Math. J. Okayama Univ. 37(1995), 105–111

GENERALIZED n-POTENT RINGS

Dedicated to the memory of Professor Hisao Tominaga

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In [1], [2], [3] and [4], some generalizations of Jacobson's theorem which states that a ring R in which for every $a \in R$ there exists an integer $n(a) > 1$, depending on a, such that $a^{n(a)} = a$, is necessarily commutative have been studied in various directions. In this note, these results will be partially generalized to a wider class of rings, namely generalized n-potent rings.

Throughout, R denotes an associative ring, N the set of nilpotent elements of R, C the center of R, J the Jacobson radical of R, $C(R)$ the commutator ideal of R, and $\mathbb{Z}$ the ring of rational integers. For x, y in R, $[x,y] = xy - yx$ denotes the commutator.

We now introduce the following definitions.

Definition 1. Let n be a fixed integer, $n > 1$. A ring R is called a *generalized n-potent ring* if

$$(*) \qquad x^n - x \in N \cap C \quad \text{for all} \quad x \in R \backslash (N \cup C).$$

Definition 2. If the set E of all idempotents of R is contained in C then R will be called a *ring with all idempotents central.* If N is contained in C then R will be called a *ring with all nilpotents central.*

Our main result is the following: *A generalized n-potent ring R with all idempotents central is commutative if it satisfies two conditions*:

(i) For all $a, b \in N$, $[a,b] = [a,b]^q$ for some integer $q > 1$.

(ii) $(n-1)[a,x] = 0$ implies $[a,x] = 0$ for all $a \in N$, $x \in R$.

It is further shown that all of the hypotheses of this theorem are essential.

We also prove a following structure theorem for generalized n-potent rings: If R is a generalized n-potent ring with all idempotents central which satisfies the above condition (ii), then $R = C$ or $R = N$.

Remark. A ring R is a generalized n-potent ring if and only if one of the following conditions holds:

(1) $R = N \cup C$.

(2) $R \neq N \cup C$ and $(*)$ is satisfied.

Now, we take a closer look at case (1). In this case, for any $a, b \in N$ with $a - b \in C$, we have $ab = ba$, and so $a - b \in N$. Moreover, for any $a \in N$ and $x \in R$ with $ax \in C$, an easy induction shows that $(ax)^k = a^k x^k$ for all positive integers k, and so $ax \in N$. Similarly, $xa \in N$. Hence, if $R = N \cup C$, then N is an ideal of R, whence $R = N$ or $R = C$. Next, for any elements $a, b \in R$ with

$$[a, b] = [a, b]^q \quad \text{for some integer} \quad q > 1$$

we have

$$[a, b] = ([a, b]^q)^q = [a, b]^{q^2} = [a, b]^{q^k}$$

for all positive integer k. Hence, if $R = N$ and R satisfies (i) above, then $R = N = C$. Therefore, it follows that, if $R = N \cup C$ and R satisfies (i) also, then $R = C$.

We have thus shown the following:

(a) If $R = N \cup C$, then $R = N$ or $R = C$.

(b) If $R = N \cup C$ and R satisfies (i) above, then $R = C$.

We now prove the following two lemmas.

Lemma 1. *If R is a generalized n-potent ring, then $J \subseteq N \cup C$.*

Proof. Suppose not, and let $x \in J$, $x \notin N$, $x \notin C$. Then, since R is generalized n-potent, $x^n - x \in N \cap C$. So $(x^n - x)^m = 0$ for some $m \in \mathbb{Z}^+$, and thus

$$x^m = x^m(xg(x)), \quad \text{for some} \quad g(\lambda) \text{ in } \mathbb{Z}[\lambda].$$

From this equation we readily obtain the relation $x^m = x^m(xg(x))^m$; so we see that $e = (xg(x))^m = x^m g(x)^m$ is idempotent. Thus, $e = (xg(x))^m$ is an idempotent element of J (since $x \in J$). Therefore $e = 0$. This implies that $x^m = 0$, which contradicts $x \notin N$. This proves the lemma.

Lemma 2. *Let R be a generalized n-potent ring with all idempotents central. Then,*

(i) *$ax \in N$ for all $a \in N$ and $x \in R$;*

(ii) *N is an ideal of R;*

(iii) $C(R) \subseteq N \subseteq J \subseteq N \cup C$.

Proof. Suppose that (i) is false and let $a \in N$ and $x \in R$, with $ax \notin N$. As seen in above Remark, we note that if $ax \in C$, then $ax \in N$, a contradiction. This shows that $ax \notin C$. Thus, $ax \notin N \cup C$. So, because R is generalized n-potent, we have $(ax)^n - ax \in N$. Then, as in the proof of Lemma 1, $(ax)^m = (ax)^m e$, where $e = ((ax)g(ax))^m$ is idempotent, and $g(\lambda)$ in $\mathbb{Z}[\lambda]$.

By hypothesis, e is central so $e = ee = e((ax)g(ax))^m = eat = aet$, for some $t \in R$. Thus, $e = aet$, and as we noted in above Remark, since $aet \in C$, $e = e^q = (aet)^q = a^q(et)^q = a^q et^q$, for all positive integers q. But since $a \in N$, for some q, $a^q = 0$; so $e = 0$. Hence, $(ax)^m = 0$, a contradiction, since $ax \notin N$. This proves part (i).

For (ii), let $a \in N$ and $x \in R$. Then, from (i), $ax \in N$, and hence ax is right quasiregular for all $x \in R$. Thus $a \in J$, and hence $N \subseteq J$. Combining this with Lemma 1 we see that $N \subseteq J \subseteq N \cup C$. Now suppose that $a, b \in N$. Then both a and $b \in J$, so $a - b \in J \subseteq N \cup C$. Thus, $a - b \in N$ or $a - b \in C$. But if $a - b \in C$, then $ab = ba$ and, therefore, $a - b \in N$, for all $a, b \in N$ in any case. Next suppose that $a \in N$ and $r \in R$. Then, from part (i), $ar \in N$, say $(ar)^q = 0$. Hence, $(ra)^{q+1} = 0$ and so $ra \in N$. Thus, N is an ideal of R.

Finally for (iii), since N is an ideal, the factor ring R/N exists. Since generalized n-potency is inherited by homomorphic images of R, R/N is also generalized n-potent and we readily obtain that for every $y \in R/N$, $y^n - y$ is in the center of R/N. Therefore, by Herstein's Theorem [1], R/N is commutative, and thus $C(R) \subseteq N$ [$C(R)$ is the commutator ideal of R.] Combining this result with $N \subseteq J \subseteq N \cup C$, we have part (iii).

We next prove the following theorems.

Theorem 1. *A generalized n-potent ring with all nilpotents central is commutative.*

Proof. Since the set N of nilpotent elements of the ring R is contained in the center C of R, this implies at once that $x^n - x \in C$ for all $x \in R$. Hence, by Herstein's Theorem [1], R is commutative.

Theorem 2. *Suppose that R is a generalized n-potent ring with identity. Suppose further that for all $a \in N$, $x \in R$,*

$$(**) \qquad (n-1)[a, x] = 0 \quad \textit{implies} \quad [a, x] = 0.$$

Then R is commutative.

Proof. We claim that in this case $N \subseteq C$. Suppose, not, and let $a \in N$, $a \notin C$. Suppose that

(1) $[a^\sigma, x] = 0$, for all $\sigma \geq \sigma_0$, σ_0 minimal, $x \in R$ arbitrary.

We claim that $\sigma_0 = 1$. Suppose not. Then, $1 + a^{\sigma_0 - 1} \notin N \cup C$, and hence,

$$(1 + a^{\sigma_0-1})^n - (1 + a^{\sigma_0-1}) = b \in C. \tag{2}$$

Combining (1) and (2), we see that $(n-1)[a^{\sigma_0-1}, x] = [b, x] = 0$ for all $x \in R$, and thus by (**), $[a^{\sigma_0-1}, x] = 0$, for all $x \in R$. This, however, contradicts the minimality of σ_0 in equation (1), so $\sigma_0 = 1$. Therefore, by (1), $[a, x] = 0$ for all $x \in R$, which contradicts $a \notin C$. This contradiction proves $N \subseteq C$, and the theorem now follows from Theorem 1.

Theorem 3. *Suppose that R is a generalized n-potent ring with all idempotents central. Suppose further that for all $a \in N$, $x \in R$,*

$$(n-1)[a, x] = 0 \quad \text{implies} \quad [a, x] = 0.$$

Then $R = C$ or $R = N$.

Proof. By Lemma 2, N is an ideal of R. Suppose $R \supsetneqq C \cup N$, and let x be an arbitrary element of $R \backslash (C \cup N)$. Then, by (*), $x^n - x \in C \cap N$. Hence, as in the proof of Lemma 1, $x^m = x^m e$ for some positive integer m and some idempotent e of R. By hypothesis, we have $e \in C$. Since $x \notin N$, e is nonzero. Now, we consider the Peirce decomposition

$$R = Re \oplus A,$$

where $A = \{a - ae; a \in R\} = \{a \in R; ae = 0\}$, which will be denoted by $R(1-e)$. Obviously

$$C = Ce \oplus C(1-e), \quad N = Ne \oplus N(1-e)$$

and, Ce (resp. Ne) coincides with the set of all central (resp. nilpotent) elements of Re. Further

$$C \cap N = (Ce \cap Ne) \oplus (C(1-e) \cap N(1-e)),$$
$$R \backslash (C \cup N) \supseteq Re \backslash (Ce \cup Ne).$$

From this, an easy computation enables us to see that Re is a generalized n-potent ring with the identity e. Moreover, for all $a \in N$, $b \in R$, we have $ae \in N$, and hence by hypothesis,

$$(n-1)[ae, be] = 0 \quad \text{implies} \quad [ae, be] = 0.$$

Now by Theorem 2, we see that Re is a commutative ring, whence $Re \subseteq C$. We write $x = xe + (x - xe)$. Then $x^m = x^m e + (x - xe)^m$. Since $x^m = x^m e$, we have $(x - xe)^m = 0$. Hence $x - xe \in N$.

We set $a = x - xe$. Since $x \notin C$ and $xe \in C$ (since $Re \subseteq C$), we have $a \notin C$. Suppose that

$$a^\sigma \in C \quad \text{for all} \quad \sigma \geq \sigma_0, \quad \sigma_0 \text{ minimal}.$$

Then $\sigma_0 > 1$, $a^{\sigma_0 - 1} \notin C$, and $e + a^{\sigma_0 - 1} \notin N \cup C$. Hence

$$(e + a^{\sigma_0 - 1})^n - (e + a^{\sigma_0 - 1}) \in C.$$

Since $ea^{\sigma_0 - 1} \in eR \subseteq C$ and $(a^{\sigma_0 - 1})^m \in C$ for all $m \geq 2$, we have,

$$(e + a^{\sigma_0 - 1})^n \in C, \quad \text{and so} \quad e + a^{\sigma_0 - 1} \in C.$$

This implies $a^{\sigma_0 - 1} \in C$, which is a contradiction. Therefore, it follows that $R = C \cup N$. Thus, we obtain that $R = C$ or $R = N$ (see above Remark (a)).

We are now in a position to prove our main result.

Main Theorem. *Suppose that R is a generalized n-potent ring with all idempotents central. Suppose further that*

(i) *for all $a, b \in N$, $[a, b] = [a, b]^q$ for some integer $q > 1$; and*

(ii) *$(n-1)[a, x] = 0$ implies $[a, x] = 0$, for all $a \in N$, $x \in R$.*

Then R is commutative (and conversely).

Proof. Let $a, b \in N$. By Lemma 2(iii), the commutator $[a, b] \in N$, and thus for some positive integer r, $[a, b]^r = 0$. Moreover, by hypothesis (i), $[a, b] = [a, b]^q$ for some integer $q > 1$. Hence, as is seen in above Remark, we have $[a, b] = 0$ and, therefore, N is *commutative*. By Theorem 3, $R = C$ or $R = N$, and the theorem thus follows.

Jacobson's Theorem for fixed n, is an immediate corollary of our main theorem, since in that case $N = \{0\}$. So all idempotents e in R are central. This follows because $(ex - exe)^2 = 0 = (xe - exe)^2$. We state this result as

Corollary 1. *Suppose that R is a ring such that for all $x \in R$, $x^n = x$, $n > 1$ a fixed integer. Then R is commutative.*

We also have the following additional corollary:

Corollary 2. *Suppose R is a generalized n-potent ring with all idempotents central and with commuting nilpotents. Suppose further that the set N of nilpotents is $(n-1)$-torsion-free. Then R is commutative.*

Proof. By Lemma 2(ii), N is an ideal of R, and so $[a, x] \in N$, for all $a \in N$ and $x \in R$. Therefore, hypothesis (ii) of the main theorem is satisfied (since N is $n-1$-torsion-free), and the corollary follows.

We conclude with the following examples which show that our main theorem need not be true if any one of the hypotheses is deleted.

Example 1. Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : 0, 1 \in GF(2) \right\};$$

and let $n = 2$.

This example shows that the hypothesis that all of the idempotents are central cannot be deleted from the main theorem.

Example 2. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in GF(4) \right\};$$

and let $n = 2$.

This example shows that the hypothesis that the ground ring R is generalized n-potent cannot be deleted from the main theorem.

Example 3. The ring of all *strictly* upper triangular 3×3 matrices over $GF(3)$, with $n = 3$, shows that hypothesis (i) of the main theorem cannot be deleted.

Example 4. The ring in Example 2, but now with $n = 7$, shows that hypothesis (ii) of the main theorem cannot be deleted.

Acknowledgement. In conclusion, we would like to express our gratitude to Professor Takasi Nagahara for his valuable suggestions and helpful comments, which resulted in strengthening the results.

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(*Received April 6, 1994*)