defined in [4] (see also [2]). This properties will be used to prove theorem 1.2. We begin with a result about spherical fibrations and recall that for any sphere bundle $S^k \xrightarrow{i} E \xrightarrow{p} B$ there is long exact sequence of $H^*(B;Z)$-modules called the Gysin sequence (see [8] p.143 or [9] p.356)

$$\cdots \longrightarrow H^q(B;Z) \xrightarrow{p^*} H^q(E;Z) \xrightarrow{\phi} H^{q-k}(B;Z) \xrightarrow{\cup e} H^{q+1}(B;Z) \longrightarrow \cdots$$

where e is the Euler-class of the fibration. In particular we have:

Lemma 2.1. *If in the above spherical fibration, B is connected and the Euler-class e is zero, then*

$$H^*(E;Z) \cong H^*(B;Z) \oplus a \cup H^*(B;Z)$$

as an $H^*(B;Z)$-module, where a is an element of $H^k(E;Z)$ such that $\phi(a)$ is a generator of $H^0(B;Z) \cong Z$.

The proof of this lemma is straightforward.

We now turn to the space $B(n,k)$ which by definition is the pull-back space of the diagram

$$\begin{array}{ccc} & & BSO(k) \\ & & \downarrow \\ BU(n) & \longrightarrow & BSO(2n) \end{array}$$

Inductively we can identify the space $B(n,k)$ with the pull-back of the diagram

$$\begin{array}{ccc} & & BSO(k) \\ & & \downarrow \\ B(n,k+1) & \longrightarrow & BSO(k+1) \end{array}$$

Let be $V_{2n,2n-2j}$ the Stiefel manifold $SO(2n)/SO(2j)$, and let

(2.1) $$V_{2n,2n-2j} \xrightarrow{i_1} B(n,2j) \xrightarrow{p} BU(n)$$

be the fibration induced from

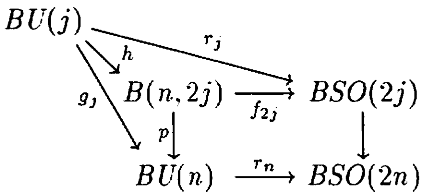
$$V_{2n,2n-2j} \xrightarrow{i_2} BSO(2j) \longrightarrow BSO(2n)$$

by the canonical map $BU(n) \xrightarrow{r_n} BSO(2n)$.

Let u_j be the generator of $H^{2j}(V_{2n,2n-2j}; Z) \cong Z$ such that

$$i_2^*(e_j) = -2u_j,$$

where $e_j \in H^{2j}(BSO(2j); Z)$ is the universal Euler-Poincaré class. By the pull-back property, there is a map $BU(j) \xrightarrow{h} B(n, 2j)$ and a commutative diagramm



where all the others maps are canonical maps.

Lemma 2.2. *For every* $n \geq 1$ *and* $1 \leq j \leq n-1$, *there is an element* a_j *in the abelian group* $H^{2j}(B(n, 2j); Z)$ *such that*

$$f_{2j}^*(e_j) = p^*(c_j) - 2a_j, \quad i_1^*(a_j) = u_j, \quad h^*(a_j) = 0.$$

Proof. There is an exact sequence coming from the Serre spectral sequence of the fibration (2.1)

$$0 \longrightarrow H^{2j}(BU(n); Z) \xrightarrow{p^*} H^{2j}(B(n, 2j); Z) \xrightarrow{i_1^*} H^{2j}(V_{2n,2n-2j}; Z) \longrightarrow 0$$

since $V_{2n,2n-2j}$ is $(2j-1)$-connected and $BU(n)$ is 1-connected without cohomology in odd degree. Let $x \in H^{2j}(B(n, 2j); Z)$ be such that $i^*(x) = u_j$. Since the map g_j^* is an isomorphism in degree $\leq 2j$ we can replace x by $a_j = x - p^*((g_j^*)^{-1}(h^*(x)))$ so that $h^*(a_j) = 0$. The above exact sequence splits and we have an isomorphism

$$H^{2j}(B(n, 2j); Z) \cong \mathrm{im}(p^*) \oplus Za_j \cong H^{2j}(BU(n); Z) \oplus Za_j.$$

On the other hand $h^*(f_{2j}^*(e_j)) = r_j^*(e_j) = c_j$ so $f_{2j}^*(e_j) = p^*(c_j) + ma_j$. As $i_1^*((f_{2j}^*(e_j))) = i_2^*(e_j) = -2u_j$ we see that $m = -2$ and $f_{2j}^*(e_j) = p^*(c_j) - 2a_j$.

Let now

$$S^{r-1} \longrightarrow B(n,r-1) \xrightarrow{p_{r-1}} B(n,r) \tag{2.2}$$

be the spherical fibration induced from

$$S^{r-1} \longrightarrow BSO(r-1) \longrightarrow BSO(r)$$

by the map $B(n,r) \xrightarrow{f_r} BSO(r)$. We consider the Gysin sequence of (2.2) which becomes, using lemma 2.2,

$$\cdots \longrightarrow H^0(B(n,2j);Z) \xrightarrow{\cup(p^*(c_j)-2a_j)} H^{2j}(B(n,2j);Z) \xrightarrow{p^*_{2j-1}} H^{2j}(B(n,2j-1);Z) \longrightarrow \cdots.$$

By exactness, $p^*_{2j-1}(p^*(c_j)) = 2p^*_{2j-1}(a_j)$.

In the following, we note $p^*_{2j-1}(a_j) = b_j$, more generally, $(p^*_k \circ p^*_{k+1} \circ \cdots \circ p^*_{2j-1})(a_j) = b_j$ and for simpicity $(p^*_k \circ p^*_{k+1} \circ \cdots \circ p^*_{2j-1})(p^*(c_j)) = c_j$. So we have for every space $B(n,k)$ a family of elements b_i, $[k/2]+1 \le i \le n-1$, such that $2b_i = c_i$. We can now give the additive structure of $H^*(B(n,k);Z)$. This result already appears in [2] and [4].

Theorem 2.1. *$H^*(B(n,k);Z)$ is a free Z-module determined by the isomorphism*

$$H^*(B(n,k);Z) \cong \begin{cases} Z[c_1,\ldots,c_t] \otimes \Delta(a_t, b_{t+1},\ldots,b_{n-1}) & \text{if } k = 2t \\ Z[c_1,\ldots,c_t] \otimes \Delta(b_{t+1},\ldots,b_{n-1}) & \text{if } k = 2t+1 \end{cases}$$

where $\Delta(x_1,\cdots,x_m)$ is the free abelian group generated by the elements

$$x_{i_1}x_{i_2}\ldots x_{i_s}, \quad 1 \le i_1 < i_2 < \cdots < i_s \le m.$$

Proof of theorem 2.1. We proceed by induction descending over k, beginning with $k = 2n-1$. In this case the result is valid since $B(n,2n-1) = BU(n-1)$. Next we examine the case $k = 2n-2$. Here we consider the spherical fibration (2.2) with $r = 2n-1$. As $H^{2n-1}(BU(n-1);Z) = 0$, the Euler class of this fibration is 0 and the Gysin sequence splits into short exact sequences

$$0 \longrightarrow H^{2q}(BU(n-1);Z) \xrightarrow{p^*_{2n-2}} H^{2q}(B(n,2n-2);Z) \xrightarrow{\phi} H^{2q-2n+2}(BU(n-1);Z) \longrightarrow 0.$$

In lemma 2.2 we have seen that

$$H^{2n-2}(B(n,2n-2);Z) \cong \mathrm{im}(p^*) \oplus Z a_{n-1}.$$

But $\mathrm{im}(p^*) \subset \mathrm{im}(p^*_{2n-2}) = \ker(\phi)$, so the element $\phi(a_{n-1})$ is a generator of $H^0(BU(n-1);Z) = Z$. Under the map p^*_{2n-2} we have a $H^*(BU(n-1);Z)$-module structure over $H^*(B(n,2n-2);Z)$. With the help of lemma 2.1, we can see that this structure is given by the isomorphism

$$\begin{aligned} H^*(B(n,2n-2);Z) &\cong H^*(BU(n-1);Z) \oplus a_{n-1} \cup H^*(BU(n-1);Z) \\ &\cong Z[c_1,\cdots,c_{n-1}] \otimes \Delta(a_{n-1}) \end{aligned}$$

this achieves the proof in this case. Moreover the multiplicative structure is well-known in this case, since

$$\begin{aligned} (c_{n-1} - 2a_{n-1})^2 &= f^*_{2n-2}(e^2_{n-1}) = f^*_{2n-2}(P_{n-1}) \\ &= p^*(r^*_n(P_{n-1})) = p^*(c^2_{n-1} - 2c_{n-2}c_n) \\ &= c^2_{n-1} \end{aligned}$$

where P_{n-1} is the $(n-1)^{\text{th}}$-Pontrjagin class in $H^{4n-4}(BSO(2n-2);Z)$ or in $H^{4n-4}(BSO(2n);Z)$. The relations used here are proved for example in [8] (see also proof of lemma (2.3)). So we have $a^2_{n-1} = c_{n-1}a_{n-1}$.

Now we suppose that the result is valid for $r \leq k \leq 2n-1$. We consider the Gysin sequence of the sphere bundle (2.2):

$$\begin{aligned} \cdots \longrightarrow H^q(B(n,r);Z) &\xrightarrow{p^*_{r-1}} H^q(B(n,r-1);Z) \xrightarrow{\phi} \\ H^{q-r+1}(B(n,r);Z) &\xrightarrow{\cup e} H^{q+1}(B(n,r);Z) \longrightarrow \cdots. \end{aligned}$$

1) If r is odd, say $r = 2j+1$, we prove exactly as above that

$$\begin{aligned} H^*(B(n,2j);Z) &\cong H^*(B(n,2j+1);Z) \oplus a_j \cup H^*(B(n,2j+1);Z) \\ &\cong H^*(B(n,2j+1);Z) \otimes \Delta(a_j) \\ &\cong Z[c_1,\ldots,c_j] \otimes \Delta(a_j,b_{j+1},\ldots,b_{n-1}). \end{aligned}$$

Moreover the group homomorphism

$$\begin{aligned} \psi_n : Z[c_1,\ldots,c_{j-1},c_j - 2a_j] &\otimes \Delta(a_j,b_{j+1},\ldots,b_{n-1}) \\ &\longrightarrow H^*(B(n,2j);Z) \end{aligned}$$

defined by $\psi_n(x \otimes y) = x \cup y$, is an isomorphism.

We proceed by induction on n, beginning with $n = j+1$. In this case, the morphism ψ_n becomes

$$\psi_{j+1}\colon\ Z[c_1,\dots,c_{j-1},c_j-2a_j]\otimes\Delta(a_j) \longrightarrow H^*(B(j+1,2j);Z).$$

We have seen above that $a_j^2 = a_jc_j$ in $H^*(B(j+1,2j);Z)$, so

$$\begin{aligned} c_j &= \psi_{j+1}((c_j-2a_j)\otimes 1 + 2(1\otimes a_j)) \\ c_j^2 &= \psi_{j+1}((c_j-2a_j)^2\otimes 1) \end{aligned}$$

and ψ_{j+1} is surjective. As we have a bijection between the Z-module basis, ψ_{j+1} is an isomorphism.

Suppose now that the result is true for $n-1$, and let h: $B(n-1,2j)\to B(n,2j)$ the map induced by the pull-back property of $B(n,2j)$. Let

$$A = Z[c_1,\dots,c_{j-1},c_j]\otimes\Delta(a_j,b_{j+1},\dots,b_{n-2})$$

and

$$B = Z[c_1,\dots,c_{j-1},c_j-2a_j]\otimes\Delta(a_j,b_{j+1},\dots,b_{n-2}).$$

By definition of h^* we have a short exact sequence

$$\begin{array}{ccccc} & & A & & \\ & & \cap & & \\ \ker(h^*) & \longrightarrow & H^*(B(n,2j);Z) & \xrightarrow{h^*} & H^*(B(n-1,2j);Z) \end{array}$$

where $\ker(h^*) = A\cup b_{n-1}$. We also have a Z-modules isomorphism

$$H^*(B(n,2j);Z)\cong A\oplus A\cup b_{n-1}.$$

If $(x_q)_{q\geq 1}$ is the canonical Z-module basis of A, $(x_q\cup b_{n-1})_{q\geq 1}$ is a basis of $A\cup b_{n-1}$. Since h^* is a ring homomorphism, we have the commutative diagramm

$$\begin{array}{ccc} & B & \\ {}^{\psi_n|B}\swarrow & & \searrow{}^{\psi_{n-1}|B} \\ H^*(B(n,2j);Z) & \xrightarrow{h^*} & H^*(B(n-1,2j);Z) \end{array}$$

By the induction hypothesis, ψ_{n-1} is an isomorphism and so $\psi_n|B$ is a monomorphism and there is a basis $(y_q)_{q\geq 1}$ of B, such that

$$\psi_n(y_q) = x_q + z_q\cup b_{n-1}\quad\text{for } q\geq 1,\ \text{ with } z_q\in A.$$

As $b_{n-1}^2 = 0$ in $H^*(B(n,2j);Z)$, $\psi_n| B \cup b_{n-1}$ is injective and $\psi_n(B \cup b_{n-1}) = A \cup b_{n-1}$.

2) If r is even, say $r = 2j$, we know by lemma 2.2 that the Euler-class of the spherical fibration (2.2) is the element $c_j - 2a_j$ and since ψ_n is injective, we can say that the multiplication by the Euler-class is injective, so the map $\phi = 0$ in the Gysin sequence of (2.2) and we have the group isomorphisms

$$\begin{aligned} H^*(B(n,2j-1);Z) &\cong H^*(B(n,2j);Z)/\langle c_j - 2a_j\rangle \\ &\cong Z[c_1,\ldots,c_{j-1}] \otimes \Delta(a_j, b_{j+1},\ldots,b_{n-1}). \end{aligned}$$

We can now describe the multiplicative structure of $H^*(B(n,2j);Z)$ as follows.

Lemma 2.3. *For every $n \geq 1$ and $1 \leq j \leq n-1$, the element a_j in the abelian group $H^*(B(n,2j);Z)$ satisfies the relation*

$$a_j^2 = a_j c_j + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r b_r c_{2j-r}. \tag{2.3}$$

Proof. Recall that the universal Euler-Poincaré class $e_j \in H^{2j}(BSO(2j);Z)$, satisfies the relation

$$e_j^2 = P_j$$

where P_j is the j^{th} universal Pontrjagin class in $H^{4j}(BSO(2j);Z)$, and that

$$r_n^*(P_j) = c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n)} (-1)^r 2c_r c_{2j-r}$$

in $H^{4j}(BU(n);Z)$, here P_j is the j^{th} universal Pontrjagin class in $H^{4j}(BSO(2n);Z)$, (see [8]). From the definition of a_j and the above relations, we see that

$$f_{2j}^*(e_j^2) = c_j^2 - 4a_jc_j + 4a_j^2$$

and

$$\begin{aligned} p^*(r_n^*(P_j)) &= c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r 2c_r c_{2j-r} \\ &= c_j^2 + (-1)^j \sum_{r=j+1}^{\min(2j,n-1)} (-1)^r 4b_r c_{2j-r}. \end{aligned}$$

Since $H^*(B(n,2j);\boldsymbol{Z})$ has no torsion, the relation (2.3) is valid.

Using the relation (2.3) we can now give the Steenrod squares of the mod 2 reduction of the elements a_j in $H^{2j}(B(n,2j);\boldsymbol{Z}/2)$.

Theorem 2.2. *For every $n \geq 1$, every $1 \leq j \leq n-1$ and every $0 \leq k \leq j$ the following relation is valid in $H^*(B(n,2j);\boldsymbol{Z}/2)$.*

$$Sq^{2k}(a_j) = \sum_{r=\max(0,k+j+1-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-r}c_r + a_j c_k. \tag{2.4}$$

Proof. We proceed by an induction argument over n. We begin with the case $n = 1$ where all relations are empty. For $n = 2$, $j = 1$ and $k = 0$ or 1, so the only non trivial relation in $H^*(B(2,2);\boldsymbol{Z}/2)$ is $Sq^2(a_1) = a_1^2 = a_1c_1$ which is compatible with (2.4).

Now we suppose the result is valid for $n \geq 2$. First we observe that (2.4) is still true for $k+j \leq n-1$ in $H^*(B(n+1,2j);\boldsymbol{Z}/2)$ since

$$H^q(B(n+1,2j);\boldsymbol{Z}/2) \cong H^q(B(n,2j);\boldsymbol{Z}/2) \qquad q < 2n.$$

If $k+j \geq n$, we consider the following diagram, where all the arrows are canonical.

$$\begin{array}{ccccc}
B(n,2j-2)\times \boldsymbol{CP}^\infty & & \longrightarrow & & BSO(2j-2)\times \boldsymbol{CP}^\infty \\
\Big\downarrow & & & & \Big\downarrow \\
 & & B(n+1,2j) & \longrightarrow & BSO(2j) \\
 & & \Big\downarrow & & \Big\downarrow \\
BU(n)\times \boldsymbol{CP}^\infty & \longrightarrow & BU(n+1) & \longrightarrow & BSO(2n+2)
\end{array}$$

In particular the next square is homotopy commutative

$$\begin{array}{ccc}
B(n,2j-2)\times \boldsymbol{CP}^\infty & \longrightarrow & BSO(2j) \\
\Big\downarrow & & \Big\downarrow \\
BU(n+1) & \longrightarrow & BSO(2n+2)
\end{array} \tag{2.5}$$

and replacing if necessary $B(n,2j-2)\times \boldsymbol{CP}^\infty \to BSO(2j)$ by a map homotopy equivalent, we can suppose that the diagramm (2.5) is commutative since the map $BSO(2j) \to BSO(2n+2)$ is a fibration.

So there is a map f: $B(n, 2j-2) \times CP^\infty \to B(n+1, 2j)$ such that the squares

$$\begin{array}{ccc} B(n,2j-2) \times CP^\infty & \longrightarrow & BSO(2j-2) \times CP^\infty \\ {\scriptstyle f}\downarrow & & \downarrow \\ B(n+1,2j) & \longrightarrow & BSO(2j) \end{array}$$

and

$$\begin{array}{ccc} B(n,2j-2) \times CP^\infty & \xrightarrow{f} & B(n+1,2j) \\ \downarrow & & \downarrow \\ BU(n) \times CP^\infty & \longrightarrow & BU(n+1) \end{array}$$

are still commutative. We can easy see that

$$\begin{aligned} f^*(c_i) &= c_i + c_{i-1}z \quad (1 \le i \le j), & f^*(a_j) &= b_j + a_{j-1}z, \\ f^*(b_i) &= b_i + b_{i-1}z \quad (j+1 \le i \le n-1), & f^*(b_n) &= b_{n-1}z \end{aligned}$$

in $H^*(B(n,2j-2)\times CP^\infty; Z) \cong H^*(B(n,2j-2); Z) \otimes H^*(CP^\infty; Z)$, where z is the canonical generator of $H^2(CP^\infty; Z)$.

Let be $G = Z/2[c_1, \ldots, c_{j-1}] \otimes (Z/2\langle a_j\rangle \oplus Z/2\langle b_{j+1}\rangle \oplus \cdots \oplus Z/2\langle b_n\rangle)$, where $Z/2\langle x\rangle$ is the group of order two with generator x. It is clear that G is a subgroup of $H^*(B(n+1, 2j); Z/2)$ and we can easy see that the restriction of f^* to G is injective. Let h: $B(n, 2j) \to B(n+1, 2j)$ be the canonical map as in the proof of theorem 2.1. For $j > 1$, $k < j$ and $k + j \ge n$,

$$\begin{aligned} h^*(Sq^{2k}(a_j)) = Sq^{2k}(h^*(a_j)) &= Sq^{2k}(a_j) \\ &= \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} b_{k+j-r}c_r + a_j c_k \end{aligned}$$

by the induction hypothesis, and since $\ker(h^*) = b_n \cup H^*(B(n, 2j); Z/2)$, we have

$$Sq^{2k}(a_j) = \sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} b_{k+j-r}c_r + a_j c_k + b_n p(c_1, \cdots, c_{j+k-n})$$

where $p(c_1, \ldots, c_{j+k-n}) \in Z/2[c_1, \ldots, c_{j-1}]$. Then, the element $Sq^{2k}(a_j)$ is in G and we can give its image under f^*

$$\begin{aligned} f^*(Sq^{2k}(a_j)) = Sq^{2k}(f^*(a_j)) &= Sq^{2k}(b_j + a_{j-1}z) \\ &= Sq^{2k}(b_j) + Sq^{2k}(a_{j-1})z + Sq^{2k-2}(a_{j-1})z^2. \end{aligned}$$

Applying once more the induction hypothesis,

$$\begin{aligned} f^*(Sq^{2k}(a_j)) = &\sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r} c_r \\ &+ \sum_{r=k+j-n}^{k-1} \binom{j-1-r}{k-r} b_{k+j-1-r} c_r z \\ &+ \sum_{r=\max(0,k+j-1-n)}^{k-2} \binom{j-1-r}{k-1-r} b_{k+j-2-r} c_r z^2 \\ &+ a_{j-1} c_k z + a_{j-1} c_{k-1} z^2 \end{aligned}$$

and since $\binom{j-1-r}{k-r} \equiv \binom{j-1-r}{k-1-r} + \binom{j-r}{k-r} \pmod 2$,

$$\begin{aligned} f^*(Sq^{2k}(a_j)) = &\sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r}(c_r + c_{r-1} z) \\ &+ \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_r z \\ &+ \sum_{r=\max(1,k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} c_{r-1} z^2 \\ &+ a_{j-1} z(c_k + c_{k-1} z). \end{aligned}$$

If $k + j > n$ we have

$$\begin{aligned} f^*(Sq^{2k}(a_j)) = &\sum_{r=k+j+1-n}^{k} \binom{j-r}{k-r} b_{k+j-r}(c_r + c_{r-1} z) \\ &+ \sum_{r=k+j-n}^{k-1} \binom{j-r}{k-r} b_{k+j-1-r} z(c_r + c_{r-1} z) \\ &+ a_{j-1} z(c_k + c_{k-1} z) \\ = &\sum_{r=k+j+1-n}^{k-1} \binom{j-r}{k-r} (b_{k+j-r} + b_{k+j-1-r} z)(c_r + c_{r-1} z) \\ &+ (b_j + a_{j-1} z)(c_k + c_{k-1} z) \\ &+ \binom{n-k}{n-j} b_{n-1} z(c_{j+k-n} + c_{j+k-1-n} z) \\ = &f^*(\sum_{r=\max(0,k+j-n)}^{k-1} \binom{j-r}{k-r} b_{k+j-r} c_r + a_j c_k) \end{aligned}$$

as expected since $f^*|G$ is injective. If $k + j = n$, we proceed exactly as above. It remains two cases, the first is for $j = 1$, but the only non trivial

Steenrod operations are $Sq^0(a_1) = a_1$ and $Sq^2(a_1) = a_1^2 = a_1c_1 + b_2$ by lemma 2.3. The second is for $k = j$ but in this case

$$\begin{aligned} Sq^{2j}(a_j) = a_j^2 &= a_jc_j + \sum_{r=2j-n}^{j-1} b_{2j-r}c_r \\ &= \sum_{r=2j-n}^{j-1} \binom{j-r}{j-r} b_{2j-r}c_r + a_jc_j \end{aligned}$$

always by lemma 2.3.

3. Proof of Theorem 1.2. The integral cohomology and the mod 2 cohomology of $L^n(2^m)$ are well known, they are given by the isomorphisms of abelian groups:

$$H^q(L^n(2^m); Z) \cong \begin{cases} Z & \text{if } q = 0,\ 2n+1 \\ Z/2^m & \text{if } q = 2i,\ 0 < i \leq n \\ 0 & \text{otherwise,} \end{cases}$$

$$H^q(L^n(2^m); Z/2) \cong \begin{cases} Z/2 & \text{if } 0 \leq q \leq 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Let be $\pi\colon L^n(2^m) \to CP^n$ the natural projection, μ the canonical complex line bundle over CP^n, and let denote $z = c_1(\pi^*(\mu)) = \pi^*(c_1(\mu)) \in H^2(L^n(2^m); Z)$. We observe that z^i is an additive generator of $H^{2i}(L^n(2^m); Z)$ for every $1 \leq i \leq n$.

Let us still write z^i for the mod 2 reduction of the additive generator above, we see readily that

$$Sq^2(z^i) = iz^{i+1} \tag{3.1}$$

$$Sq^4(z^i) = \binom{i}{2} z^{i+2}. \tag{3.2}$$

Finally, let $l(n)$ denote the integer

$$l(n) = \max\left\{0 \leq i \leq n-1 \text{ such that } \binom{n+i+1}{n} \not\equiv 0 \pmod 4\right\}.$$

Recall the 2-divisibility of $\binom{n+i+1}{n}$:

$$\nu_2(\binom{n+i+1}{n}) = \alpha(n) + \alpha(i+1) - \alpha(n+i+1).$$

We observe that for $\alpha(n) = 1$ and $i = n-1$, we get $\nu_2(\binom{n+i+1}{n}) = \alpha(n) = 1$ and so $l(n) = n-1$. For $\alpha(n) = 2$ we obtain, likewise, $l(n) = n-2$.

For $\alpha(n) \geq 3$, we have the next result where we relate $l(n)$ with the dyadic expansion of n.

Lemma 3.1. *If* $n = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_k}$ *with* $s_1 > s_2 > \cdots > s_k \geq 0$ *and* $k \geq 3$, $l(n) = 2^{s_1} + 2^{s_2} - 2 - 2^{s_3} - \cdots - 2^{s_k}$.

Proof. The 2-divisibility of $\binom{n+i+1}{n}$ is 0 or 1 if n and $i+1$ have at most one common term in their dyadic expansion. So $i+1$ is greatest possible, if there is one common term of highest 2-valuation, here 2^{s_1}. The rest of the expansion of $i+1$ contains all powers 2^r with $r < s_2$, except $r = s_3, \ldots, s_k$.

This description of $l(n)$ gives for $\alpha(n) \geq 3$:

$$l(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \equiv 0 \pmod 4 \\ 1 \pmod 4 & \text{if } n \equiv 1 \pmod 4 \\ 0 \pmod 4 & \text{if } n \equiv 2 \pmod 4 \\ 3 \pmod 4 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

We come back to the immersion problem for $L^n(2^m)$. We know that the stable class of the tangent bundle of $L^n(2^m)$ is $r(n+1)\sigma$ (see [10]), where r denotes the realification. So, if $L^n(2^m)$ immerses in $\boldsymbol{R}^{2n+1+k}$, the stable class of the normal bundle of this immersion is $-r(n+1)\sigma$ and its classifying map

$$-r(n+1)\sigma\colon\ L^n(2^m) \longrightarrow BSO(2n+2)$$

lifts to $BU(n+1)$ and to $BSO(k)$. Therefore, this map also lifts to $B(n+1,k)$, and we obtain the commutative diagram

$$\begin{array}{ccc} & & B(n+1,k) \\ & {\scriptstyle \tilde{f}_k}\nearrow & \downarrow{\scriptstyle p} \\ L^n(2^m) & \xrightarrow{\ g\ } & BU(n+1) \end{array}$$

where $g\colon L^n(2^m) \to BU(n+1)$ denotes a lifting of $-r(n+1)\sigma$ to $BU(n+1)$ and $\tilde{f}_k$ a lifting of g in $B(n+1,k)$. We also note that

$$\begin{aligned} g^*(c_i) &= c_i(-(n+1)\sigma) = \binom{-n-1}{i} z^i \\ &= (-1)^i \binom{n+i}{n} z^i \end{aligned}$$

since for the total Chern class of $-(n+1)\sigma$ we find

$$c(-(n+1)\sigma) = c(\sigma)^{-n-1} = (1+c_1(\sigma))^{-n-1} = (1+z)^{-n-1}$$
$$= \sum_{i\geq 0} \binom{-n-1}{i} z^i.$$

For $i \geq [k/2]+1$, we have $p^*(c_i) = 2b_i$ in $H^*(B(n+1,k);Z)$, hence

$$2\tilde{f}_k^*(b_i) = \tilde{f}_k^*(2b_i) = \tilde{f}_k^*(p^*(c_i)) = g^*(c_i)$$

and therefore, if $\binom{n+i}{i} \not\equiv 0 \pmod{2^m}$,

$$\tilde{f}_k^*(b_i) = \frac{1}{2}\binom{-n-1}{i} z^i = \pm\frac{1}{2}\binom{n+i}{n} z^i$$

Now, if $k = 2i$, and $a_i \in H^{2i}(B(n+1,2i);Z)$ as in the previous section, $\tilde{f}_{2i}^*(a_i)$ is an element $\lambda_i z^i$ of $H^{2i}(L^n(2^m);Z) \cong Z/2^m$ where $\lambda_i \in Z/2^m$.

The Steenrod squares are natural and so with the help of relations (2.4), (3.1) and (3.2), we deduce for $i \leq n-2$

$$i\lambda_i = (n+1)\lambda_i + i\frac{1}{2}\binom{n+i+1}{n} \pmod 2, \tag{3.4}$$

$$\binom{i}{2}\lambda_i = \binom{n+2}{2}\lambda_i + (i-1)(n+1)\frac{1}{2}\binom{n+i+1}{n} + \binom{i}{2}\frac{1}{2}\binom{n+i+2}{n} \pmod 2. \tag{3.5}$$

We shall note that (3.4) is still valid for $i = n-1$. In the following we shall take $i = l(n)$ and $m \geq 2$.

First we suppose $n = 2^s$ with $s \geq 1$. In this case, $i = l(n) = n-1$ and (3.4) becomes

$$\frac{1}{2}\binom{n+i+1}{n} \equiv 0 \pmod 2$$

which is impossible.

When n is even with $\alpha(n) \geq 2$, $i = l(n)$ is even and $\leq n-2$, so by (3.4)

$$\lambda_i \equiv 0 \pmod 2.$$

Using (3.5) we deduce

$$0 \equiv \frac{1}{2}\binom{n+i+1}{n} + \binom{i}{2}\frac{1}{2}\binom{n+i+2}{n}$$
$$\equiv \frac{1}{2}\binom{n+i+1}{n} \pmod 2,$$

since $i+1 > l(n)$, which is in contradiction with the definition of $l(n)$.

When n is odd with $\alpha(n) \geq 3$, $i = l(n) < n-3$ is odd, the relation (3.4) becomes

$$\lambda_i \equiv \frac{1}{2}\binom{n+i+1}{n} \pmod 2.$$

Now, using (3.5) and (3.3) we obtain

$$\frac{1}{2}\binom{n+i+1}{n} \equiv \begin{cases} 0 & \pmod 2 \quad \text{if } n \equiv 1 \pmod 4 \\ \frac{1}{2}\binom{n+i+2}{n} & \pmod 2 \quad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

As before we have a contradiction since $i+1 > l(n)$ and so we have proved part a) of theorem 1.2.

Finally, if $n = 2^s + 1$ with $s \geq 1$, and if $L^n(2^m)$ immerses in $\boldsymbol{R}^{2n+1+2(n-2)-1}$, the classifying map g of $-(n+1)\sigma$ lifts to $B(n+1, 2(n-2)-1)$, and also to $B(n+1, 2(n-2))$. With the same notations, relation (3.4) becomes in this case

$$\lambda_{n-2} \equiv \frac{1}{2}\binom{2n-1}{n} \pmod 2$$

and so

$$\lambda_{n-2} \equiv 1 \pmod 2.$$

However, if the map g lifts to $B(n+1, 2(n-2)-1)$, we have

$$\begin{aligned} \lambda_{n-2} z^{n-2} &= \tilde{f}^*_{2(n-2)}(a_{n-2}) \\ &= \tilde{f}^*_{2(n-2)-1}(p^*_{2n-5}(a_{n-2})) \\ &= \tilde{f}^*_{2(n-2)-1}(b_{n-2}) \\ &= \frac{1}{2}\binom{2n-2}{n} z^{n-2} \\ &\equiv 0 \pmod 2 \end{aligned}$$

where p_{2n-5} denotes the canonical map $B(n+1, 2(n-2)-1) \to B(n+1, 2(n-2))$. So, we have proved part b) of theorem 1.2.

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