DUAL RINGS AND COGENERATOR RINGS

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Let R be a ring with identity. We call R *a dual ring* if $l_R r_R(I) = I$ for every left ideal I of R and $r_R l_R(J) = J$ for every right ideal J of R. As is well-known, cogenerator rings are just two-sided self-injective dual rings. Basic properties of dual rings are investigated in [2]. We say that a class **P** of rings is *Morita stable* if $A \in$ **P** implies that all rings Morita equivalent to A also belong to **P** (see [7, p.109]). Obviously, the class of cogenerator rings is Morita stable. Thus it seems natural to ask whether the class of dual rings is Morita stable. In this note, we shall characterize cogenerator rings as dual rings for which all rings Morita equivalent to itself are dual rings and deduce that the class of dual rings is not Morita stable.

Throughout this note, R denotes an associative ring with identity and modules are unitary modules. We denote by R-Mod the category of left R-modules and by $(\ )^*$ the R-dual functors $\mathrm{Hom}_R(-, R)$. Let X be a one-sided R-module. We denote by $E(X)$ the injective hull of X and by $\epsilon_X \colon X \to X^{**}$ the usual evaluation map. Recall that X is *torsionless* in case ϵ_X is a monomorphism. Moreover we denote by $\tau(X)$ the Lambek torsion submodule of X. Therefore $\tau(X) = \bigcap \{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}_R(X, E(R))\}$.

For notations, definitions and familiar results concerning the ring theory we shall mainly follow [1] and [7].

First we note the following.

Propsosition 1. $l_R r_R(I) = I$ *for every left ideal* I *of* R *if and only if every cyclic left* R-*module is torsionless. In particular,* R *is a dual ring if and only if every cyclic left and every cyclic right* R-*module is torsionless.*

Proof. This follows from [4, Lemma 1.2].

A ring R is called *left QF-3″* in case every finitely generated submodules of $E(_RR)$ is torsionless (see [5]). The following fact is essential in this note.

Lemma 2 ([3, Theorem 1.2]). *For a ring* R, *the following are equivalent:*
  (1) R *is left QF-3″.*
  (2) $\tau(X) = \mathrm{Ker}(\epsilon_X)$ *for every finitely generated left* R-*module* X.

(3) $\mathrm{Ext}^1_R(X,R)_R$ *is τ-torsion for every finitely generated left R-module X.*

Recall that a ring R is *left Kasch* in case every simple left R-module is isomorphic to a minimal left ideal of R (see [7]).

Corollary 3. *If R is left QF-3'' right Kasch, then R is left self-injective.*

Proof. Let X be a cyclic left R-module. Then $\mathrm{Ext}^1_R(X,R)_R$ is τ-torsion by the lemma above. On the other hand, since R is right Kasch, all τ-torsion right R-modules are zero. Thus $\mathrm{Ext}^1_R(X,R)_R = 0$, so R is left self-injective.

A module X is called *a self-cogenerator* if every submodule of factor modules of X^n is cogenerated by X for all integers $n > 0$ (see [6]).

Theorem 4. *For a ring R, the following are equivalent*:
(1) $_RR$ *is a self-cogenerator.*
(2) *Every finitely generated left R-module is torsionless.*
(3) *Every cyclic left $M_n(R)$-module is torsionless for all integers $n > 0$, where $M_n(R)$ denotes the ring of $n \times n$-matrices over R.*
(4) *Every cyclic left S-module is torsionless for all rings S Morita equivalent to R.*
(5) R *is left QF-3'' and every cyclic left R-module is torsionless.*
(6) R *is left QF-3'' left Kasch.*

Proof. (1)$\Longleftrightarrow$(2). Obvious.
(2)$\Longleftrightarrow$(3). Let $n > 0$, $S = M_n(R)$ and $P = R^n$. Then P becomes a faithfully balanced (R,S)-bimodule and $_RP$ is a finitely generated projective generator. Thus the functor $F = \mathrm{Hom}_R(P,-)\colon R\text{-Mod} \to S\text{-Mod}$ is an equivalence. Since $F(R)^n \cong F(P) \cong S$, every factor module of $_RP$ is torsionless iff every factor module of $_SS$ is cogenerated by $F(R)$ iff every cyclic left S-module is torsionless. Thus the equivalence of (2) and (3) is proved.
(2)$\Longrightarrow$(4). The condition (2) for R is categorical. Therefore, if S is Morita equivalent to R, then every finiteley genereted (particularly cyclic) left S-module is torsionless.
(4)$\Longrightarrow$(3) and (2)$\Longrightarrow$(5)$\Longrightarrow$(6). Obvious.
(6)$\Longrightarrow$(2). Let X be a finitely generated left R-module. Then $\mathrm{Ker}(\epsilon_X) = \tau(X)$ by Lemma 2. On the other hand, all τ-torsion left

R-modules are zero, because R is left Kasch. So $\mathrm{Ker}(\epsilon_X) = 0$, i.e., X is torsionless.

Now we present the following as the two-sided version of the theorem above.

Theorem 5. *For a ring R, the following are equivalent*:

(0) *R is a cogenerator ring.*

(1) *$_RR$ and R_R are self-cogenerators.*

(2) *Every finitely generated left and every finitely generated right R-module is torsionless.*

(3) *$M_n(R)$ are dual rings for all integers $n > 0$.*

(4) *All rings Morita equivalent to R are dual rings.*

(5) *R is a two-sided QF-3″ dual ring.*

(6) *R is two-sided QF-3″ two-sided Kasch.*

Proof. By Proposition 1 and Theorem 4, (1) to (6) are equivalent.

(0) $\Longrightarrow$ (1). Obvious.

(6) $\Longrightarrow$ (0). By Corollary 3, R is two-sided self-injective. Thus R is a cogenerator ring since R is two-sided Kasch.

Remark 6. (1) It is easy to see that a class **P** of rings is Morita stable if and only if the following two conditions hold:

(a) If $R \in \mathbf{P}$, then $M_n(R) \in \mathbf{P}$ for all $n > 0$.

(b) If $R \in \mathbf{P}$ and $e = e^2 \in R$ with $ReR = R$, then $eRe \in \mathbf{P}$.

Theorem 5 and Example 7 show that the class of dual rings does not satisfy the condition (a).

(2) By the well-known characterization of cogenerator rings, the condition that R being a cogenerator on both sides implies the two-sided self-injectivity of R (see [8, Theorem 5.4]). The condition (3) in Theorem 5 is more weaker than this.

Example 7. Let p be a prime number, $R = Z_{(p)} = \{a/b \in Q \mid (a,b) = 1$ and $p \nmid b\}$, the localization of Z at p, and $M = Q/R$, where Z is the ring of integers and Q is the field of rational numbers. Then, by [2, Example 6.1] the trivial extension T of R by M is a dual ring, but not self-injective. Thus the preceding theorem shows that there exists a ring S which is Morita equivalent to the dual ring T but not a dual ring.

The following theorem states that the class of dual rings satisfies the condition (b) in Remark 6.

Theorem 8.   *Let R be a dual ring and e an idempotent of R such that $ReR = R$. Then eRe is a dual ring.*

Proof.   Since $ReR = R$, $_RRe$ is a finitely generated projective generator. Thus, by a similar way to prove the equivalence of (2) and (3) in Theorem 4, we can show that every factor module of $_{eRe}eR$ is cogenerated by eR. Then, since $_{eRe}eR$ is a finitely generated projective generator, every cyclic left eRe-module is torsionless. By symmetry, every cyclic right eRe-module is torsionless. Thus eRe is a dual ring by Proposition 1.

Example 9.   The condition that $ReR = R$ in Theorem 8 is necessary. Let R be the algebra of matricies, over a field k, of the form

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}.$$

It is easy to see that R is a QF ring. Thus R is a dual ring. Set $e = e_{11} + e_{22} + e_{44} + e_{55} \in R$, where e_{ij} are matrix units. Then e is an idempotent of R such that eRe is isomorphic to the ring of 2×2 upper triangular matrices over k and that $ReR \neq R$. We can easily verify that eRe is not QF. Therefore R is a dual ring, but it has an idempotent e such that eRe is not a dual ring.

By [2, Theorem 3.9], dual rings are semiperfect. Thus dual ring has a basic idempotent.

Corollary 10.   *Let R be a dual ring and e its basic idempotent. Then the basic ring eRe is a dual ring.*

Example 11.   Let T and S be the rings as in Example 7. Since T is Morita equivalent to S and S is commutative, the basic ring of T is isomorphic to S. Thus the converse of Corollary 10 does not hold.

REFERENCES

[1]   F. W. ANDERSON and K. R. FULLER: Rings and Categories of Modules 2nd ed., Springer-verlag, Berlin, Heidelberg and New York. 1992.

[2] C. R. HAJANAVIS and N. C. NORTON: On dual rings and their modules, J. Algebra
       93 (1985), 253–266.
[3] M. HOSHINO: On Lambek torsion theories III, preprint.
[4] Y. KURATA and K. HASHIMOTO: On dual-bimodules, Tsukuba J. Math. **16** (1992),
       85–105.
[5] J. L. GOMEZ PARDO and P. A. GUIL ASENSIO: Morita dualities associated with
       the R-dual functors, preprint.
[6] F. L. SANDOMIERSKI: Linearly compact modules and local Morita duality. Proc.
       Conf. Ring Theory Utah 1971, Academic Press, London and New York, 1972,
       333–346.
[7] B. STENSTROM: Rings of Quotients, Springer-Verlag, Berlin, Heidelberg and New
       York, 1975.
[8] W. XUE: Rings with Morita Duality, Lecture Notes in Math. 1523, Springer-
       Verlag, Berlin, Heidelberg and New York, 1992.

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Note added in proof.   After the submission of this manuscript, the author has found the following paper, which is related to our present work: [a] T. KATO: Torsionless modules, Tohoku Math. J. **20** (1968), 234–243. It follows from [a, Corollary] that the conditions (2) and (3) in Theorem 5 can be weakened.