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ON HARPER'S TORSION MOLECULE

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Introduction. Harper's torsion molecule [1] for an odd prime p is a finite simply connected Hopf space $K(p)$ whose cohomology in F_p-coefficients is given by $H^*(K(p); F_p) = E(x_3, x_{2p+1}) \otimes F_p[x_{2p+2}]/(x_{2p+2}^p)$ and $\wp^1(x_3) = x_{2p+1}$, $\beta(x_{2p+1}) = x_{2p+2}$. Here we study homotopy groups and BP-Hurewicz homomorphism of $K(p)$.

The organization of this note is as follows. In the first section, we compute the $Z_{(p)}$-homology and the BP-homology of the Harper's torsion molecule $K(p)$ [1] and determine the Thom map $T_{Z_{(p)}}$: $BP_*(K(p)) \to H_*(K(p); Z_{(p)})$, by using the Adams spectral sequence.

In section 2, we observe that the space F obtained by killing the 3-dimensional homotopy group of $K(p)$ is homotopy equivalent to the Toda's spectrum $V(1/2)$ in the stable range. By making use of this fact and applying the Adams-Novikov spectral sequence, we examine the unstable homotopy groups of $K(p)$ for dimension less than $4p^2-1$. We also compute the stable homotopy group of the $(2p^2+2p-2)$-skeleton of $K(p)$ in dimension $2p^2+2p-2$, where the attaching map of the $(2p^2+2p-1)$-cell of $K(p)$ lives.

In section 3, we determine the BP-Hurewicz homomorphism h^{BP}: $\pi_*(K(p)) \to BP_*(K(p))$ for dimension less than $4p^2-1$ and show that it is a split monomorphism in this range.

In the last section, by showing that the image of the homology suspension coincides with the set of the diagonal primitive elements of $BP_*(K(p))$, we completely determine the BP-Hurewicz homomorphism, that is, it is a trivial map in dimension other than 3 and $2p^2+2p-1$.

This work is motivated by G. Moreno's paper [4], where the BP-Hurewicz homomorphism of the Harper's torsion molecule is first studied. Our result improves the main result of [4].

1. Computations in the Adams spectral sequences.

First we fix the notations. Let p be an odd prime number. We denote by $\mathcal{A}_p$ be the mod p Steenrod algebra generated by the Bockstein operator β and the reduced power operators $\wp^i$ for $i = 1, 2, \ldots$. $\mathcal{A}_{p*}$ denotes the dual of the Steenrod algebra which is isomorphic to $E(s_0, s_1, s_2, \ldots) \otimes \boldsymbol{F}_p[t_1, t_2, \ldots]$. Here, s_i and t_i are the conjugates by the canonical anti-automorphism of τ_i and ξ_i in Milnor's paper [2]. In particular, $-s_i$ and $-t_1$ are the duals of Q_i and $\wp^1$ with respect to the Milnor basis. Then, the coproduct ϕ_* of $\mathcal{A}_{p*}$ is given by

$$(1.1)\qquad \begin{aligned} \phi_*(t_i) &= \sum_{j=0}^{i} t_j \otimes t_{i-j}^{p^j}, \\ \phi_*(s_i) &= \sum_{j=0}^{i} s_j \otimes t_{i-j}^{p^j} + 1 \otimes s_i. \end{aligned}$$

For an abelian group G, we denote by $K(G)$ the Eilenberg-MacLane spectrum for the group G. We only deal with the cases $G = \boldsymbol{F}_p$ and $\boldsymbol{Z}_{(p)}$. In these cases, $K(G)$ is a commutative ring spectrum. There is a cofiber sequence

$$(1.2)\qquad K(\boldsymbol{Z}_{(p)}) \xrightarrow{p} K(\boldsymbol{Z}_{(p)}) \xrightarrow{\rho} K(\boldsymbol{F}_p) \xrightarrow{\delta} \Sigma K(\boldsymbol{Z}_{(p)}),$$

where p denotes the p times of the identity map, ρ the map induced by the mod p reduction $\boldsymbol{Z}_{(p)} \to \boldsymbol{F}_p$. Then ρ is a map of ring spectra and the composition $K(\boldsymbol{F}_p) \xrightarrow{\delta} \Sigma K(\boldsymbol{Z}_{(p)}) \xrightarrow{\rho} \Sigma K(\boldsymbol{F}_p)$ is the Bockstein operator. It is easy to show (also well-known) that $\rho^*\colon \mathcal{A}_p = H^*(K(\boldsymbol{F}_p); \boldsymbol{F}_p) \to H^*(K(\boldsymbol{Z}_{(p)}); \boldsymbol{F}_p)$ is an epimorphism of left $\mathcal{A}_p$-modules with kernel $\mathcal{A}_p\beta$. By dualizing this, we have the following.

Proposition 1.3. $\rho_*\colon H_*(K(\boldsymbol{Z}_{(p)}); \boldsymbol{F}_p) \to H_*(K(\boldsymbol{F}_p); \boldsymbol{F}_p) = \mathcal{A}_{p*}$ *is a monomorphism of left* $\mathcal{A}_{p*}$*-comodule algebras onto a subalgebra* $E(s_1, s_2, \ldots) \otimes \boldsymbol{F}_p[t_1, t_2, \ldots]$.

Thus we identify $H_*(K(\boldsymbol{Z}_{(p)}); \boldsymbol{F}_p)$ with the image of ρ_*.

Let E and X be connective spectra. We consider the Adams spectral sequence

$$(1.4)\qquad E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_{p*}}^{s,t}(\boldsymbol{F}_p, H_*(E \wedge X; \boldsymbol{F}_p)) \Rightarrow E_{t-s}(X).$$

If E is a ring spectrum and there is a map of ring spectra $f\colon E \to K(\boldsymbol{F}_p)$ such that $f_*\colon H_*(E; \boldsymbol{F}_p) \to H_*(K(\boldsymbol{F}_p); \boldsymbol{F}_p) = \mathcal{A}_{p*}$ is injective, by applying the change-of-rings isomorphism ([6] A1.3.13), E_2-term of the above spectral sequence is identified with $\mathrm{Ext}_{\mathcal{A}_{p*}/I_f}^{s,t}(\boldsymbol{F}_p, H_*(X; \boldsymbol{F}_p))$, where I_f is an ideal of $\mathcal{A}_{p*}$ generated by $f_*(\sum_{i>0} H_i(E; \boldsymbol{F}_p))$.

In the case $E = K(\boldsymbol{Z}_{(p)})$, by (1.3) we have a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{E(s_0)}^{s,t}(\boldsymbol{F}_p, H_*(X;\boldsymbol{F}_p)) \Rightarrow H_{t-s}(X;\boldsymbol{Z}_{(p)}). \tag{1.5}$$

In the case $E = BP$, since the Thom reduction map $T_{\boldsymbol{F}_p}\colon BP \to K(\boldsymbol{F}_p)$ induces a monomorphism $T_{\boldsymbol{F}_p *}\colon H_*(BP;\boldsymbol{F}_p) \to H_*(K(\boldsymbol{F}_p);\boldsymbol{F}_p) = \mathcal{A}_{p*}$ onto $\boldsymbol{F}_p[t_1, t_2, \ldots]$, we have

$$E_2^{s,t} = \mathrm{Ext}_{E(s_0,s_1,\ldots)}^{s,t}(\boldsymbol{F}_p, H_*(X;\boldsymbol{F}_p)) \Rightarrow BP_{t-s}(X). \tag{1.6}$$

Remark 1.7. We note that the Thom map $T_{\boldsymbol{Z}_{(p)}}\colon BP \to K(\boldsymbol{Z}_{(p)})$ induces a morphism of spectral sequences from (1.6) to (1.5). Since $\rho T_{\boldsymbol{Z}_{(p)}} = T_{\boldsymbol{F}_p}$, the map

$$\begin{aligned} T_{\boldsymbol{Z}(p)*}\colon &\mathrm{Ext}_{\mathcal{A}_{p*}}(\boldsymbol{F}_p, H_*(BP \wedge X;\boldsymbol{F}_p)) \\ &\longrightarrow \mathrm{Ext}_{\mathcal{A}_{p*}}(\boldsymbol{F}_p, H_*(K(\boldsymbol{Z}_{(p)}) \wedge X;\boldsymbol{F}_p)) \end{aligned}$$

induced by $T_{\boldsymbol{Z}_{(p)}}$ is identified with the map

$$\mathrm{Ext}_{E(s_0,s_1,\ldots)}(\boldsymbol{F}_p, H_*(X;\boldsymbol{F}_p)) \longrightarrow \mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(X;\boldsymbol{F}_p))$$

induced by the projection $E(s_0, s_1, \ldots) \to E(s_0, s_1, \ldots)/(s_1, \ldots) = E(s_0)$ through the change-of-rings isomorphisms.

Now we compute the E_2-terms of the spectral sequences (1.5) and (1.6) for $X = K(p)$ the Harper's torsion molecule. The structure of $H^*(K(p);\boldsymbol{F}_p)$ is given by

$$\begin{aligned} &H^*(K(p);\boldsymbol{F}_p) = E(x_3, x_{2p+1}) \otimes \boldsymbol{F}_p[x_{2p+2}]/(x_{2p+2}^p), \\ &\wp^1(x_3) = x_{2p+1}, \quad \beta(x_{2p+1}) = x_{2p+2}. \end{aligned} \tag{1.8}$$

Let b_i $(i = 3,\ 2p+1,\ 2p+2)$ be the dual of x_i, then the structure of $H_*(K(p);\boldsymbol{F}_p)$ as a left $\mathcal{A}_{p*}$-comodule algebra is given as follows.

$$\begin{aligned} &H_*(K(p);\boldsymbol{F}_p) = E(b_3, b_{2p+1}) \otimes \boldsymbol{F}_p[b_{2p+2}]/(b_{2p+2}^p), \\ &\varphi(b_3) \quad = 1 \otimes b_3, \\ &\varphi(b_{2p+1}) = 1 \otimes b_{2p+1} - t_1 \otimes b_3, \\ &\varphi(b_{2p+2}) = 1 \otimes b_{2p+2} + s_0 \otimes b_{2p+1} - s_1 \otimes b_3 + s_0 t_1 \otimes b_3 \end{aligned} \tag{1.9}$$

$E(s_0)$ is a quotient Hopf algebra of $\mathcal{A}_{p*}$ by an ideal generated by $s_1, s_2, \ldots, t_1, t_2, \ldots$ and $H_*(K(p); \boldsymbol{F}_p)$ is regarded as a left $E(s_0)$-comodule via projection $\mathcal{A}_{p*} \to E(s_0)$. It follows from (1.9) that the $E(s_0)$-comodule structure of $H_*(K(p); \boldsymbol{F}_p)$ is given by

$$\begin{aligned} \varphi_0(b_3) &= 1 \otimes b_3, \\ \varphi_0(b_{2p+1}) &= 1 \otimes b_{2p+1}, \\ \varphi_0(b_{2p+2}) &= 1 \otimes b_{2p+2} + s_0 \otimes b_{2p+1}. \end{aligned} \tag{1.10}$$

Then $H_*(K(p); \boldsymbol{F}_p)$ is a direct sum of subcomodules $M = E(b_3, b_{2p+1}b_{2p+2}^{p-1})$ and $M_{\varepsilon,i} = \boldsymbol{F}_p\{b_3^\varepsilon b_{2p+1} b_{2p+2}^{i-1}, b_3^\varepsilon b_{2p+2}^i\}$ for $\varepsilon = 0, 1$, $i = 1, 2, \ldots, p-1$. Here we denote by $R\{a, b, c, \ldots\}$ the free R-module generated by elements $a, b, c, \ldots$.

Since $M_{\varepsilon,i}$ is isomorphic to $E(s_0)$ with shifting degree, we have $\mathrm{Ext}^{0,*}_{E(s_0)}(\boldsymbol{F}_p, M_{\varepsilon,i}) = \boldsymbol{F}_p\{b_3^\varepsilon b_{2p+1} b_{2p+2}^{i-1}\}$, $\mathrm{Ext}^{s,*}_{E(s_0)}(\boldsymbol{F}_p, M_{\varepsilon,i}) = 0$ if $s \neq 0$. On the other hand, since M consists of primitive elements and $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, \boldsymbol{F}_p) = \boldsymbol{F}_p[\bar{v}_0]$, we have $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, M) = \mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, \boldsymbol{F}_p) \otimes M = \boldsymbol{F}_p[\bar{v}_0] \otimes M$, where $\bar{v}_0$ is an element of $\mathrm{Ext}^{1,1}_{E(s_0)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ represented by $[s_0]$ in the cobar complex for $E(s_0)$. Note that $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ has a structure of $\boldsymbol{F}_p[\bar{v}_0]$-module. (1.10) implies a relation $\bar{v}_0 b_3^\varepsilon b_{2p+1} b_{2p+2}^{i-1} = 0$.

Let z_3, $z_{2i(p+1)-1}$ and $z_3 z_{2i(p+1)-1}$ be the elements of $\mathrm{Ext}^{0,*}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ represented by cocycles $[\,]b_3$, $[\,]b_{2p+1}b_{2p+2}^{i-1}$ and $[\,]b_3 b_{2p+1} b_{2p+2}^{i-1}$ of the cobar complex, respectively. The above argument shows

Proposition 1.11. *As an $\boldsymbol{F}_p[\bar{v}_0]$-module,* $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ *is isomorphic to*

$$\begin{aligned} &\boldsymbol{F}_p[\bar{v}_0]/(\bar{v}_0)\{z_{2i(p+1)-1}, z_3 z_{2i(p+1)-1} \mid 1 \le i \le p-1\} \\ &\quad \oplus \boldsymbol{F}_p[\bar{v}_0]\{1, z_3, z_{2p^2+2p-1}, z_3 z_{2p^2+2p-1}\}. \end{aligned}$$

Since $\bar{v}_0$ is a permanent cycle representing p, the preceding result immediately implies the following.

Proposition 1.12. *The Adams spectral sequence* (1.5) *collapses and* $H_*(K(p); \boldsymbol{Z}_{(p)})$ *is isomorphic to*

$$\begin{aligned} &\boldsymbol{F}_p\{z_{2i(p+1)-1}, z_3 z_{2i(p+1)-1} \mid 1 \le i \le p-1\} \\ &\quad \oplus \boldsymbol{Z}_{(p)}\{1, z_3, z_{2p^2+2p-1}, z_3 z_{2p^2+2p-1}\} \end{aligned}$$

as a $Z_{(p)}$-module, where $\deg z_i = i$.

We put $E = E(s_0, s_1, \dots)$ for short. Since E is the quotient Hopf algebra of $\mathcal{A}_{p*}$ by an ideal generated by $t_1, t_2, \dots$, it follows from (1.1) that each s_i is primitive in E. By (1.9), E-comodule structure of $H_*(K(p); \boldsymbol{F}_p)$ is given by

$$\begin{aligned} \varphi_1(b_3) &= 1 \otimes b_3, \\ \varphi_1(b_{2p+1}) &= 1 \otimes b_{2p+1}, \\ \varphi_1(b_{2p+2}) &= 1 \otimes b_{2p+2} + s_0 \otimes b_{2p+1} - s_1 \otimes b_3. \end{aligned} \tag{1.13}$$

Thus $H_*(K(p); \boldsymbol{F}_p)$ can be regarded as an $E(s_0, s_1)$-comodule. Put $E' = E(s_2, s_3, \dots)$, then $E = E(s_0, s_1) \otimes E'$ as a Hopf algebra and the external product

$$\begin{aligned} &\mathrm{Ext}_{E(s_0,s_1)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p)) \otimes \mathrm{Ext}_{E'}(\boldsymbol{F}_p, \boldsymbol{F}_p) \\ &\qquad \longrightarrow \mathrm{Ext}_E(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p)) \end{aligned} \tag{1.14}$$

is an isomorphism by the Künneth theorem.

Let $\bar{v}_i$ be the element of $\mathrm{Ext}^{1,2p^i-1}_{E(s_i)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$ represented by a cocycle $[s_i]$. Then,

$$\mathrm{Ext}_{E'}(\boldsymbol{F}_p, \boldsymbol{F}_p) = \boldsymbol{F}_p[\bar{v}_2, \bar{v}_3, \dots, \bar{v}_i, \dots], \tag{1.15}$$

and $\mathrm{Ext}_{E(s_0,s_1)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ has a structure of $\boldsymbol{F}_p[\bar{v}_1, \bar{v}_2]$-module. In order to compute $\mathrm{Ext}_{E(s_0,s_1)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$, we apply the Cartan-Eilenberg spectral sequence to an extension of Hopf algebras $E(s_1) \to E(s_0, s_1) \to E(s_0)$ ([6] A1.3.14)

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^s_{E(s_1)}(\boldsymbol{F}_p, \mathrm{Ext}^t_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))) \\ &\Rightarrow \mathrm{Ext}^{s+t}_{E(s_0,s_1)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p)). \end{aligned}$$

It follows from (1.11) that the $E(s_1)$-comodule structure of $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ is given by

$$\begin{aligned} \tilde{\varphi}_1(z_{2i(p+1)-1}) &= 1 \otimes z_{2i(p+1)-1} - (i-1)s_1 \otimes z_3 z_{2(i-1)(p+1)-1}, \\ \tilde{\varphi}_1(z_3 z_{2i(p+1)-1}) &= 1 \otimes z_3 z_{2i(p+1)-1}. \end{aligned} \tag{1.16}$$

We set

$$\begin{aligned} N &= \boldsymbol{F}_p\{z_{2p+1}\} \oplus \boldsymbol{F}_p[\bar{v}_0]\{1, z_3, \bar{v}_0 z_{2p^2+2p-1}, z_3 z_{2p^2+2p-1}\}, \\ N_i &= \boldsymbol{F}_p\{z_3 z_{2i(p+1)-1}, z_{2(i+1)(p+1)-1}\} \end{aligned}$$

for $i = 1, 2, \ldots, p-1$. Then N and N_i are subcomodules of $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$. N consists of primitive elements and N_i is isomorphic to $E(s_1)$ with shifting degrees.

Since $\mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ is a direct sum of N and N_i for $i = 1, 2, \ldots, p-1$, $\mathrm{Ext}_{E(s_1)}(\boldsymbol{F}_p, \mathrm{Ext}_{E(s_0)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p)))$ is isomorphic to

$$\boldsymbol{F}_p\{X_{2i(p+1)+2} \mid 1 \le i \le p-1\} \oplus \boldsymbol{F}_p[\bar{v}_1]\{z_{2p+1}\} \oplus \boldsymbol{F}_p[\bar{v}_0, \bar{v}_1]\{1, z_3, z'_{2p^2+2p-1}, X_{2p^2+2p+2}\},$$

where $X_{2i(p+1)+2}$ and z'_{2p^2+2p-1} are the classes of $[\,]z_3 z_{2i(p+1)-1}$ and $[\,]\bar{v}_0 z_{2p^2+2p-1}$, respectively.

We note that $X_{2i(p+1)+2}$, z_{2p+1}, z_3 and z'_{2p^2+2p-1} belong to $E_2^{0,0}$, hence the spectral sequence collapses. We use the same symbols as X_j, z_{2p+1}, z_3 and z'_{2p^2+2p-1} to denote the elements of $\mathrm{Ext}_{E(s_0,s_1)}(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ corresponding to them. The cocycles representing $X_{2i(p+1)+2}$, z_{2p+1}, z_3 and z'_{2p^2+2p-1} in the cobar complex are given by $[\,]b_3 b_{2p+1} b_{2p+2}^{i-1}$, $[\,]b_{2p+1}$, $[\,]b_3$ and $[s_0]b_{2p+1}b_{2p+2}^{p-1} + [s_0 s_1]b_3 b_{2p+1} b_{2p+2}^{p-1} - [s_1]b_3 b_{2p+2}^{p-1}$, respectively.

By (1.13), $[s_0]b_{2p+1}$ is cohomologous to $[s_1]b_3$, thus we have a relation $\bar{v}_0 z_{2p+1} = \bar{v}_1 z_3$. Similarly, $[s_j]b_3 b_{2p+1} b_{2p+2}^{i-1}$ ($j = 0, 1$, $i = 1, 2, \ldots, p-1$) are cohomologous to 0 in the cobar complex, hence $X_{2i(p+1)+2}$ is annihilated by $\bar{v}_0$ and $\bar{v}_1$ if $i < p$.

We put $R = \mathrm{Ext}_E(\boldsymbol{F}_p, \boldsymbol{F}_p) = \boldsymbol{F}_p[\bar{v}_0, \bar{v}_1, \ldots]$. By virtue of the isomorphism (1.14) we obtain the following result.

Proposition 1.17. *$\mathrm{Ext}_E(\boldsymbol{F}_p, H_*(K(p); \boldsymbol{F}_p))$ is isomorphic to*

$$R/(\bar{v}_0, \bar{v}_1)\{X_{2i(p+1)+2} \mid 1 \le i \le p-1\} \oplus R\{z_3, z_{2p+1}\}/(\bar{v}_1 z_3 - \bar{v}_0 z_{2p+1}) \oplus R\{1, z'_{2p^2+2p-1}, X_{2p^2+2p+2}\}$$

as an R-module.

Since $X_j \in E_2^{0,j}$, $z_j \in E_2^{0,j}$ and $z'_{2p^2+2p-1} \in E_2^{1,2p^2+2p}$, the Adams spectral sequence (1.6) collapses for dimensional reason. Moreover, the extension problem of the BP_*-module structure is trivial also for dimensional reason.

Consider the morphism of spectral sequences $T_{\boldsymbol{Z}_{(p)}*}$ from (1.6) to (1.5) induced by the Thom map $T_{\boldsymbol{Z}_{(p)}}\colon BP \to K(\boldsymbol{Z}_{(p)})$. It follows from (1.7) that

$T_{Z_{(p)}}$ maps $X_{2i(p+1)+2}$ to $z_3 z_{2i(p+1)-1}$, z_3 to z_3, z_{2p+1} to z_{2p+1} and z'_{2p^2+2p-1} to $\bar{v}_0 z_{2p^2+2p-1}$ in the E_2-terms.

Proposition 1.18. *$BP_*(K(p))$ is isomorphic to*

$$\begin{aligned} &BP_*/(p, v_1)\{X_{2i(p+1)+2} \mid 1 \le i \le p-1\} \\ &\quad \oplus BP_*\{z_3, z_{2p+1}\}/(v_1 z_3 - p z_{2p+1}) \\ &\quad \oplus BP_*\{1, z'_{2p^2+2p-1}, X_{2p^2+2p+2}\} \end{aligned}$$

as an BP_*-module, where $\deg X_j = j$, $\deg z_j = j$ and $\deg z'_{2p^2+2p-1} = 2p^2 + 2p - 1$. $T_{Z_{(p)}*}$: $BP_*(K(p)) \to H_*(K(p); Z_{(p)})$ maps $X_{2i(p+1)+2}$ to $z_3 z_{2i(p+1)-1}$, z_3 to z_3, z_{2p+1} to z_{2p+1} and z'_{2p^2+2p-1} to $p z_{2p^2+2p-1}$.

Remark 1.19. (1) Since $K(p)$ is a Hopf space, the spectral sequences (1.5) and (1.6) are multiplicative. Thus, in $BP_*(K(p))$, we have relations $z_3 z_{2p+1} = X_{2p+4}$ and $z_3 z'_{2p^2+2p-1} = pX_{2p^2+2p+2} + \gamma v_2 X_{2p+4}$ for some $\gamma \in Z_{(p)}$. (See (4.14).)

(2) Let us denote by ψ: $BP_*(K(p)) \to BP_*BP \otimes_{BP_*} BP_*(K(p))$ the BP_*BP-comodule structure map. Then, it follows from (1.9) and (1.18) that $X_{2i(p+1)+2}$ $(1 \le i \le p)$ and z_3 are primitive and that $\psi(z_{2p+1}) = 1 \otimes z_{2p+1} - t_1 \otimes z_3$. We note that $\widetilde{BP}_*(K(p))$ is the direct sum of subcomodules $BP_*/(p, v_1)\{X_{2i(p+1)+2}\}$ $(1 \le i \le p-1)$, $BP_*\{X_{2p^2+2p+2}\}$ and $BP_*\{z_3, z_{2p+1}\}/(v_1 z_3 - p z_{2p+1}) \oplus BP_*\{z'_{2p^2+2p-1}\}$. Let us denote by $K(p)^n$ the n-skeleton of $K(p)$, then it is shown that the inclusion map $K(p)^{2p^2+2p-2} \hookrightarrow K(p)$ induces a monomorphism $\widetilde{BP}_*(K(p)^{2p^2+2p-2}) \to \widetilde{BP}_*(K(p))$ onto the direct sum of $BP_*\{z_3, z_{2p+1}\}/(v_1 z_3 - p z_{2p-1})$ and $BP_*/(p, v_1)\{X_{2i(p+1)+2}\}$ for $1 \le i \le p-1$.

2. The homotopy groups of $K(p)$. We calculate $H^*(K(p); Z_{(p)})$ by using the Bockstein long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^i(K(p); Z_{(p)}) &\xrightarrow{p\times} H^i(K(p); Z_{(p)}) \xrightarrow{\rho} \\ H^i(K(p); F_p) &\xrightarrow{\delta} H^{i+1}(K(p); Z_{(p)}) \longrightarrow \cdots. \end{aligned}$$

We can easily show the following.

Proposition 2.1. *We put $y_{2p+2} = \delta(x_{2p+1})$ and there are elements y_3 and y_{2p^2+2p-1} of $H^*(K(p); Z_{(p)})$ such that $\rho(y_3) = x_3$ and*

$\rho(y_{2p^2+2p-1}) = x_{2p+1}x_{2p+2}^{p-1}$. $H^*(K(p); Z_{(p)})$ *is isomorphic to*

$$F_p\{y_{2p+2}^i, y_3y_{2p+2}^i \mid 1 \leq i \leq p-1\}$$
$$\oplus Z_{(p)}\{1, y_3, y_{2p^2+2p-1}, y_3y_{2p^2+2p-1}\}.$$

Let F be the homotopy fiber of y_3: $K(p) \to K(Z_{(p)}, 3)$. f: $F \to K(p)$ denotes the inclusion map of the fiber. Applying the Serre spectral sequence to the fibration $F \xrightarrow{f} K(p) \xrightarrow{y_3} K(Z_{(p)}, 3)$, a routine argument shows

Proposition 2.2. *For degree* $\leq 2p^3$, $H^*(F; F_p)$ *is isomorphic to* $E(u_{2p^2+1}, u_{2p^2+2p-1}) \otimes F_p[u_{2p^2}]$ *with* A_p*-action* $\beta(u_{2p^2}) = u_{2p^2+1}$ *and* $\wp^1(u_{2p^2+1}) = u_{2p^2+2p-1}$.

Corollary 2.3. *The* $(4p^2-1)$*-skeleton of* F *is homotopy equivalent to a 3-cell complex* $X = S^{2p^2} \cup e^{2p^2+1} \cup e^{2p^2+2p-1}$ *such that the subcomplex* $Y = S^{2p^2} \cup e^{2p^2+1}$ *is the* mod p *Moore space and the top cell* e^{2p^2+2p-1} *is attached to* Y *by the map* $\bar{\alpha}_1 \in \pi_{2p^2+2p-2}(Y)$ *which maps to* $\alpha_1 \in \pi_{2p^2+2p-2}(S^{2p^2+1}) \cong \pi_{2p-3}^S(S^0)$ *by the map induced by* $Y \to Y/S^{2p^2} = S^{2p^2+1}$.

Namely, X is nothing but the $2p^2$-fold suspension of the Toda's spectrum $V(1/2)$ [7].

Corollary 2.4. $\pi_i(K(p)) = 0$ *for* $i < 3$ *or* $3 < i < 2p^2$ *and there are isomorphisms*

$$\pi_3(K(p)) \cong Z_{(p)},$$
$$\pi_i(K(p)) \cong \pi_{i-2p^2}^S(V(1/2)) \quad \textit{for } 2p^2 \leq i \leq 4p^2-2.$$

Hence it suffices to know $\pi_i^S(V(1/2))$ for $i \leq 2p^2-2$ to know $\pi_i(K(p))$ for $i \leq 4p^2-2$.

We denote by M_p the mod p Moore spectrum $S^0 \cup_p e^1$. Consider the long exact sequences of stable homotopy groups associated with the cofibrations $S^0 \xrightarrow{p} S^0 \xrightarrow{\iota_0} M_p \xrightarrow{\partial_0} S^1$ and $S^{2p-2} \xrightarrow{\bar{\alpha}_1} M_p \xrightarrow{\iota_1} V(1/2) \xrightarrow{\partial_1} S^{2p-1}$.

$$(2.5) \qquad \cdots \longrightarrow \pi_i^S(S^0) \xrightarrow{p} \pi_i^S(S^0) \xrightarrow{\iota_{0*}} \pi_i^S(M_p) \xrightarrow{\partial_{0*}} \pi_{i-1}^S(S^0) \longrightarrow \cdots,$$

$$(2.6)\qquad \cdots \longrightarrow \pi_i^S(S^{2p-2}) \xrightarrow{\bar\alpha_{1*}} \pi_i^S(M_p) \xrightarrow{\iota_{1*}} \pi_i^S(V(1/2)) \xrightarrow{\partial_{1*}} \pi_{i-1}^S(S^{2p-2}) \longrightarrow \cdots.$$

We recall several stable homotopy groups of the sphere.

Theorem 2.7. *For $i \le 2p^2$, the p-component of $\pi_i^S(S^0)$ is trivial except for the following cases.*

$$\begin{aligned}
\pi_0^S(S^0) &= Z\{1\},\\
\pi_{2r(p-1)-1}^S(S^0) &= Z/p\{\alpha_r\} \quad \text{for } 1 \le r < p \text{ or } r = p+1,\\
\pi_{2p^2-2p-2}^S(S^0) &= Z/p\{\beta_1\},\\
\pi_{2p^2-2p-1}^S(S^0) &= Z/p^2\{\alpha_{p/2}\},\\
\pi_{2p^2-5}^S(S^0) &= Z/p\{\alpha_1\beta_1\}.
\end{aligned}$$

Let α: $\Sigma^{2(p-1)}M_p \to M_p$ denote the Adams map. We put $\bar\alpha_r = \alpha^r\iota_0$, then $\alpha_r = \partial_0\bar\alpha_r$. Since every element of $\pi_*^S(M_p)$ is order p, the long exact sequence (2.5) splits into split short exact sequences

$$(2.8)\qquad 0 \longrightarrow \pi_i^S(S^0)/p\pi_i^S(S^0) \longrightarrow \pi_i^S(M_p) \longrightarrow \operatorname{Ker}\{p\colon \pi_{i-1}^S(S^0) \to \pi_{i-1}^S(S^0)\} \longrightarrow 0.$$

If $x \in \pi_{i-1}^S(S^0)$ is an element of order p, we denote by $\bar x$ an element of $\pi_i^S(M_p)$ such that $\partial_0\bar x = x$. Theorem (2.7) immediately implies

Proposition 2.9. *For $i \le 2p^2$, $\pi_i^S(M_p) = 0$ except for the following cases.*

$$\begin{aligned}
\pi_0^S(M_p) &= Z/p\{\iota_0\},\\
\pi_{2r(p-1)-1}^S(M_p) &= Z/p\{\iota_0\alpha_r\} \quad \text{for } 1 \le r < p \text{ or } r = p+1,\\
\pi_{2r(p-1)}^S(M_p) &= Z/p\{\bar\alpha_r\} \quad \text{for } 1 \le r < p \text{ or } r = p+1,\\
\pi_{2p^2-2p-2}^S(M_p) &= Z/p\{\iota_0\beta_1\},\\
\pi_{2p^2-2p-1}^S(M_p) &= Z/p\{\iota_0\alpha_{p/2}, \bar\beta_1\},\\
\pi_{2p^2-2p}^S(M_p) &= Z/p\{\overline{p\alpha_{p/2}}\},\\
\pi_{2p^2-5}^S(M_p) &= Z/p\{\iota_0\alpha_1\beta_1\},\\
\pi_{2p^2-4}^S(M_p) &= Z/p\{\bar\alpha_1\beta_1\}.
\end{aligned}$$

We need to determine the map $\bar\alpha_{1*}$: $\pi_{i-2p+2}^S(S^0) \to \pi_i^S(M_p)$ for $i \le 2p^2 - 2$. Consider the Adams-Novikov spectral sequences for BP-theory converging to $\pi_*^S(S^0)$ and $\pi_*^S(M_p)$.

From now on, for a BP_*BP-comodule M, we set $H^{s,t}(M) = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M)$.

Recall from [3] that the cocycles representing α_r and $\alpha_{p/2}$ in $H^{1,*}(BP_*)$ are the following elements of the cobar complex $\Omega^*(BP_*)$.

$$(2.10)\qquad \begin{aligned} &r[t_1]v_1^{r-1} + \sum_{j=2}^{r} \binom{r}{j}(-p)^{j-1}[t_1^j]v_1^{r-j}, \\ &[t_1]v_1^{p-1} + \sum_{j=2}^{p} \binom{p}{j}(-p)^{j-2}[t_1^j]v_1^{p-j} \end{aligned}$$

Here we regard BP_* as a *left* BP_*BP-comodule.

We summarize the structure of $H^{s,t}(BP_*)$, $H^{s,t}(BP_*/(p))$ and $H^{s,t}(BP_*/(p, v_1))$.

Proposition 2.11. (1) *For $t - s < 2p^2 + 2p$, $H^{s,t}(BP_*) = 0$ except for the following cases.*

$$\begin{aligned} H^{0,0}(BP_*) &= Z_{(p)}\{1\}, \\ H^{1,2r(p-1)}(BP_*) &= Z/p\{\alpha_r\} \quad \text{for } 1 \le r < p \text{ or } r = p+1, \\ H^{1,2p(p-1)}(BP_*) &= Z/p^2\{\alpha_{p/2}\}, \\ H^{2,2p(p-1)}(BP_*) &= Z/p\{\beta_1\}, \\ H^{3,2p^2-2}(BP_*) &= Z/p\{\alpha_1\beta_1\}, \end{aligned}$$

where β_1 is represented by a cocycle $\sum_{i=1}^{p-1} \binom{p}{i}[t_1^i|t_1^{p-i}]/p$.

(2) *For $t - s < 2p^2$, $H^{s,t}(BP_*/(p)) = 0$ except for the following cases.*

$$\begin{aligned} H^{0,0}(BP_*/(p)) &= Z/p\{1\}, \\ H^{0,2r(p-1)}(BP_*/(p)) &= Z/p\{v_1^r\} \quad \text{for } 1 \le r \le p+1, \\ H^{1,2r(p-1)}(BP_*/(p)) &= Z/p\{v_1^{r-1}\alpha_1\} \quad \text{for } 1 \le r < p \text{ or } r = p+1, \\ H^{1,2p(p-1)}(BP_*/(p)) &= Z/p\{v_1^{p-1}\alpha_1, h_1\}, \\ H^{2,2p(p-1)}(BP_*/(p)) &= Z/p\{\beta_1\}, \\ H^{2,2p^2-2}(BP_*/(p)) &= Z/p\{v_1\beta_1\}, \\ H^{3,2p^2-2}(BP_*/(p)) &= Z/p\{\alpha_1\beta_1\}. \end{aligned}$$

h_1 is the element represented by a cocycle $[t_1^p]$, α_1 and β_1 here are the mod p *reduction of the elements of* (1) *with the same symbol.*

(3) *For* $t - s < 2p^2$, $H^{s,t}(BP_*/(p,v_1)) = 0$ *except for the following cases.*

$$\begin{aligned} H^{0,0}(BP_*/(p,v_1)) &= Z/p\{1\}, \\ H^{0,2p^2-2}(BP_*/(p,v_1)) &= Z/p\{v_2\}, \\ H^{1,2p-2}(BP_*/(p,v_1)) &= Z/p\{\alpha_1\}, \\ H^{1,2p(p-1)}(BP_*/(p,v_1)) &= Z/p\{h_1\}, \\ H^{2,2p(p-1)}(BP_*/(p,v_1)) &= Z/p\{\beta_1\}, \\ H^{3,2p^2-2}(BP_*/(p,v_1)) &= Z/p\{\alpha_1\beta_1\}. \end{aligned}$$

h_1, α_1 *and* β_1 *are the* mod (p, v_1) *reduction of the elements of* (2) *with the same symbol.*

Since ι_{0*}: $BP_*(S^0) \to BP_*(M_p) = BP_*/(p)$ is the mod p reduction map, in the E_2-level, ι_{0*}: $H^{*,*}(BP_*(S^0)) \to H^{*,*}(BP_*(M_p))$ maps α_r and $\alpha_{p/2}$ to the elements represented by $r[t_1]v_1^{r-1}$ and $[t_1]v_1^{p-1}$, respectively. On the other hand, since α: $\Sigma^{2(p-1)}M_p \to M_p$ induces the v_1-multiplication map on $BP_*(M_p)$, $\bar{\alpha}_1 = \alpha\iota_0$ induces $\bar{\alpha}_{1*}$: $H^{*,*}(BP_*(S^0)) \to H^{*,*}(BP_*(M_p))$ which maps α_{r-1} and $\alpha_{p/2}$ to the elements represented by $(r-1)[t_1]v_1^{r-1}$ and $[t_1]v_1^p$, respectively. Thus $\bar{\alpha}_{1*}(\alpha_{r-1}) = ((r-1)/r)\iota_{0*}(\alpha_r)$ for $2 \le r < p$ or $r = p+2$, $\bar{\alpha}_{1*}(\alpha_{p-1}) = -\iota_{0*}(\alpha_{p/2})$ and $\bar{\alpha}_{1*}(\alpha_{p/2}) = \iota_{0*}(\alpha_{p+1})$ hold in the E_2-term.

It follows from (2.11) that $H^{1,2p^2-2p}(BP_*(M_p))$ is a 2-dimensional vector space over F_p and $H^{s,2p^2-2p+s}(BP_*(M_p)) = 0$ for $s > 1$, hence both elements $\iota_0\alpha_{p/2}$ and $\bar{\beta}_1$ of $\pi^S_{2p^2-2p-1}(M_p)$ are in the same Adams filtration.

These arguments show the following.

Proposition 2.12. $\bar{\alpha}_{1*}$: $\pi^S_{i-2p+2}(S^0) \to \pi^S_i(M_p)$ *is given as follows.*

$$\begin{aligned} \bar{\alpha}_{1*}(1) &= \bar{\alpha}_1, \\ \bar{\alpha}_{1*}(\alpha_{r-1}) &= \frac{r-1}{r}\iota_0\alpha_r \quad \text{for } 2 \le r < p \text{ or } r = p+2, \\ \bar{\alpha}_{1*}(\alpha_{p-1}) &= -\iota_0\alpha_{p/2}, \\ \bar{\alpha}_{1*}(\alpha_{p/2}) &= \iota_0\alpha_{p+1}, \\ \bar{\alpha}_{1*}(\beta_1) &= \bar{\alpha}_1\beta_1, \\ \bar{\alpha}_{1*}(\alpha_1\beta_1) &= 0. \end{aligned}$$

Applying the above result to (2.6) we obtain

Proposition 2.13. *For $i < 2p^2-2$, $\pi_i^S(V(1/2)) = 0$ except for the following cases.*

$$\begin{aligned}
\pi_0^S(V(1/2)) &= Z/p\{\iota_1\iota_0\},\\
\pi_{2p-3}^S(V(1/2)) &= Z/p\{\iota_1\iota_0\alpha_1\},\\
\pi_{2p-1}^S(V(1/2)) &= Z_{(p)}\{w_{2p-1}\},\\
&\qquad \text{where } w_{2p-1} \text{ maps to } p \in \pi_0^S(S^0) \text{ by } \partial_{1*},\\
\pi_{2r(p-1)}^S(V(1/2)) &= Z/p\{\bar\alpha_r\} \quad \text{for } 2 \le r < p,\\
\pi_{2p^2-2p-2}^S(V(1/2)) &= Z/p\{\iota_1\iota_0\beta_1\},\\
\pi_{2p^2-2p-1}^S(V(1/2)) &= Z/p\{\iota_1\bar\beta_1\},\\
\pi_{2p^2-2p}^S(V(1/2)) &= Z/p\{\iota_1\overline{p\alpha_{p/2}}\},\\
\pi_{2p^2-5}^S(V(1/2)) &= Z/p\{\iota_1\iota_0\alpha_1\beta_1\}.
\end{aligned}$$

Furthermore, there is a short exact sequence

$$0 \longrightarrow \pi_{2p^2-2}^S(M_p) \xrightarrow{\iota_{1*}} \pi_{2p^2-2}^S(V(1/2)) \xrightarrow{\partial_{1*}} p\pi_{2p^2-2p-1}^S(S^0) \longrightarrow 0.$$

Next, we solve the extension problem in $\pi_{2p^2-2}^S(V(1/2))$. Consider the long exact sequence of BP-homology associated with cofibration $S^{2p-1} \xrightarrow{\eta} V(1/2) \xrightarrow{\iota_2} V(1) \xrightarrow{\partial} S^{2p}$. Since $BP_*(S^{2p-1})$ is concentrated in odd dimensions and $BP_*(V(1)) = BP_*/(p,v_1)$ is concentrated in even dimensions, the long exact sequence splits and give an isomorphism of BP_*-modules,

$$BP_*(V(1/2)) \cong BP_*\{w_{2p-1}\} \oplus BP_*/(p,v_1)\{1\}. \tag{2.14}$$

Here we put $w_{2p-1} = \eta_*(id_{S^{2p-1}}) \in BP_{2p-1}(V(1/2))$ and $1 \in BP_0(V(1/2))$ is the unique element that maps to $1 \in BP_0(V(1))$ by ι_{2*}. Both of them are primitive and the above isomorphism is an isomorphism of BP_*BP-comodules.

We consider the Adams-Novikov spectral sequence

$$E_2^{s,t} = H^{s,t}(BP_*(V(1/2))) \Rightarrow \pi_{t-s}^S(V(1/2)). \tag{2.15}$$

By (2.14), we have an isomorphism

$$E_2^{s,t} \cong H^{s,t-2p+1}(BP_*)\{w_{2p-1}\} \oplus H^{s,t}(BP_*/(p,v_1)). \tag{2.16}$$

Then, there are non trivial elements $\beta_1 w_{2p-1} \in E_2^{2,2p^2-1}$, $v_2 \in E_2^{0,2p^2-2}$ of order p, and $\alpha_{p/2} w_{2p-1} \in E_2^{1,2p^2-1}$ of order p^2. Since $\pi_{2p^2-3}^S(V(1/2)) = 0$ by (2.13), $\beta_1 w_{2p-1}$ is killed by v_2 and $\alpha_{p/2} w_{2p-1}$ is a permanent cycle representing an element of order p^2. This solves the extension problem.

Proposition 2.17.

$$\pi_{2p^2-2}^S(V(1/2)) \cong Z/p^2.$$

Proposition 2.18. *For $i \le 4p^2-2$, $\pi_i(K(p)) = 0$ except for the following cases.*

$$\begin{aligned}
&\pi_3(K(p)) &&\cong Z_{(p)},\\
&\pi_{2p^2}(K(p)) &&\cong \pi_{2p^2+2p-3}(K(p)) \cong Z/p,\\
&\pi_{2p^2+2p-1}(K(p)) &&\cong Z_{(p)},\\
&\pi_{2p^2+2r(p-1)}(K(p)) &&\cong Z/p \quad \text{for } 2 \le r < p,\\
&\pi_{4p^2-2p-2}(K(p)) &&\cong \pi_{4p^2-2p-1}(K(p)) \cong Z/p,\\
&\pi_{4p^2-2p}(K(p)) &&\cong \pi_{4p^2-5}(K(p)) \cong Z/p,\\
&\pi_{4p^2-2}(K(p)) &&\cong Z/p^2
\end{aligned}$$

The rest of this section is devoted to show the following lemma which is needed in the next section.

Lemma 2.19.

$$\pi_{2p^2+2p-2}^S(K(p)^{2p^2+2p-2}) \cong Z/p.$$

To show this lemma, we use the Adams-Novikov spectral sequence

$$(2.20) \quad E_2^{s,t} = H^{s,t}(BP_*(K(p)^{2p^2+2p-2})) \Rightarrow \pi_{t-s}^S(K(p)^{2p^2+2p-2}).$$

Set $L_i = BP_*/(p, v_1)\{X_{2i(p+1)+2}\}$ $(i < p)$, $L_p = BP_*\{X_{2p^2+2p+2}\}$, $M_0 = BP_*\{z_3\}$, $M_1 = BP_*\{z_3, z_{2p+1}\}/(v_1 z_3 - p z_{2p+1})$, $M_2 = BP_*\{z_3, z_{2p+1}\}/(v_1 z_3 - p z_{2p+1}) \oplus BP_*\{z'_{2p^2+2p-1}\}$. They are subcomodules of $BP_*(K(p))$ and $BP_*(K(p)^{2p^2+2p-2})$ is the direct sum of L_i $(1 \le i \le p-1)$ and M_1 ((1.19)).

Consider the long exact sequence associated with a short exact sequence of BP_*BP-comodules $0 \to M_0 \to M_1 \to M_1/M_0 \to 0$.

$$(2.21)\quad \cdots \longrightarrow H^{s,t}(M_0) \longrightarrow H^{s,t}(M_1) \longrightarrow H^{s,t}(M_1/M_0) \xrightarrow{\delta} H^{s+1,t}(M_0) \longrightarrow \cdots$$

We denote by $\bar{z}_{2p+1} \in M_0/M_1$ represented by z_{2p+1}. Since $M_1/M_0 = BP_*/(p)\{\bar{z}_{2p+1}\}$, $H^{s,t}(M_1/M_0)$ is identified with $H^{s,t-2p-1}(BP_*/(p))\{\bar{z}_{2p+1}\}$. We can verify the formula

$$(2.22)\quad \psi(v_1^r z_{2p+1}) = 1 \otimes v_1^r z_{2p+1} - \sum_{i=1}^{r+1} \binom{r+1}{i} (-p)^{i-1} t_1^i \otimes v_1^{r+1-i} z_3.$$

Then, it follows from (2.11) that $\delta\colon H^{s,t}(M_1/M_0) \to H^{s+1,t}(M_0)$ is given by $\delta(v_1^r \bar{z}_{2p+1}) = \alpha_{r+1} z_3$, $\delta(v_1^{r-1}\alpha_1 \bar{z}_{2p+1}) = 0$, $\delta(\beta_1 \bar{z}_{2p+1}) = \alpha_1\beta_1 z_3$, $\delta(v_1\beta_1\bar{z}_{2p+1}) = \delta(\alpha_1\beta_1\bar{z}_{2p+1}) = 0$.

We put $\xi_r = \sum_{j=1}^{r}((-p)^{j-1})/(j+1))\binom{r}{j}[t_1^{j+1}]v_1^{r-j} \in \Omega^1(BP_*)$. Then, we have $\alpha_1\alpha_r = d(-\xi_r)$ in $\Omega^2(BP_*)$ and $[t_1]v_1^{r-1}z_{2p+1} - \xi_r z_3$ is a cocycle of $\Omega^1(M_1)$. Let ω_r be the element of $H^{1,2r(p-1)+2p+1}(M_1)$ represented by $[t_1]v_1^{r-1}z_{2p+1} - \xi_r z_3$. It is easy to see that ω_r maps to $v_1^{r-1}\alpha_1\bar{z}_{2p+1}$ by $H^{*,*}(M_1) \to H^{*,*}(M_1/M_0)$ and that $p\omega_{p-1} = -\alpha_{p/2} z_3$ holds in the cobar complex of M_1. Therefore we obtain

Proposition 2.23. *$H^{s,t}(M_1) = 0$ if $t - s \le 2p^2 + 2p$ except for the following cases.*

$$\begin{aligned}
H^{0,3}(M_1) &= Z_{(p)}\{z_3\},\\
H^{1,2r(p-1)+2p+1}(M_1) &= Z/p\{\omega_r\} \quad \text{for } r < p-1 \text{ or } r = p+1,\\
H^{1,2p^2-2p+3}(M_1) &= Z/p^2\{\omega_{p-1}\},\\
H^{1,2p^2+1}(M_1) &= Z/p\{\omega_p, h_1 z_{2p+1}\},\\
H^{2,2p^2-2p+3}(M_1) &= Z/p\{\beta_1 z_3\},\\
H^{2,2p^2+2p-1}(M_1) &= Z/p\{v_1\beta_1 z_{2p+1}\},\\
H^{3,2p^2+2p-1}(M_1) &= Z/p\{\beta_1\omega_1\}.
\end{aligned}$$

(2.11) implies that $H^{s,t}(L_i) \cong H^{s,t-2i(p+1)-2}(BP_*/(p,v_1)) = 0$ for $1 \le i \le p-1$ if $t - s = 2p^2 + 2p - 2$. Since $H^{s,t}(BP_*(K(p)^{2p^2+2p-2})) \cong H^{s,t}(M_1) \oplus \sum_{i=1}^{p-1} H^{s,t}(L_i)$, (2.19) follows from (2.23).

Remark 2.24. It is easy to verify from (2.21) that $H^{0,t}(M_1) = 0$ unless $t = 3$.

3. The Hurewicz homomorphism. To begin with, we examine the Hurewicz homomorphisms $h^{BP}: \pi_i(F) \to BP_i(F)$ and $h: \pi_i(F) \to H_i(F; Z_{(p)})$ for $i \le 4p^2 - 2$.

Since $E_2^{0,t} = H^{0,t}(BP_*(V(1/2))) = 0$ if $t \ne 2p-1, 2r(p^2-1)$ $(r = 0, 1, \ldots)$ in (2.14) and v_2 supports a differential, $h^{BP}: \pi_i^S(V(1/2)) \to BP_i(V(1/2))$ is trivial if $i \ne 0$, $2p-1$ and $i \le 2p^2 - 2$, and it is an isomorphism if $i = 0$ or $2p - 1$.

Proposition 3.1. *For $i \le 4p^2 - 2$, the Hurewicz homomorphism $h^{BP}: \pi_i(F) \to BP_i(F)$ is trivial if $i \ne 2p^2$, $2p^2 + 2p - 1$, and it is an isomorphism if $i = 2p^2$ or $2p^2 + 2p - 1$.*

Remark 3.2. It follows from (2.13) that, for degree $\le 4p^2 - 2$, $\widetilde{BP}_*(F)$ is isomorphic to $BP_*/(p, v_1)\{Y_{2p^2}\} \oplus BP_*\{Y_{2p^2+2p-1}\}$.

Consider the long exact sequence of ordinary homology theory of $Z_{(p)}$-coefficients associated with a cofibration $S^{2p-2} \xrightarrow{\bar{\alpha}_1} M_p \xrightarrow{\iota_1} V(1/2) \xrightarrow{\partial_1} S^{2p-1}$.

$$\text{(3.3)} \quad \longrightarrow H_i(S^{2p-2}; Z_{(p)}) \xrightarrow{\bar{\alpha}_{1*}} H_i(M_p; Z_{(p)}) \xrightarrow{\iota_{1*}} H_i(V(1/2); Z_{(p)}) \xrightarrow{\partial_{1*}} H_{i-1}(S^{2p-2}; Z_{(p)}) \longrightarrow$$

Then, we see the following fact

Proposition 3.4. (1) *$H_i(V(1/2); Z_{(p)}) = 0$ if $i \ne 0$, $2p-1$,*

$$\begin{aligned} \iota_{1*}: H_0(M_p; Z_{(p)}) &\longrightarrow H_0(V(1/2); Z_{(p)}) \quad \text{and} \\ \partial_{1*}: H_{2p-1}(V(1/2); Z_{(p)}) &\longrightarrow H_{2p-2}(S^{2p-2}; Z_{(p)}) \end{aligned}$$

are isomorphisms, where $H_0(M_p; Z_{(p)}) \cong F_p$, $H_{2p-2}(S^{2p-2}; Z_{(p)}) \cong Z_{(p)}$.

(2) *$H_i(F; Z_{(p)}) = 0$ if $i \ne 0$, $2p^2$, $2p^2 + 2p - 1$ for $i \le 4p^2 - 2$, and there are isomorphisms $H_0(F; Z_{(p)}) \cong H_{2p^2+2p-1}(F; Z_{(p)}) \cong Z_{(p)}$ and $H_{2p^2}(F; Z_{(p)}) \cong F_p$.*

The Hurewicz homomorphisms give a morphism of the long exact sequences from (2.6) to (3.3). It follows from (2.12) and (3.4) that $h: \pi_{2p-1}^S(V(1/2)) \to H_{2p-1}(V(1/2); Z_{(p)})$ maps injectively onto $pH_{2p-1}(V(1/2); Z_{(p)})$.

Proposition 3.5. *For $i \leq 4p^2 - 2$, the Hurewicz homomorphism $h\colon \pi_i(F) \to H_i(F; Z_{(p)})$ is trivial if $i \neq 2p^2,\ 2p^2 + 2p - 1$, it is an isomorphism if $i = 2p^2$, and it is an injection onto $pH_{2p^2+2p-1}(F; Z_{(p)})$ if $i = 2p^2 + 2p - 1$.*

Remark 3.6. (3.4) and (3.5) imply that the Thom map $T_{Z_{(p)}}\colon BP_i(F) \to H_i(F; Z_{(p)})$ is a bijection if $i = 0, 2p^2$, and it is an injection onto $pH_{2p^2+2p-1}(F; Z_{(p)})$ if $i = 2p^2 + 2p - 1$.

Recall that $H_*(K(Z_{(p)}, 3); Z_{(p)})$ for degree $\leq 2p^2 + 2p + 1$ is isomorphic to

$$(3.7) \qquad Z_{(p)}\{\theta_0, \theta_3\} \oplus F_p\{\theta_{2r(p+1)-1} | r \leq p\} \oplus F_p\{\theta_{2r(p+1)+2} | r < p\} \oplus F_p\{\theta_{2p^2+1}, \theta_{2p^2+4}\},$$

where $\deg \theta_i = i$.

Consider the homology Serre spectral sequence associated with the fibration $F \xrightarrow{f} K(p) \xrightarrow{y_3} K(Z_{(p)}, 3)$;

$$E^2_{s,t} = H_s(K(Z_{(p)}, 3); H_t(F; Z_{(p)})) \Rightarrow H_{s+t}(K(p); Z_{(p)}).$$

By (1.12) and (3.4), $\theta_{2p^2+1} \in E^2_{2p^2+1,0}$ supports a differential and kills a generator of $E^2_{0,2p^2} = H_{2p^2}(F; Z_{(p)}) \cong Z/p$. Since $E^2_{s,t} = 0$ if $s + t = 2p^2 + 2p - 2$ or $2p^2 + 2p$, elements of total degree $2p^2 + 2p - 1$ are permanent cycles. We also note the fact $E^2_{s,t} = 0$ if $s + t = 2p^2 + 2p - 1$ and $s, t > 0$. Hence we have a short exact sequence

$$0 \longrightarrow H_{2p^2+2p-1}(F; Z_{(p)}) \xrightarrow{f_*} H_{2p^2+2p-1}(K(p); Z_{(p)}) \xrightarrow{y_{3*}} H_{2p^2+2p-1}(K(Z_{(p)}, 3); Z_{(p)}) \longrightarrow 0.$$

By virtue of (1.12), (3.4) and (3.7), we obtain

Proposition 3.8. *$f_*\colon H_i(F; Z_{(p)}) \to H_i(K(p); Z_{(p)})$ is a bijection if $i = 0$, an injection onto $pH_{2p^2+2p-1}(K(p); Z_{(p)})$ if $i = 2p^2 + 2p - 1$. Otherwise f_* is a zero map.*

Combining (3.5) and (3.8), we have the following result by the naturality of the Hurewicz homomorphism

Proposition 3.9. *The Hurewicz homomorphism $h\colon \pi_i(K(p)) \to H_i(K(p); Z_{(p)})$ is trivial if $i \neq 3, 2p^2 + 2p - 1$, an isomorphism if $i = 3$ and an injection onto $p^2 H_{2p^2+2p-1}(K(p); Z_{(p)})$ if $i = 2p^2 + 2p - 1$.*

It follows from (1.18) and the above result that the BP-Hurewicz homomorphism h^{BP}: $\pi_{2p^2+2p-1}(K(p)) \to BP_{2p^2+2p-1}(K(p))$ maps a generator to an element, say ζ, of the form $pz'_{2p^2+2p-1} + (\lambda v_1^{p+1} + \mu v_2)z_{2p+1}$. Since ζ is primitive, $(\lambda v_1^{p+1} + \mu v_2)z_{2p+1}$ is primitive modulo p. Recall that $v_1 z_3 = pz_{2p+1}$, $\eta_R(v_1) = v_1 + pt_1$ and $\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1$ modulo p [5], then we can easily verify from (1.19) that $v_1^{p+1} z_{2p+1}$ is primitive modulo p and that $\psi(v_2 z_{2p+1}) \equiv 1 \otimes v_2 z_{2p+1} + t_1 \otimes v_1^p z_{2p+1} - t_1 \otimes v_2 z_3$ modulo p. This implies that $\mu \equiv 0$ modulo p, hence we may assume $\mu = 0$ by replacing $z'_{2p^2+2p-1} + (\mu/p)v_2 z_{2p+1}$ by z'_{2p^2+2p-1}.

We set $\psi(z'_{2p^2+2p-1}) = 1 \otimes z'_{2p^2+2p-1} + A \otimes z_{2p+1} + B \otimes z_3$ for $A \in BP_{2p^2-2}BP$ and $B \in BP_{2p^2+2p-4}BP$. We may assume $B = bt_1^{p+2} + cv_2 t_1 + dt_1 t_2$ ($b, c, d \in Z_{(p)}$) and put $A = \sum_{i=0}^{p+1} a_i t_1^i \eta_R(v_1)^{p+1-i} + at_2 + a'v_2$. Then, $(\varepsilon \otimes 1)\psi(z'_{2p^2+2p-1}) = z'_{2p^2+2p-1}$ implies that $a_0 = a' = 0$. An easy calculation shows that the equality $\psi(\zeta) = 1 \otimes \zeta$ forces $a_i = \lambda(-p)^{i-1}\binom{p+2}{i}$, $b = \lambda p^p$ and $a = c = d = 0$. Then we have

$$(3.10) \qquad \psi(z'_{2p^2+2p-1}) = 1 \otimes z'_{2p^2+2p-1} + \lambda(p+2)t_1 \otimes v_1^p z_{2p+1} - \lambda \sum_{i=0}^{p} (-p)^i \binom{p+2}{i+2} t_1^{i+2} \otimes v_1^{p-i} z_3$$

Consider the long exact sequence associated with a short exact sequence of BP_*BP-comodules $0 \to M_1 \to M_2 \to M_2/M_1 \to 0$.

$$(3.11) \quad \cdots \longrightarrow H^{s,t}(M_1) \longrightarrow H^{s,t}(M_2) \longrightarrow H^{s,t}(M_2/M_1) \xrightarrow{\delta} H^{s+1,t}(M_1) \longrightarrow \cdots .$$

We denote by $\bar{z}'_{2p^2+2p-1} \in M_2/M_1$ the class of $z'_{2p^2+2p-1} \in M_2$. Then $\bar{z}'_{2p^2+2p-1}$ is primitive in M_2/M_1 and $H^{s,t}(M_2/M_1)$ is identified with $H^{s,t-2p^2-2p+1}(BP_*)\{\bar{z}'_{2p^2+2p-1}\}$. It follows from (2.23) and (3.10) that δ: $H^{0,2p^2+2p-1}(M_2/M_1) \to H^{1,2p^2+2p-1}(M_1)$ maps $\bar{z}'_{2p^2+2p-1}$ to $(p+2)\lambda\omega_{p+1}$. From (2.11), (2.23) and (3.11), we obtain

Lemma 3.12. *If $\lambda \in pZ_{(p)}$, we have $H^{1,2p^2+2p-1}(M_2) = Z/p\{\omega_{p+1}\}$. Thus $H^{1,2p^2+2p-1}(BP_*(K(p))) = Z/p\{\omega_{p+1}\}$ and ω_{p+1} represents a nontrivial element of $\pi^S_{2p^2+2p-2}(K(p))$.*

Let g: $S^{2p^2+2p-2} \to K(p)^{2p^2+2p-2}$ denote the attaching map of the $(2p^2 + 2p - 1)$-cell of $K(p)$. Consider a cofiber sequence $S^{2p^2+2p-2} \xrightarrow{g}$

$K(p)^{2p^2+2p-2} \xrightarrow{\iota} K(p)^{2p^2+2p-1}$. Since $\wp^1(b_3 b_{2p+2}^{p-1}) = b_{2p+1} b_{2p+2}^{p-1}$ in $H^*(K(p)^{2p^2+2p-1}; F_p) = H^*(K(p); F_p)/(b_3 b_{2p+1} b_{2p+2}^{p-1})$, $K(p)^{2p^2+2p-1}$ can not be stably homotopy equivalent to $K(p)^{2p^2+2p-2} \vee S^{2p^2+2p-1}$. Hence g is stably non-trivial and generates $\pi_{2p^2+2p-2}^S(K(p)^{2p^2+2p-2})$ ((2.19)).

Suppose that $\lambda \in pZ_{(p)}$ and let us denote by ω_{p+1} the element of $\pi_{2p^2+2p-2}^S(K(p))$ corresponding to $\omega_{p+1} \in H^{1,2p^2+2p-1}(BP_*(K(p)))$. There is a map ω'_{p+1}: $S^{2p^2+2p-2} \to K(p)^{2p^2+2p-2}$ such that ω_{p+1} is a composition $S^{2p^2+2p-2} \xrightarrow{\omega'_{p+1}} K(p)^{2p^2+2p-2} \hookrightarrow K(p)$. By the preceding argument, $\omega'_{p+1} = cg$ for some $c \in Z/p$. However, since $S^{2p^2+2p-2} \xrightarrow{g} K(p)^{2p^2+2p-2} \hookrightarrow K(p)$ is trivial, this contradicts the non-triviality of ω_{p+1}. Thus λ is a unit in $Z_{(p)}$, and $\zeta = pz'_{2p^2+2p-1} + \lambda v_1^{p+1} z_{2p+1}$ generates a direct summand of $BP_{2p^2+2p-1}(K(p)) = Z_{(p)}\{z'_{2p^2+2p-1}, v_1^{p+1} z_{2p+1}, v_2 z_{2p+1}\}$. Together with (1.18), (3.1), (3.5) and (3.8), we showed

Proposition 3.13. *For $i \leq 4p^2 - 2$, the BP-Hurewicz homomorphism h^{BP}: $\pi_i(K(p)) \to BP_i(K(p))$ is an isomorphism if $i = 3$, a split monomorphism onto the set of primitive elements if $i = 2p^2 + 2p - 1$ and otherwise a zero map.*

Remark 3.14. Put $PBP_t(K(p)) = \{x \in BP_t(K(p)) \mid \psi(x) = 1 \otimes x\} = H^{0,t}(BP_*(K(p)))$. Then, it follows from (2.24) and (3.11) that $PBP_t(K(p)) = 0$ except for the following cases.

$$\begin{aligned}
PBP_3(K(p)) &= Z_{(p)}\{z_3\}, \\
PBP_{2p^2+2p-1}(K(p)) &= Z_{(p)}\{\zeta\}, \\
PBP_{2i(p+1)+2j(p^2-1)+2}(K(p)) &= Z/p\{v_2^j X_{2i(p+1)+2}\} \\
&\qquad \text{for } 1 \leq i < p,\ j \geq 0, \\
PBP_{2p^2+2p+2}(K(p)) &= Z_{(p)}\{X_{2p^2+2p+2}\}.
\end{aligned}$$

4. The homology suspension. Consider the Serre spectral sequence for mod p cohomology associated with the path fibration $\Omega K(p) \to PK(p) \to K(p)$.

$$(4.1) \qquad E_2^{s,t} = H^s(K(p); H^t(\Omega K(p); F_p)) \Rightarrow H^{s+t}(PK(p); F_p)$$

There exists an element $\bar{x}_2 \in H^2(\Omega K(p); F_p) = E_3^{0,2}$ which maps to $x_3 \in E_3^{3,0}$ by d_3. The transgression theorem implies that $d_{2p+1}(\bar{x}_2^p) = x_{2p+1}$ and $d_{2p+2}(x_3 \otimes \bar{x}_2^{p-1}) = x_{2p+2}$.

Lemma 4.2. $H^{2p+1}(\Omega K(p); \boldsymbol{F}_p) = 0$.

We apply the Eilenberg-Moore spectral sequence associated with the above fibration.

$$E^2_{s,t} = \mathrm{Cotor}^{H_*(K(p);\boldsymbol{F}_p)}_{s,t}(\boldsymbol{F}_p, \boldsymbol{F}_p) \Rightarrow H_{s+t}(\Omega K(p); \boldsymbol{F}_p)$$

The E^2-term is given by $E^2 = \boldsymbol{F}_p[a_2, a_{2p}, a_{2p^2+2p-2}] \otimes E(a_{2p+1})$, where $a_2 \in E^2_{-1,3}$, $a_{2p} \in E^2_{-1,2p+1}$, $a_{2p+1} \in E^2_{-1,2p+2}$ and $a_{2p^2+2p-2} \in E^2_{-2,2p^2+2p}$ are the elements represented by $[b_3]$, $[b_{2p+1}]$, $[b_{2p+2}]$ and $\sum_{i=1}^{p-1}(1/p)\binom{p}{i}$ $[b^i_{2p+2}|b^{p-i}_{2p+2}]$, respectively, in the cobar complex $\Omega_*(H_*(K(p); \boldsymbol{F}_p))$. Since $H_{2p+1}(\Omega K(p); \boldsymbol{F}_p) = 0$ by (4.2), a_{2p+1} should support a differential. The only possible differential is $d_{p-1}(a_{2p+1}) = \nu a_2^p$ ($\nu \in \boldsymbol{F}_p$, $\nu \neq 0$) for dimensional reasons. Hence $E^p = E^\infty = \boldsymbol{F}_p[a_2, a_{2p}, a_{2p^2+2p-2}]/(a_2^p)$ and the extension is trivial.

Proposition 4.3. (1) $H_*(\Omega K(p); \boldsymbol{F}_p) = \boldsymbol{F}_p[a_2, a_{2p}, a_{2p^2+2p-2}]/(a_2^p)$. *Let* φ *be the* $\mathcal{A}_{p*}$*-comodule structure map, then* $\varphi(a_{2p}) = 1 \otimes a_{2p} - t_1 \otimes a_2$.

(2) *The homology suspension map* σ_*: $H_t(\Omega K(p); \boldsymbol{F}_p) \to H_{t+1}(K(p);$ $\boldsymbol{F}_p)$ *is given by* $\sigma_*(a_2) = b_3$, $\sigma_*(a_{2p}) = b_{2p+1}$, $\sigma_*(a_{2p^2+2p-2}) = 0$.

The equality $\varphi(a_{2p}) = 1 \otimes a_{2p} - t_1 \otimes a_2$ follows from the fact that a_2 and a_{2p} are the duals of $\bar{x}_2$ and $\bar{x}_2^p$ and that $\wp^1(\bar{x}_2) = \bar{x}_2^p$ in $H^*(\Omega K(p); \boldsymbol{F}_p)$. The last statement follows from $d_3(\bar{x}_2) = x_3$ and $d_{2p+1}(\bar{x}_2^p) = x_{2p+1}$ in (4.1).

Since $H_t(\Omega K(p); \boldsymbol{F}_p) = 0$ if t is odd, it follows from the Bockstein exact sequence that $H_*(\Omega K(p); \boldsymbol{Z}_{(p)})$ is torsion free and that there are elements $\tilde{a}_2$, $\tilde{a}_{2p}$ and $\tilde{a}_{2p^2+2p-2}$ of $H_*(\Omega K(p); \boldsymbol{Z}_{(p)})$ such that $\rho(\tilde{a}_i) = a_i$ ($i = 2, 2p, 2p^2+2p-2$), where ρ is the mod p reduction map. Then we have $H_*(\Omega K(p); \boldsymbol{Z}_{(p)}) = \boldsymbol{Z}_{(p)}[\tilde{a}_2, \tilde{a}_{2p}, \tilde{a}_{2p^2+2p-2}]/(\tilde{a}_2^p - \kappa p \tilde{a}_{2p})$ for some $\kappa \in \boldsymbol{Z}_{(p)}$.

Since $H_*(\Omega K(p); \boldsymbol{Z}_{(p)})$ is torsion free, it has a structure of Hopf algebra. Let Δ_* denote the diagonal map. We set $\Delta_*(\tilde{a}_{2p}) = 1 \otimes \tilde{a}_{2p} + \sum_{i=1}^{p-1} l_i \tilde{a}_2^i \otimes \tilde{a}_2^{p-i} + \tilde{a}_{2p} \otimes 1$ for some $l_i \in \boldsymbol{Z}_{(p)}$. On the other hand, we have $\Delta_*(\tilde{a}_2^p) = \sum_{i=0}^p \binom{p}{i} \tilde{a}_2^i \otimes \tilde{a}_2^{p-i}$ for $\tilde{a}_2$ is primitive. Applying Δ_* to the both sides of $\tilde{a}_2^p = \kappa p \tilde{a}_{2p}$, we have $l_i = (1/\kappa p)\binom{p}{i}$, therefore

$$(4.4) \qquad \Delta_*(\tilde{a}_{2p}) = 1 \otimes \tilde{a}_{2p} + \sum_{i=1}^{p-1} \frac{1}{\kappa p}\binom{p}{i} \tilde{a}_2^i \otimes \tilde{a}_2^{p-i} + \tilde{a}_{2p} \otimes 1.$$

Let us denote by $\langle\ \rangle$: $H_*(\Omega K(p);\boldsymbol{F}_p)\otimes H^*(\Omega K(p);\boldsymbol{F}_p)\to\boldsymbol{F}_p$ the canonical pairing and by Δ_*^{p-1} the $(p-1)$-fold diagonal map

$$\Delta_*^{p-1}=(\Delta_*\otimes\overbrace{1\otimes\cdots\otimes 1}^{p-2\text{ times}})\cdots(\Delta_*\otimes 1\otimes 1)(\Delta_*\otimes 1)\Delta_*.$$

Then, $\langle\Delta_*^{p-1}(a_{2p}),\bar{x}_2\otimes\cdots\otimes\bar{x}_2\rangle=\langle a_{2p},\bar{x}_2^p\rangle=1$ and by (4.4),

$$\Delta_*^{p-1}(a_{2p})=\frac{(p-1)!}{\kappa}\overbrace{a_2\otimes\cdots\otimes a_2}^{p\text{ times}}+\cdots.$$

Thus we have $\kappa\equiv(p-1)!\equiv -1$ modulo p and we may replace $-\kappa\tilde{a}_{2p}$ by $\tilde{a}_{2p}$.

Proposition 4.5. *$H_*(\Omega K(p);\boldsymbol{Z}_{(p)})=\boldsymbol{Z}_{(p)}[\tilde{a}_2,\tilde{a}_{2p},\tilde{a}_{2p^2+2p-2}]/(\tilde{a}_2^p+p\tilde{a}_{2p})$ with $\rho(\tilde{a}_i)=a_i$ $(i=2,\ 2p,\ 2p^2+2p-2)$, and $\Delta_*(\tilde{a}_{2p})=1\otimes\tilde{a}_{2p}-\sum_{i=1}^{p-1}(1/p)\binom{p}{i}\tilde{a}_2^i\otimes\tilde{a}_2^{p-i}+\tilde{a}_{2p}\otimes 1$.*

To determine the homology suspension σ_*: $H_t(\Omega K(p);\boldsymbol{Z}_{(p)})\to H_{t+1}(K(p);\boldsymbol{Z}_{(p)})$, we consider the Serre spectral sequence for $\boldsymbol{Z}_{(p)}$-homology associated with the path fibration $\Omega K(p)\to PK(p)\to K(p)$; $E^2_{s,t}=H_s(K(p);\ H_t(\Omega K(p);\boldsymbol{Z}_{(p)}))\Rightarrow H_{s+t}(PK(p);\boldsymbol{Z}_{(p)})$. The routine argument shows that the differentials are given as follows (See (1.12)).

$$\begin{aligned}
&d^3(z_3) &&=\tilde{a}_2, &&\\
&d^3(z_3z_{2i(p+1)-1}) &&=\tilde{a}_2z_{2i(p+1)-1} &&(1\le i\le p),\\
&d^{2p-1}(z_{2(i+1)(p+1)-1}) &&=\tilde{a}_2^{p-1}z_3z_{2i(p+1)-1} &&(1\le i\le p-1),\\
&d^{2p+1}(z_{2p+1}) &&=\tilde{a}_{2p}, &&\\
&d^{2p^2+2p-1}(pz_{2p^2+2p-1}) &&=\tilde{a}_{2p^2+2p-1} &&
\end{aligned}$$

This implies that pz_{2p^2+2p-1} is transgressible though z_{2p^2+2p-1} is not.

Proposition 4.6. *The homology suspension σ_*: $H_t(\Omega K(p);\boldsymbol{Z}_{(p)})\to H_{t+1}(K(p);\boldsymbol{Z}_{(p)})$ is given by $\sigma_*(\tilde{a}_2)=z_3$, $\sigma_*(\tilde{a}_{2p})=z_{2p+1}$ and $\sigma_*(\tilde{a}_{2p^2+2p-1})=pz_{2p^2+2p-1}$.*

Consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t}=H_s(\Omega K(p);BP_t)\Rightarrow BP_{s+t}(\Omega K(p)).$$

It collapses for dimensional reason and $E^{\infty} = BP_*[\tilde{a}_2, \tilde{a}_{2p}, \tilde{a}_{2p^2+2p-2}]/(\tilde{a}_2^p + p\tilde{a}_{2p})$. Let $\hat{a}_2$, $\hat{a}_{2p}$ and $\hat{a}_{2p^2+2p-2}$ be the elements of $BP_*(\Omega K(p))$ corresponding to $\tilde{a}_2$, $\tilde{a}_{2p}$ and $\tilde{a}_{2p^2+2p-2}$ in the E^{∞}-term. There exists $\tau \in Z_{(p)}$ such that $\hat{a}_2^p + p\hat{a}_{2p} = \tau v_1 \hat{a}_2$. By (4.3), we have $\psi(\hat{a}_{2p}) = 1 \otimes \hat{a}_{2p} - t_1 \otimes \hat{a}_2$, where ψ: $BP_*(\Omega K(p)) \to BP_*BP \otimes_{BP_*} BP_*(\Omega K(p))$ is the BP_*BP-comodule structure map. Then, $\psi(\hat{a}_2^p + p\hat{a}_{2p}) = 1 \otimes \hat{a}_2^p + 1 \otimes p\hat{a}_{2p} - t_1 \otimes p\hat{a}_2$. On the other hand, $\psi(\tau v_1 \hat{a}_2) = \tau v_1 \otimes \hat{a}_2 = 1 \otimes \tau v_1 \hat{a}_2 - \tau t_1 \otimes p\hat{a}_2$. Thus we have $\tau = 1$.

By the naturality of the Thom map, it follows from (1.18) that the homology suspension σ_*: $\widetilde{BP}_t(\Omega K(p)) \to \widetilde{BP}_{t+1}(K(p))$ maps $\sigma_*(\hat{a}_2) = z_3$, $\sigma_*(\hat{a}_{2p}) = \nu z_{2p+1}$ and $\sigma_*(\hat{a}_{2p^2+2p-2}) = z'_{2p^2+2p-1} + \mu_1 v_1^{p+1} z_{2p+1} + \mu_2 v_2 z_{2p+1}$ for some ν, μ_1, $\mu_2 \in Z_{(p)}$, $\nu \equiv 1$ modulo p.

Proposition 4.7. $BP_*(\Omega K(p)) = BP_*[\hat{a}_2, \hat{a}_{2p}, \hat{a}_{2p^2+2p-2}]/(\hat{a}_2^p + p\hat{a}_{2p} - v_1\hat{a}_2)$, $T_{Z_{(p)}}(\hat{a}_i) = \tilde{a}_i$ *for* $i = 2$, $2p$, $2p^2 + 2p - 2$. *The image of the homology suspension* σ_*: $\widetilde{BP}_*(\Omega K(p)) \to \widetilde{BP}_{*+1}(K(p))$ *is a* BP_**-submodule of* $BP_*(K(p))$ *generated by* z_3, z_{2p+1} *and* z'_{2p^2+2p-1}, *that is, it coincides with* $\sum_{i \geq 0} BP_{2i+1}(K(p))$.

If X is a topological space with a base point x_0 and $E_*(-)$ is a generalized homology theory, then the set of diagonal primitive elements is defined to be the kernel of the reduced diagonal map $\tilde{\Delta}$: $E_*(X) \to E_*(X \times X)$ which maps $x \in E_*(X)$ to $\Delta_*(x) - (i_{1*}(x) + i_{2*}(x))$. Here Δ: $X \to X \times X$ denotes the diagonal map and i_1, i_2: $X \to X \times X$ are the maps given by $i_1(x) = (x, x_0)$, $i_2(x) = (x_0, x)$. Note that this definition does not depend on the choice of the base point if X is path connected.

Let us denote by $P_dE_*(X)$ the set of diagonal primitive elements of $E_*(X)$. Then, it is known that $P_dE_*(X)$ contains the image of the homology suspension σ_*: $\tilde{E}_*(\Omega X) \to \tilde{E}_{*+1}(X)$ and if a map η: $S^0 \to E$ is given, $P_dE_*(X)$ also contains the image of the unstable Hurewicz homomorphism h^E: $\pi_*(X) \to E_*(X)$.

To show that $P_dBP_*(K(p)) = \operatorname{Im}\sigma_*$, we consider a homology theory $P(2)_*(-)$ with coefficient ring $BP_*/(p, v_1)$. (It is also denoted by $BPI_{2*}(-)$.) There are canonical maps of ring spectra T_2: $BP \to P(2)$ and ρ_2: $P(2) \to K(F_p)$ such that T_2 induces the natural projection $BP_* \to BP_*/(p, v_1)$ of the coefficient rings and composition $\rho_2 T_2$ coincides with the Thom map. It is also known that ρ_{2*}: $H_*(P(2); F_p) \to H_*(K(F_p); F_p) = \mathcal{A}_{p*}$ is an injection onto $E(s_0, s_1) \otimes F_p[t_1, t_2, \ldots]$.

We apply the Adams spectral sequence (1.4) for $E = P(2)$, $X = K(p)$.

(4.8) $$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(s_2,s_3,\dots)}(\boldsymbol{F}_p, H_*(K(p);\boldsymbol{F}_p)) \Rightarrow P(2)_{t-s}(K(p))$$

It follows from (1.9) that every element of $H_*(K(p);\boldsymbol{F}_p)$ is primitive as an $E(s_2,s_3,\dots)$-comodule, thus $E_2 = \boldsymbol{F}_p[\bar{v}_2,\bar{v}_3,\dots] \otimes H_*(K(p);\boldsymbol{F}_p)$. Therefore this spectral sequence collapses for dimensional reason. Note that $T_{2*}\colon BP_*(K(p)) \to P(2)_*(K(p))$ induces morphism of the spectral sequences from (1.6) to (4.8), and that, as in (1.7), the maps between the E_2-terms coincides with the maps induced by the natural projection $E(s_0,s_1,\dots) \to E(s_2,s_3,\dots)$. We obtain an analog of (1.18).

Proposition 4.9. *$P(2)_*(K(p)) = BP_*/(p,v_1) \otimes E(b_3,b_{2p+1}) \otimes \boldsymbol{F}_p[b_{2p+2}]/(b_{2p+2}^p)$ and $T_{2*}\colon BP_*(K(p)) \to P(2)_*(K(p))$ is given by*

$$\begin{aligned} T_{2*}(X_{2i(p+1)+2}) &= b_3 b_{2p+1} b_{2p+2}^{i-1} \quad (1 \le i \le p), \\ T_{2*}(z_3) &= b_3, \\ T_{2*}(z_{2p+1}) &= b_{2p+1}, \\ T_{2*}(z'_{2p^2+2p-1}) &= 0. \end{aligned}$$

Moreover, $\rho_{2*}\colon P(2)_*(K(p)) \to H_*(K(p);\boldsymbol{F}_p)$ maps b_i to b_i.

The above result implies that $P(2)_*(K(p))$ is a free $BP_*/(p,v_1)$-module, thus it has a structure of Hopf algebra. Hence $\rho_{2*}\colon P(2)_*(K(p)) \to H_*(K(p);\boldsymbol{F}_p)$ is a map of a Hopf algebras, and b_i ($i = 3, 2p+1, 2p+2$) are primitive. If $cX_{2i(p+1)+2} \in L_i \cap P_d BP_*(K(p))$ ($c \in BP_*/(p,v_1)$, $1 \le i \le p-1$), then $T_{2*}(cX_{2i(p+1)+2}) \in P_d P(2)_*(K(p))$. On the other hand, $T_{2*}(cX_{2i(p+1)+2}) = cb_3 b_{2p+1} b_{2p+2}^{i-1}$, which is not diagonal primitive unless $c = 0$.

Lemma 4.10.

$$L_i \cap P_d BP_*(K(p)) = 0 \quad \textit{if } 1 \le i \le p-1.$$

Consider the canonical map $BP_*(K(p)) \to BP_*(K(p)) \otimes \boldsymbol{Q}$ instead of T_{2*}. Since $X_{2p^2+2p+2} = p^{-1} z_3 z'_{2p^2+2p-1}$ in $BP_*(K(p)) \otimes \boldsymbol{Q}$ ((1.19)) and both z_3 and z'_{2p^2+2p-1} are primitive, the similar argument shows

Lemma 4.11.

$$L_p \cap P_d BP_*(K(p)) = 0.$$

Suppose that there exists a non-zero element x of $P_dBP_*(K(p))$ of even degree, say $2n$. Then $x = \sum_{i=1}^{p} c_i X_{2i(p+1)+2}$ for some $c_i \in BP_{2n-2i(p+1)-2}$ and it follows from (4.10) and (4.11) that $c_i \neq 0$ for at least two i's. If $c_j \neq 0$ and $c_k \neq 0$ $(1 \leq j < k \leq p)$, $2n - 2j(p+1) - 2 = 2l(p-1)$ and $2n - 2k(p+1) - 2 = 2m(p-1)$ for some integers l, m. Hence we have $(k-j)(p+1) = (l-m)(p-1)$ and this implies that $j = 1$ and $k = p$. Thus $x = c_1 X_{2p+4} + c_p X_{2p^2+2p+2}$. Since X_{2p+4} is a p-torsion element, we have $px = pc_p X_{2p^2+2p+2} \in L_p \cap P_dBP_*(K(p))$, hence $pc_pX_{2p^2+2p+2} = 0$ by (4.11). But X_{2p^2+2p+2} is torsion free, this contradicts $c_p \neq 0$. Therefore we have shown $P_dBP_*(K(p)) \cap \sum_{i \geq 0} BP_{2i}(K(p)) = 0$ and by (4.7), this implies the following result.

Proposition 4.12.

$$P_dBP_*(K(p)) = \operatorname{Im}\{\sigma_*\colon \widetilde{BP}_*(\Omega K(p)) \to \widetilde{BP}_{*+1}(K(p))\}.$$

By (3.14) and (4.7), we have $P_dBP_*(K(p)) \cap PBP_*(K(p)) = Z_{(p)}\{z_3, \zeta\}$. The left hand side contains the Hurewicz image and we already showed that the Hurewicz image contains the right hand side. This implies $\operatorname{Im}\{h^{BP}\colon \pi_*(K(p)) \to BP_*(K(p))\} = Z_{(p)}\{z_3, \zeta\}$. Finally we obtain

Theorem 4.13. *The BP-Hurewicz homomorphism $h^{BP}\colon \pi_i(K(p)) \to BP_i(K(p))$ is an isomorphism if $i = 3$, a split monomorphism onto the set of primitive elements if $i = 2p^2 + 2p - 1$ and otherwise a zero map.*

Remark 4.14. If we apply T_{2*} to the both sides of a relation $z_3 z'_{2p^2+2p-1} = pX_{2p^2+2p+2} + \gamma v_2 X_{2p+4}$ ((1.19)), it follows that $\gamma \equiv 0$ modulo p. Since X_{2p+4} is a p-torsion element, we have $z_3 z'_{2p^2+2p-1} = pX_{2p^2+2p+2}$ in $BP_*(K(p))$.

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