Math. J. Okayama Univ. **37**(1995), 91–98

DUAL-BIMODULES AND FINITELY COGENERATED MODULES

In memory of Professor Hisao Tominaga

YOSHIKI KURATA and KAZUTOSHI KOIKE

Let R and S be rings with identity and ${}_RQ_S$ an (R,S)-bimodule. We shall call Q a left dual-bimodule provided that $\ell_R r_Q(A) = A$ for every left ideal A of R and $r_Q \ell_R(Q') = Q'$ for every S-submodule Q' of Q (see [5]).

In this note, first we shall show that a left dual-bimodule ${}_RQ_S$ defines a duality between the finitely generated left ideals of R and the finitely cogenerated factor modules of Q_S. Then, as an application of this duality, we shall give necessary and sufficient conditions for R to be left semihereditary or left coherent.

For notations and definitions we shall follow [1] and [5].

1. Let R and S be rings with identity and ${}_RQ_S$ an (R,S)-bimodule. Suppose that Q_S is quasi-injective and the natural homomorphism λ: $R \to \mathrm{End}(Q_S)$ is an isomorphism. Then by [7, Theorem 2.1], for each S-module N_S, there is a bijection between the finitely generated submodules of the Q-dual ${}_RN^*$ of N and the finitely closed submodules of N_S given by

$$L \longrightarrow r_N(L)$$

with the inverse $K \to \ell_{N^*}(K)$. Here, a submodule K of N_S is said to be finitely closed (with respect to Q_S) if there exists an integer $m > 0$ such that

$$0 \longrightarrow N/K \longrightarrow Q^m$$

is exact, or equivalently, there exist $f_1, f_2, \ldots, f_m$ in N^* such that

$$\bigcap_{j=1}^{m} \mathrm{Ker} f_j = K.$$

In case Q_S is finitely cogenerated, K finitely closed means that N/K is finitely cogenerated Q-torsionless.

Using this theorem, Miller and Turnidge pointed out that, under the same assumption as above, R is left Noetherian (right perfect) if and only if Q_S has DCC (ACC) on finitely closed submodules.

If, in particular, ${}_RQ_S$ is a left dual-bimodule with Q_S quasi-injective and λ surjective, then the bijection yields one between the finitely generated left ideals of R and the finitely closed submodules of Q_S given by

$$A \longrightarrow r_Q(A)$$

with the inverse $Q' \to \ell_R(Q')$. Hence, in this case, R is left Noetherian (right perfect) if and only if Q_S has DCC (ACC) on the submodules $r_Q(A)$ of Q_S with A a finitely generated left ideal of R and R is regular if and only if every submodule of Q_S of the above form is a direct summand of Q_S (cf. [4, Proposition 4.2 and Theorem 4.3]). On the other hand, since the mapping $Ra \to r_Q(a)$ is a bijection between the principal left ideals of R and the submodules $r_Q(a)$ of Q_S with a in R, it follows that R is right perfect if and only if Q_S has ACC on the submodules $r_Q(a)$ of Q_S with a in R and that R is regular if and only if every submodule of Q_S of the last form is a direct summand of Q_S (cf. [4, Theorem 3.1 and Proposition 4.1]).

2. For an (R,S)-bimodule ${}_RQ_S$, as was shown in [5, Theorem 3.3], if Q_S is quasi-injective and λ is surjective, then the pair (H', H'') of functors

$$H' = \mathrm{Hom}_R(-,Q) : {}_R\underline{\mathrm{M}} \longrightarrow \underline{\mathrm{N}}_S,$$
$$H'' = \mathrm{Hom}_S(-,Q) : \underline{\mathrm{N}}_S \longrightarrow {}_R\underline{\mathrm{M}}$$

defines a duality between the full subcategory ${}_R\underline{\mathrm{M}}$ of R-mod of finitely generated Q-torsionless R-modules and the full subcategory $\underline{\mathrm{N}}_S$ of mod-S whose objects are all the S-modules N_S such that

$$0 \longrightarrow N \longrightarrow Q^n \longrightarrow Q^I$$

is exact for some integer $n > 0$ and a set I. Assume further that Q_S is finitely cogenerated, then by [5, Proposition 3.4]

$${}_R\underline{\mathrm{M}} = \{{}_RM \mid M \text{ is finitely generated and } Q\text{-reflexive}\}$$

and

$$\underline{\mathrm{N}}_S = \{N_S \mid N \text{ is finitely cogenerated and } Q\text{-reflexive}\}.$$

If, in addition, Q_S is a self-cogenerator, then by [6, Proposition 4]

$$\underline{\mathrm{N}}_S = \{N_S \mid 0 \to N \to Q^n \text{ is exact for some } n > 0\}.$$

Using the bijection in Section 1, we shall now show that (H', H'') defines a duality between more restricted subcategories of ${}_R\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}_S$.

Theorem 1. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then (H', H'') defines a duality between the finitely generated left ideals of R and the finitely cogenerated factor modules of Q_S.*

Proof. Let A be a finitely generated left ideal of R. Then A belongs to ${}_R\underline{\mathrm{M}}$ and A^* is isomorphic to a finitely cogenerated factor module $Q/r_Q(A)$ of Q_S by [5, Lemma 1.13]. On the other hand, for each finitely cogenerated factor module Q/Q' of Q_S, Q' is finitely closed and hence $\ell_R(Q')$ is finitely generated and is Q-reflexive. Again by [5, Lemma 1.13], $Q/Q' \cong \ell_R(Q')^*$. Thus, Q/Q' is in $\underline{\mathrm{N}}_S$ and $(Q/Q')^*$ is isomorphic to $\ell_R(Q')$.

Corollary 2. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. If R is left Noetherian, then (H', H'') defines a duality between the left ideals of R and the factor modules of Q_S.*

In contrast with Corollary 2, (H', H'') always defines a duality between the factor modules of ${}_RR$ and the submodules of Q_S under the same assumption of Corollary 2. Indeed, for each left ideal A of R, R/A is Q-reflexive by [5, Theorem 3.2] and $(R/A)^* \cong r_Q(A)$. On the other hand, for each submodule Q' of Q_S, $Q' = r_Q(\ell_R(Q')) \cong (R/\ell_R(Q'))^*$. Hence, Q' is Q-reflexive by [1, Proposition 20.14] and $Q'^* \cong R/\ell_R(Q')$.

If ${}_RQ_S$ is a dual-bimodule with both ${}_RQ$ and Q_S injective, then using [1, Exercise 23.7] (H', H'') defines a duality between the left R-modules of finite length and the right S-modules of finite length by [5, Theorem 2.1]. However, we have

Theorem 3. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then (H', H'') defines a duality between the Q-torsionless left R-modules of finite length and the Q-torsionless right S-modules of finite length.*

Proof. Let ${}_RM$ be a Q-torsionless R-module of finite length and let

$$M = M_0 > M_1 > \cdots > M_n = 0$$

be a composition series of M. Then

$$0 = r_{M^*}(M_0) \leq r_{M^*}(M_1) \leq \cdots \leq r_{M^*}(M_n) = M^*$$

is a series of S-submodules of M^*, where $r_{M^*}(M_i) = \{f\colon M \to Q \mid M_i \leq \operatorname{Ker} f\}$ (see [1, p.281]). For each i, each element of $r_{M^*}(M_{i+1})$ induces an R-homomorphism from M_i/M_{i+1} to Q and hence $r_{M^*}(M_{i+1})/r_{M^*}(M_i)$ can be seen as an S-submodule of $(M_i/M_{i+1})^*$. Since M_i/M_{i+1} is simple, $(M_i/M_{i+1})^*$ is isomorphic to a simple submodule of Q_S, as is seen from the proof of [5, Theorem 2.1]. Hence, $r_{M^*}(M_{i+1})/r_{M^*}(M_i)$ is zero or simple. Thus, M_S^* is a module of finite length and $c(M^*) \leq c(M)$, where $c(\text{-})$ denotes the composition length. Furthermore, by [1, Proposition 20.14], M_S^* is Q-torsionless.

Using [1, Exercise 16.18], for a Q-torsionless S-module N_S of finite length, ${}_RN^*$ is a Q-torsionless R-module of finite length and $c(N^*) \leq c(N)$ holds.

Clearly each Q-torsionless R-module ${}_RM$ of finite length is Q-reflexive and we have $c(M) = c(M^*)$. On the other hand, each Q-torsionless S-module N_S of finite length is finitely cogenerated. Hence it is Q-reflexive. Thus, we have $c(N) = c(N^*)$.

Corollary 4. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then (H', H'') defines a duality between the simple left R-modules and the Q-torsionless simple right S-modules.*

In case ${}_RQ_S$ is a dual-bimodule, however, (H', H'') defines a duality between the simple left R-modules and the simple right S-modules, as is seen from [5, Theorem 2.1].

3. It is shown by [5, Proposition 1.12] that for a left dual-bimodule ${}_RQ_S$, R is semisimple if and only if Q_S is semisimple. On the other hand, we have

Theorem 5. *Let ${}_RQ_S$ be a left dual-bimodule with λ surjective. Then R is simple Artinian if and only if $Q_S \cong Q_1^n$ for some integer $n > 0$ and some simple right S-module Q_1.*

Proof. Suppose that R is simple Artinian. Then Q_S is semisimple and is finitely generated by [5, Proposition 1.8]. Let Q_1 be a simple submodule of Q_S. Then $\ell_R(RQ_1)$ is a proper ideal of R and hence it must be zero by assumption. Therefore, $RQ_1 = r_Q\ell_R(RQ_1) = Q$. However, $RQ_1 = \sum_{a\in R} aQ_1$ and each aQ_1 is either zero or isomorphic to Q_1. Thus we have $Q \cong Q_1^n$ for some integer $n > 0$.

Conversely, suppose that $Q_S \cong Q_1^n$ for some integer $n > 0$ and some simple right S-module Q_1. Then since $R \cong^{\lambda} \mathrm{End}(Q_S)$, R is isomorphic to the ring of all $n \times n$ matrices over the division ring $\mathrm{End}(Q_1)$. Thus, R is simple Artinian.

Now, using Theorem 1, we shall give a necessary and sufficient condition for R to be left semihereditary (cf. [2, Corollary 2.4] and [8, Proposition 2.1]).

Theorem 6. *Let ${}_RQ_S$ be a left dual-bimodule with Q_S quasi-injective and λ surjective. Then the following conditions are equivalent:*

(1) *R is left semihereditary.*

(2) *Every finitely cogenerated factor module of Q_S is Q-injective.*

(3) *For every finitely generated left ideal A of R, A^* is Q-injective.*

Proof. Let A be a finitely generated left ideal of R and let $R^n \to A \to 0$ be exact for some integer $n > 0$. Then the sequence

$$0 \longrightarrow A^* \longrightarrow Q^n \qquad (*)$$

is also exact. Since A is Q-reflexive and $R \cong^{\lambda} \mathrm{End}(Q_S)$, A is projective if and only if $(*)$ is split exact and this is so if and only if A^* is Q-injective. Thus, (1) and (3) are equivalent. By Theorem 1, (2) and (3) are also equivalent.

Theorem 7. *For a dual ring R the following conditions are equivalent:*

(1) *R is left semihereditary.*

(2) *R is semisimple.*

Proof. (1)$\Longrightarrow$(2). Suppose that R is left semihereditary. Since $R/\mathrm{rad}(R)$ is semisimple by [5, Theorem 1.10], $0 \to R/\mathrm{rad}(R) \to \mathrm{soc}(R)^n$ is split exact for some integer $n > 0$. By [5, Proposition 1.8] $\mathrm{soc}(R)$ is projective. Hence, $R/\mathrm{rad}(R)$ is also projective. Thus, $\mathrm{rad}(R)$ must be a direct summand of R, from which it follows that $\mathrm{rad}(R) = 0$ and R is semisimple. (2)$\Longrightarrow$(1) is trivial.

As is easily seen, a ring R is left coherent if and only if for every integer $n > 0$ and every R-homomorphism $f\colon {}_RR^n \to {}_RR$ there exist an integer $m > 0$ and an R-homomorphism $g\colon {}_RR^m \to {}_RR^n$ such that

$$R^m \xrightarrow{g} R^n \xrightarrow{f} R$$

is exact. For a left dual-bimodule ${}_RQ_S$, using Q_S instead of R, a similar characterization for R to be left coherent can be obtained (cf. [2, Theorem 2.6 and Corollary 2.7]).

Theorem 8. *For a left dual-bimodule ${}_RQ_S$ with Q_S quasi-injective and λ surjective, the following conditions are equivalent:*

(1) *R is left coherent.*

(2) *For every finitely cogenerated factor module Q/Q' of Q_S, there exist integers $n, m > 0$ such that*

$$0 \longrightarrow Q/Q' \longrightarrow Q^n \longrightarrow Q^m$$

is exact.

(3) *For every finitely generated left ideal A of R, there exist integers $n, m > 0$ such that*

$$0 \longrightarrow A^* \longrightarrow Q^n \longrightarrow Q^m$$

is exact.

(4) *For every integer $n > 0$ and every S-homomorphism $f: Q \to Q^n$ there exist an integer $m > 0$ and an S-homomorphism $g: Q^n \to Q^m$ such that*

$$Q \xrightarrow{f} Q^n \xrightarrow{g} Q^m$$

is exact.

Proof. It is easy to see that (1), (2) and (3) are equivalent.

(1)$\Longrightarrow$(4). Assume (1) and let $f: Q \to Q^n$ be an S-homomorphism. Then $0 \to Q/K \xrightarrow{\bar{f}} Q^n$ is exact, where $K = \operatorname{Ker} f$ and $\bar{f}$ is the homomorphism induced by f. Hence Q/K is finitely cogenerated Q-reflexive. By Theorem 1, $(Q/K)^*$ is a finitely generated left ideal of R and $R^n \to (Q/K)^* \to 0$ is exact. Since R is left coherent, there exists an integer $m > 0$ such that $R^m \to R^n \to (Q/K)^* \to 0$ is exact. Thus, $0 \to Q/K \xrightarrow{\bar{f}} Q^n \xrightarrow{g} Q^m$ is exact for some S-homomorphism g, which shows that

$$Q \xrightarrow{f} Q^n \xrightarrow{g} Q^m$$

is exact.

(4)$\Longrightarrow$(2). Assume (4) and let Q/Q' be any finitely cogenerated factor module of Q_S. Then Q' is finitely closed and hence there exists an integer $n > 0$ such that $0 \to Q/Q' \xrightarrow{f} Q^n$ is exact for some S-homomorphism f. Let $\pi: Q \to Q/Q'$ be the canonical epimorphism. Then by (4) there exist an integer $m > 0$ and an S-homomorphism g such that $Q \xrightarrow{f\pi} Q^n \xrightarrow{g} Q^m$ is exact and thus so is $0 \to Q/Q' \xrightarrow{f} Q^n \xrightarrow{g} Q^m$.

It is to be noted that if R is a dual ring with R_R injective, then the bimodule ${}_RR_R$ defines a Morita duality by [1, Exercise 24.10] and [6, Corollary 6]. However, this is not the case for a left dual-bimodule in general. For example, let $R = Q = Z/(p)$, p a prime number, and $S = Z$. Then the bimodule ${}_RQ_S$ is a left dual-bimodule with Q_S quasi-injective and λ surjective, but does not define any Morita duality.

For this left dual-bimodule, R is left Noetherian, right perfect and is also regular. Furthermore, it is left semihereditary and left coherent, too.

References

[1] F. W. Anderson and K. R. Fuller: Rings and Categories of Modules, Springer-Verlag, New York Heidelberg Berlin, 1973.

[2] J. L. Garcia Hernandez and J. L. Gomez Pardo: Closed submodules of free modules over the endomorphism ring of a quasi-injective module, Comm. Algebra **16** (1988), 115–137.

[3] C. R. Hajarnavis and N. C.Norton: On dual rings and their modules, J. Algebra **93** (1985), 253–266.

[4] S. M. Khuri: Modules with regular, perfect, noetherian and artinian endomorphism rings, LN in Math. **1448**, Springer-Verlag, New York Heidelberg Berlin, 1990, 7–18.

[5] Y. Kurata and K. Hashimoto: On dual-bimodules, Tsukuba J. Math. **16** (1992), 85–105.

[6] Y. Kurata and S. Tsuboi: On linearly compact dual-bimodules, Math. J. Okayama Univ. **33** (1991), 149–154.

[7] R. W. Miller and D. R. Turnidge: Co-Artinian rings and Morita duality, Israel J. Math. **15** (1973), 12–26.

[8] R. W. Miller and D. R. Turnidge: Factors of cofinitely generated injective modules, Comm. Algebra **4** (1976), 233–243.

Y. Kurata
Department of Information Science
Kanagawa University
Tsuchiya, Hiratsuka-shi 259-12, Japan

K. Koike
Department of Computer Science
Ube College
Bunkyocho, Ube-shi 755, Japan

(*Received May 20, 1994*)