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PRIME IDEALS IN POLYNOMIAL RINGS OVER TAME ORDERS AND HEREDITARY PI-RINGS

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For a given hereditary prime PI-ring Λ and a central polynomial $f(x)$ in $\Lambda[x]$ with $f(x)\Lambda[x]$ a prime ideal in $\Lambda[x]$, it was proved in [8] and [13] that the prime factor ring $\Lambda[x]/f(x)\Lambda[x]$ is hereditary if and only if $f(x)$ is not contained in the square of any maximal ideal of $\Lambda[x]$, which is a generalization of a main result in [7].

From this result together with the fact that hereditary prime PI-rings can be tame orders by [17], first in Section 1, we investigate, in Theorem A, a condition of prime factor rings of the polynomial ring $\Lambda[x]$ to be tame orders whenever Λ is a tame order. As applications, when D is a Krull domain we give a criterion for a certain class of prime factor rings of $D[x]$ to be a Krull domain.

Furthermore, in Section 2, when Λ is a hereditary PI-ring, in Theorem B, we give a criterion for prime factor rings of $\Lambda[x]$ to be hereditary, which is a nontrivial extension of the main result in [13, Theorem] and [14, Proposition 1], thereby we can provide an answer to a question of Armendariz [1], i.e., "a characterization of prime ideals P of the polynomial ring $\Lambda[x]$ over a hereditary PI-ring Λ such that $\Lambda[x]/P$ is hereditary", which was raised from a result in [2]: every prime factor ring of a hereditary PI-ring is hereditary.

1. Prime ideals in polynomial rings over tame orders. Let Λ be an order in a simple Artinian ring Q. For a Λ-ideal A in Q, we will use the notation;

$$\begin{aligned} O_\ell(A) &= \{q \in Q \mid qA \subseteq A\}, \\ O_r(A) &= \{q \in Q \mid Aq \subseteq A\}, \\ (\Lambda : A)_\ell &= \{q \in Q \mid qA \subseteq \Lambda\}, \\ (\Lambda : A)_r &= \{q \in Q \mid Aq \subseteq \Lambda\}, \end{aligned}$$
$$A_v = (\Lambda : (\Lambda : A)_\ell)_r \quad \text{and} \quad {}_vA = (\Lambda : (\Lambda : A)_r)_\ell.$$

Clearly A_v and ${}_vA$ are again Λ-ideals containing A. We say that A is a *v-ideal* if ${}_vA = A = A_v$. A v-ideal A is *v-invertible* if $(A(\Lambda : A)_r)_v =$

$\Lambda = {}_v((\Lambda : A)_\ell A)$. As it is well known, a Λ-ideal A is projective as left and right Λ-modules if and only if $A(\Lambda : A)_\ell = O_\ell(A)$ and $(\Lambda : A)_r A = O_r(A)$.

The following is a Krull type generalization of projectivities:

(K): ${}_v(A(\Lambda : A)_\ell) = O_\ell(A)$ *for any* Λ*-ideal* A *such that* $A = {}_vA$, *that is,* A *is reflexive as a left* Λ*-module, and* $((\Lambda : A)_r A)_v = O_r(A)$ *for any* Λ*-ideal* A *such that* $A = A_v$.

An order Λ is said to be *v-hereditary* (simply, a VH-*order*) if Λ satisfies the condition (K).

For a right Λ-module X, we denote by $E_\Lambda(X)$ (simply, $E(X)$) the injective hull of X. Let $\mathcal{C} = \mathcal{C}(\Lambda)$ be the right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/\Lambda)$. Then $\mathcal{C} = \{C$: a right ideal of $\Lambda \mid (\Lambda : r^{-1}C)_\ell = \Lambda$ for any $r \in \Lambda\}$ by Proposition 5.5 of [18, p.147], where $r^{-1}C = \{\lambda \in \Lambda \mid r\lambda \in C\}$. Similarly, we can define the left Gabriel topology $\mathcal{C}'$ on Λ. Let I be any right Λ-ideal (or a right ideal of Λ). We put $\widetilde{I} = \{q \in Q \mid qC \subseteq I$ for some $C \in \mathcal{C}\}$. I is said to be $\mathcal{C}$*-closed* if $I = \widetilde{I}$. If I is a right ideal of Λ, then we note that $\widetilde{I} = \{\lambda \in \Lambda \mid \lambda C \subseteq I$ for some $C \in \mathcal{C}\}$. Also left $\mathcal{C}'$-closed ideals can be defined similarly.

In [3], Chamarie has considered the following condition to get the classical localization Λ_A of Λ at a v-ideal A in case Λ is a maximal order.

(C): Λ *satisfies the maximum condition on right* $\mathcal{C}$*-closed ideals of* Λ *and left* $\mathcal{C}'$*-closed ideals of* Λ.

Following [9], Λ is called a VHC-*order* if it is a VH-order and satisfies the condition (C).

Through this section, *we assume from now on that* D *is a Krull domain which is not a field*; K is the field of fractions of D. Let Λ be a tame D-order in a central simple K-algebra Q with finite dimension over K (see [4]). We denote by $D(\Lambda)$ the set of all v-invertible ideals in Q and by $P(\Lambda)$ the subset of $D(\Lambda)$ consisting of all principal Λ-ideals.

Now we summarize some properties of tame orders which will be necessary.

(i) Λ is a VHC-order with enough v-invertible ideals in the sense of [10] (see [9, Proposition 3.1] and [12, Proposition 1.1]).

(ii) $D(\Lambda)$ is a free abelian group generated by maximal v-invertible ideals (see [9, Theorem 1.13]).

(iii) $\Lambda[x]$ is a tame $D[x]$-order (see [4, Theorem 1.11]).

(iv) $D(\Lambda)/P(\Lambda)$ is naturally group isomorphic to $D(\Lambda[x])/P(\Lambda[x])$ (see [10, Theorem 2.19]).

(v) The set of all maximal v-invertible ideals of $\Lambda[x]$ is $\{B[x], A \mid B$ is a maximal v-invertible ideal of Λ and $A = A' \cap \Lambda[x]$ for some maximal ideal A' of $Q[x]\}$ (see [9, Theorem 3.9]).

Also throughout this section P is always a non-zero prime ideal of $\Lambda[x]$ with $P \cap \Lambda = 0$. Then in this case we have the following:

(vi) $PQ[x] \cap \Lambda[x] = P$ and $PQ[x]$ is also a prime ideal.

(vii) P is a v-ideal.

(viii) P is a minimal non-zero prime ideal of $\Lambda[x]$.

The proof of (vi) is straightforward. Since $Q[x]$ is a principal ideal ring [16], any ideal of $Q[x]$ is a v-ideal. So (vii) follows from [9, Lemma 2.3] and (vi). (viii) follows from (vi), because any non-zero prime ideal of $Q[x]$ is maximal.

We begin with local case.

Lemma 1. *Assume that D is a discrete rank one valuation domain and that Λ is hereditary. Let P be a non-zero prime ideal of $\Lambda[x]$ with $P \cap \Lambda = 0$. Then there exists $f(x) \in D[x]$ such that $P = f(x)\Lambda[x]$.*

Proof. Let $\wp = P \cap D[x]$, a minimal non-zero prime ideal of the Krull domain $D[x]$. Then, by Nagata's theorem, there exists $f(x) \in D[x]$ such that $\wp = f(x)D[x]$ (see (iv)). Now it is clear that $f(x)K[x]$ is a prime ideal with $f(x)K[x] \cap D[x] = \wp$, and K is embedded in $K[x]/f(x)K[x]$, which is a field. So it follows from [15, Theorem 7.6] that $Q[x]/f(x)Q[x]$ ($\cong Q \otimes_K K[x]/f(x)K[x]$) is a simple Artinian ring. Hence $f(x)Q[x]$ is a maximal ideal contained in $P' = PQ[x]$ and so $P' = f(x)Q[x]$. Since $f(x)\Lambda[x]$ is invertible and $f(x)\Lambda[x] \subseteq P$, we have

$$f(x)\Lambda[x] = P^n \cdot P_1^{n_1} \cdots P_k^{n_k} \cdot J^m[x],$$

where P_i is a prime ideal such that $P_i = P_i' \cap \Lambda[x]$ for some maximal ideal P_i' of $Q[x]$, and $J = J(\Lambda)$, the Jacobson radical of Λ. Note that J is the unique maximal invertible ideal of Λ, because D is a discrete rank one valuation domain. Then it follows that

$$P' = P'^n \cdot P'^{n_1}_1 \cdots P'^{n_k}_k,$$

and so we have $k = 0$ and $n = 1$. Now to prove that $m = 0$, assume to the contrary that $m \geq 1$, then $f(x) \in J[x] \cap D[x] = J(D)[x]$. In this case $J(D)$ is the unique maximal ideal of D. Thus we have $f(x)D[x] \subseteq J(D)[x]$, a minimal non-zero prime ideal of $D[x]$. So $f(x)D[x] = J(D)[x]$ and this implies that $f(x)K[x] = K[x]$. This is a contradiction and so $m = 0$, that is, $P = f(x)\Lambda[x]$. This completes the proof.

Let p be a minimal non-zero prime ideal of D. Then Λ_p is hereditary over the discrete rank one valuation domain D_p. So by Lemma 1, there exists $f_p(x) \in D_p[x]$ such that $\wp_p = f_p(x)D_p[x]$ and $P_p = f_p(x)\Lambda_p[x]$.

Lemma 2. *Let M be a prime ideal of $\Lambda[x]$ such that $M \cap (D \setminus p) = \emptyset$ for some minimal non-zero prime ideal p of D. Then* $\mathrm{rank}(M) \leq 2$ *and* $\mathrm{rank}(M \cap \Lambda) \leq 1$.

Proof. $\mathrm{rank}(M) \leq 2$ follows from the fact that the classical Krull dimension of $\Lambda_p[x]$ equals to 2. To prove that $\mathrm{rank}(M \cap \Lambda) \leq 1$, let $\mathcal{M} = M \cap \Lambda$ and suppose that $\mathcal{M}$ is not a minimal non-zero prime ideal. Then $M = \mathcal{M}[x]$, because

$$0 \subsetneq \mathcal{M}_1[x] \subsetneq \mathcal{M}[x] \subseteq M,$$

where $\mathcal{M}_1$ is a non-zero prime ideal strictly contained in $\mathcal{M}$. However, since $\mathcal{M} \not\subseteq p$, we have $M_p = \mathcal{M}_p[x] = \Lambda_p[x]$ and so $M \cap (D \setminus p) \neq \emptyset$, a contradiction. Hence $\mathrm{rank}(\mathcal{M}) \leq 1$.

For a prime ideal M of $\Lambda[x]$ we define $M^{(2)}$ as follows:

If $M \cap \Lambda = 0$, then $M^{(2)} = M^2$. If $M \cap \Lambda \neq 0$, then $m = M \cap D$ is a non-zero prime ideal and

$$M^{(2)} = \{\lambda(x) \in \Lambda[x] \mid \lambda(x)c \in M^2 \text{ for some } c \in D \setminus m\}.$$

Note that $M^{(2)} = (M_m)^2 \cap \Lambda[x]$.

Lemma 3. *The following are equivalent:*

(1) *$f_p(x) \notin (M_p)^2$ for any prime ideal M of $\Lambda[x]$ with $M \cap (D \setminus p) = \emptyset$, where p is a minimal non-zero prime ideal of D.*

(2) *$P \not\subseteq M^{(2)}$ for any prime ideal M of $\Lambda[x]$ with* $\mathrm{rank}(M \cap \Lambda) \leq 1$.

Proof. (1) implies (2). Assume to the contrary that $P \subseteq M^{(2)}$ for some prime ideal M of $\Lambda[x]$ with $\mathrm{rank}(M \cap \Lambda) \leq 1$.

Case I. If $\mathcal{M} = M \cap \Lambda = 0$, then $M = MQ[x] \cap \Lambda[x]$ and so M is a prime v-ideal which is v-invertible by [9, Lemma 3.6]. Furthermore, M is a minimal non-zero prime ideal of $\Lambda[x]$ by [11, Lemma 1.8]. If $M = P$, then, since $M \cap (D \setminus p) = \emptyset$ for any minimal non-zero prime ideal p of D, we have $P_p = f_p(x)\Lambda_p[x] = M_p \subseteq (M^{(2)})_p = (M_p)^2$, a contradiction. If $M \neq P$, then, of course, we have $P \not\subseteq M^{(2)}$, a contradiction.

Case II. If $\mathcal{M} = M \cap \Lambda$ is non-zero, then $\mathcal{M}$ is a minimal non-zero prime ideal of Λ and $m = M \cap D$ is also a minimal non-zero prime ideal of D with $M \cap (D \setminus m) = \emptyset$. So $f_m(x)\Lambda_m[x] = P_m \subseteq (M^{(2)})_m = (M_m)^2$, a contradiction.

Therefore by Cases I and II, $P \not\subseteq M^{(2)}$ for any prime ideal M of $\Lambda[x]$ with $\text{rank}(M \cap \Lambda) \leq 1$.

(2) implies (1). Suppose that $f_p(x) \in (M_p)^2$ for some prime ideal M of $\Lambda[x]$ with $M \cap (D \setminus p) = \emptyset$, where p is a minimal non-zero prime ideal of D. By Lemma 2, $\text{rank}(M \cap \Lambda) \leq 1$. Let $m = M \cap D$. Then either $m = 0$ or m is a minimal non-zero prime ideal. If $m = 0$, then M is a minimal non-zero prime ideal. First of all, if $P = M$, then $f_p(x)\Lambda_p[x] = P_p \subseteq (P_p)^2$, a contradiction. If $P \neq M$, then $P_p \neq M_p$, because $P = P_p \cap \Lambda[x]$ and $M = M_p \cap \Lambda[x]$. Hence $f_p(x) \notin M_p$ and so $f_p(x) \notin (M_p)^2$, a contradiction. Now if m is a minimal non-zero prime ideal, then $M \cap (D \setminus p) = \emptyset$ implies that $m \subseteq p$ and thus $m = p$. Since $P_p = f_p(x)\Lambda_p[x] \subseteq (M_p)^2$, we have $P = P_p \cap \Lambda[x] \subseteq (M_p)^2 \cap \Lambda[x] = M^{(2)}$, a contradiction.

Therefore $f_p(x) \notin (M_p)^2$ for any prime ideal M of $\Lambda[x]$ with $M \cap (D \setminus p) = \emptyset$, where p is a minimal non-zero prime ideal of D.

Lemma 4. *Let p be a minimal non-zero prime ideal of D. Then $(\Lambda[x]/P)_p = \Lambda_p[x]/P_p$ is hereditary if and only if $f_p(x) \notin (M_p)^2$ for any prime ideal M of $\Lambda[x]$ with $M \cap (D \setminus p) = \emptyset$.*

Proof. There is a one-to-one correspondence between prime ideals M of $\Lambda[x]$ with $M \cap (D \setminus p) = \emptyset$ and prime ideals M' of $\Lambda_p[x]$ corresponding M in $\Lambda[x]$ to $M' = M_p$ and M' in $\Lambda_p[x]$ to $M' \cap \Lambda[x]$. Hence the lemma follows from [13, Theorem].

Lemma 5. $\Lambda[x]/P = \bigcap_p(\Lambda_p[x]/P_p)$ *and* $D[x]/\wp = \bigcap_p(D_p[x]/\wp_p)$, *where p runs through all minimal non-zero prime ideals of D and $\wp = P \cap D[x]$.*

Proof. Since $\Lambda = \bigcap \Lambda_p$ and $D = \bigcap D_p$, it follows that $\Lambda[x] = \bigcap \Lambda_p[x]$ and $D[x] = \bigcap D_p[x]$. To prove that $P = \bigcap P_p$, let $z \in \bigcap P_p$. Then for any p there exists $c_p \in D \setminus p$ with $zc_p \in P$. Set $\mathcal{A} = \sum c_p\Lambda$. Then $\mathcal{A}_v = \Lambda$, because, on the contrary, assume that $\mathcal{A}_v \subsetneq \Lambda$, then there exists a maximal v-ideal P_0 with $\mathcal{A}_v \subseteq P_0$. Then $p_0 = P_0 \cap D$ is a minimal non-zero prime ideal of D and for this p_0, we have $c_{p_0} \in P_0 \cap D = p_0$, a contradiction. Now $z\mathcal{A}[x] \subseteq P$ implies that $z \in z\Lambda[x] = z\mathcal{A}_v[x] \subseteq (z\mathcal{A}[x])_v \subseteq P_v = P$. Hence $P = \bigcap P_p$ and similarly $\wp = \bigcap \wp_p$. Therefore $\Lambda[x]/P = \bigcap(\Lambda_p[x]/P_p)$ and $D[x]/\wp = \bigcap(D_p[x]/\wp_p)$.

Lemma 6. *Suppose that one of the conditions in Lemma 3 is satisfied. Then we have the following*:

(1) $D[x]/\wp$ *is a Krull domain and it is the center of* $\Lambda[x]/P$, *where* $\wp = P \cap D[x]$.

(2) $\Lambda[x]/P$ *is a tame* $D[x]/\wp$*-order.*

Proof. (1) Let p be a minimal non-zero prime ideal of D. Then Λ_p is hereditary with its center D_p and $P_p = f_p(x)\Lambda_p[x]$. So it follows from [14, Claim 4, p.1485] and Lemma 4 that $\Lambda_p[x]/P_p$ is hereditary with $Z(\Lambda_p[x]/P_p) = D_p[x]/\wp_p$, where $Z(-)$ denotes the center of a ring. In particular, $D_p[x]/\wp_p$ is a Dedekind domain. Thus $D[x]/\wp$ is a Krull domain by Lemma 5. Furthermore, we have

$$Z(\Lambda[x]/P) = \bigcap_p Z(\Lambda_p[x]/P_p) = \bigcap_p D_p[x]/\wp_p = D[x]/\wp$$

by Lemma 5.

(2) For any minimal non-zero prime ideal p, $\Lambda_p[x]/P_p$ is hereditary with $Z(\Lambda_p[x]/P_p) = D_p[x]/\wp_p$. So it follows that

$$\Lambda_p[x]/P_p = \bigcap(\Lambda_p[x]/P_p)_{q'(p)},$$

where $q'(p)$ ranges over all prime ideals of $D_p[x]/\wp_p$. There is a one-to-one correspondence between prime ideals $q'(p)$ of $D_p[x]/\wp_p$ and minimal non-zero prime ideals $q(p)$ of $D[x]/\wp$ with $q(p) \cap (D \setminus p) = \emptyset$ which is given by;

$$q'(p) \longrightarrow q(p) = q'(p) \cap (D[x]/\wp) \text{ and } q(p) \longrightarrow q(p)(D_p[x]/\wp_p).$$

Furthermore $(\Lambda[x]/P)_{q(p)} = (\Lambda_p[x]/P_p)_{q'(p)} \supseteq (D[x]/\wp)_{q(p)}$. Hence we have that

$$\Lambda[x]/P = \bigcap_p \bigcap_{q(p)} (\Lambda[x]/P)_{q(p)}$$

and $(\Lambda[x]/P)_{q(p)}$ is hereditary. To prove that $\Lambda[x]/P$ has the finite character property, let $\overline{\lambda(x)}$ be any regular element in $\overline{\Lambda[x]} = \Lambda[x]/P$. Since $\overline{\lambda(x)}\cdot\overline{\Lambda[x]}$ is an essential right ideal of $\overline{\Lambda[x]}$, we have $\overline{\lambda(x)}\cdot\overline{\Lambda[x]} \cap Z(\overline{\Lambda[x]}) \neq 0$.

Thus there exists $d(x) \in D[x]$ with $\overline{d(x)} \in \overline{\lambda(x)}\cdot\overline{\Lambda[x]}$. Since $\overline{D[x]} = D[x]/\wp$ is a Krull domain, there are only finite number of minimal non-zero prime ideals of $\overline{D[x]}$, say, $q_1, \dots, q_n$, (each $q_i = q(p)$ for some $q(p)$) such that $\overline{d(x)}\cdot\overline{D[x]}_{q_i} \subsetneq \overline{D[x]}_{q_i}$. Hence $\Lambda[x]/P$ satisfies the finite character property, because $\overline{\Lambda[x]}_{q(p)} \supseteq \overline{D[x]}_{q(p)}$ for each $q(p)$. Hence $\overline{\Lambda[x]}$ is a tame $\overline{D[x]}$-order by [5, Lemma 1.1] and [9, Proposition 3.1].

The following lemma is implicitly known. However, we could not find the place in which the proof of the lemma is given. So we give a complete proof for our convenience. We denote by *cl.K.dim* Λ the classical Krull dimension of Λ.

Lemma 7. *Let Λ be a Noetherian tame D-order with* cl.K.dim $\Lambda \leq 1$. *Then Λ is hereditary.*

Proof. If cl.K.dim $\Lambda = 0$, then Λ is a simple Artinian ring and so it is hereditary. If cl.K.dim $\Lambda = 1$, then any prime ideal M of Λ is a prime v-ideal by [11, Lemma 1.8]. Furthermore, $\Lambda = \bigcap \Lambda_p$, where p runs through all minimal non-zero prime ideals of D and Λ_p is hereditary. So we have

$$1 \in \Lambda_p \cap O_r(M_p) = \Lambda_p \cap (\Lambda_p : M_p)_r M_p = (\Lambda \cap (\Lambda : M)_r M)\Lambda_p,$$

where $O_r(M_p) = \{q \in Q \mid M_p q \subseteq M_p\}$. Thus

$$(\Lambda \cap (\Lambda : M)_r M)_v = \bigcap_p (\Lambda \cap (\Lambda : M)_r M)\Lambda_p = \Lambda.$$

This implies that $M \subsetneq \Lambda \cap (\Lambda : M)_r M$ and so $\Lambda \cap (\Lambda : M)_r M = \Lambda$, i.e., $1 \in (\Lambda : M)_r M$. Hence it follows that M is left projective and similarly right projective. Then the lemma follows from the same method as in [6, Proposition 1.3].

Now from all lemmas prepared, we have one of our main results of this note as follows;

Theorem A. *Let D be a Krull domain with the field of quotients K and let Λ be a tame D-order in a central simple K-algebra Q with finite dimension over K. Let P be a prime ideal of $\Lambda[x]$ with $P \cap \Lambda = 0$. Then $\Lambda[x]/P$ is a tame order if and only if $P \not\subseteq M^{(2)}$ for any prime ideal M of $\Lambda[x]$ with* $\operatorname{rank}(M \cap \Lambda) \leq 1$. *Furthermore, under these conditions, the center $Z(\Lambda[x]/P)$ of $\Lambda[x]/P$ is $D[x]/\wp$, where $\wp = P \cap D[x]$.*

Let M be a prime ideal of $\Lambda[x]$ with $\operatorname{rank}(M \cap \Lambda) \leq 1$ and $P \not\subseteq M$. Then it is easily checked that $P \not\subseteq M^{(2)}$. Thus we have the following:

Remark. $\Lambda[x]/P$ is a tame order if and only if $P \not\subseteq M^{(2)}$ for any prime ideal M of $\Lambda[x]$ satisfying both $\operatorname{rank}(M \cap \Lambda) \leq 1$ and $P \subsetneq M$.

Let $f(x)$ be any polynomial in $D[x]$ with $\deg f(x) \geq 1$. Then $f(x)\Lambda[x] \cap \Lambda = 0$. So we have the following corollary which extends results in [7], [8] and [13].

Corollary 1. *Let $f(x)$ be any polynomial in $D[x]$ such that $P = f(x)\Lambda[x]$ is a prime ideal of $\Lambda[x]$ and deg $f(x) \geq 1$. Then $\Lambda[x]/P$ is a tame order if and only if $f(x) \notin M^{(2)}$ for any prime ideal M of $\Lambda[x]$ with* $\operatorname{rank}(M \cap \Lambda) \leq 1$.

Noting that Krull domains are tame orders, we have following fact immediately, which is Krull domains version of a result in [7].

Corollary 2. *For a Krull domain D, let $f(x)$ be any polynomial in $D[x]$ such that $P = f(x)D[x]$ is a prime ideal and deg $f(x) \geq 1$. Then $D[x]/P$ is a Krull domain if and only if $f(x) \notin M^{(2)}$ for any prime ideal M of $D[x]$ with* $\operatorname{rank}(M \cap D) \leq 1$.

2. Prime ideals in polynomial rings over hereditary PI-rings. As we mentioned, in this section we consider hereditary prime factor rings of the polynomial ring $\Lambda[x]$ over a hereditary PI-ring Λ.

Lemma 8. *Let Λ be a prime hereditary PI-ring and let M be a prime ideal of $\Lambda[x]$ with $\mathcal{M} = M \cap \Lambda \neq 0$. Then* $\operatorname{rank}(\mathcal{M}) = 1$ *and* $M^{(2)} = M^2$.

Proof. It is clear that $\operatorname{rank}(\mathcal{M}) = 1$ and $\mathcal{M}[x] \subseteq M$. First assume that $M = \mathcal{M}[x]$ and let $\lambda(x) \in M^{(2)}$. Then there exists an element $c \in D$, but $c \notin M$ such that $\lambda(x)c \in M^2$ and so $\lambda(x) \in M$. Since $\mathcal{M}$ is a maximal ideal, we have $\mathcal{M} + c\Lambda = \Lambda$ and $\Lambda[x] = \mathcal{M}[x] + c\Lambda[x] = M + c\Lambda[x]$. Hence

$$\lambda(x) \in \lambda(x)\Lambda[x] = \lambda(x)(M + c\Lambda[x]) \subseteq M^2.$$

Next assume that $\mathcal{M}[x] \subsetneq M$, then M is a maximal ideal. So we have $M + c\Lambda[x] = \Lambda[x]$ and hence $\lambda(x) \in M^2$.

By Theorem A and Lemma 8, we answer to a question of Armendariz [1], which is a characterization of hereditary prime factor rings of polynomial rings over a hereditary PI-ring, in the following:

Theorem B. *Let Λ be a hereditary PI-ring and let P be a prime ideal of $\Lambda[x]$. Let $P_0 = P \cap \Lambda$, a prime ideal of Λ. Then we have the following:*

(1) *If $P = P_0[x]$, then $\Lambda[x]/P$ is hereditary if and only if P_0 is a maximal ideal of Λ.*

(2) *If $P_0[x] \subsetneq P$, then $\Lambda[x]/P$ is hereditary if and only if $P \not\subseteq M^2 + P_0[x]$ for any prime ideal M of $\Lambda[x]$ with $P \subsetneq M$.*

Proof. (1) If $P = P_0[x]$, then $\Lambda[x]/P \cong (\Lambda/P_0)[x]$ and Λ/P_0 is a prime hereditary by [2, Theorem]. Hence $(\Lambda/P_0)[x]$ is hereditary if and only if Λ/P_0 is a simple Artinian ring, that is, P_0 is a maximal ideal.

(2) Set $\bar{\Lambda} = \Lambda/P_0$ and consider the natural mapping f from $\Lambda[x]$ to $\bar{\Lambda}[x]$. We just write $f(P)$ by $\bar{P}$. Then $\bar{P}$ is a non-zero prime ideal with $\bar{P} \cap \bar{\Lambda} = \bar{0}$. Hence the result follows from the remark to Theorem A and Lemma 8.

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