Math. J. Okayama Univ. 37(1995), 55–58

A CHARACTERIZATION OF ANTI-INTEGRAL EXTENSIONS

JUNRO SATO, SUSUMU ODA and KEN-ICHI YOSHIDA

In this paper, we mean by a ring a commutative ring with identity and by an *integral domain* (or a *domain*) a ring which has no non-trivial zero-divisors. Our unexplained technical terms are standard and are seen in [1].

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be a non-zero element of an algebraic field extension L of the quotient field K of R and let $\pi: R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$

Then η_i $(1 \le i \le d)$ are uniquely determined by α. Let $I_{\eta_i} := R:_R \eta_i$ and $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$, the latter of which is called a *denominator ideal* of α. We say that α is an *anti-integral* element if and only if $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. The concept of anti-integralness is given in [2] in the birational case, and the higher degree case appears in [3]. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal of R generated by the coefficients of $f(X)$. For an ideal J of $R[X]$, let $C(J)$ denote the ideal generated by the coefficients of the elements in J. If α is an anti-integral element, then $C(\operatorname{Ker}\pi) = C(I_{[\alpha]}\varphi_\alpha(X)R[X]) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. Put $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$. If $J_{[\alpha]} \not\subseteq p$ for all $p \in Dp_1(R) := \{p \in \operatorname{Spec}(R) \mid \operatorname{depth} R_p = 1\}$, then α is called a *super-primitive* element. It is known that a super-primitive element is an anti-integral element (cf. [3,(1.12)]). By definition, the super-primitive is characterized by the set of $Dp_1(R)$. In this paper, we shall show that the anti-integrality is also characterized by the set $Dp_1(R)$. In fact, we prove the following:

Let R be a Noetherian domain with quotient field K and let α be an element of an algebraic field extension L of K. Then the following statements are equivalent:

(1) α is an anti-integral element over R,

(2) α is an anti-integral element over R_p for all $p \in Dp_1(R)$.

In what follows, we use the notation as above.

We start with the following theorem, which characterizes anti-integrality.

Theorem 1. *The following statements are equivalent:*

(1) *α is an anti-integral element of degree d over R,*

(2) *the ideal $I_{[\alpha]}\eta_d$ of R is generated by the set $\{g(0) \mid g(X) \in \operatorname{Ker}\pi\}$.*

Proof. (1) $\Longrightarrow$ (2): Let J be the ideal of R generated by the set $\{g(0) \mid g(X) \in \operatorname{Ker}\pi\}$. Since $I_{[\alpha]}\varphi_\alpha(X) \subseteq \operatorname{Ker}\pi$ and the constant term of $I_{[\alpha]}\varphi_\alpha(X)$ is $I_{[\alpha]}\eta_d$, it follows that $I_{[\alpha]}\eta_d \subseteq J$. Conversely take $a \in J$, and let

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a = 0$$

be a relation, where $a_i \in R$ and $a_n \neq 0$. Put $f(X) = a_nX^n + \cdots + a_1X + a$. Then $f(X) \in \operatorname{Ker}\pi$. Since α is an anti-integral element of degree d over R, we have $\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $f(X) = \sum h_i(X)g_i(X)$ for some $h_i(X) \in I_{[\alpha]}\varphi_\alpha(X)$ and $g_i(X) \in R[X]$. Thus $a = f(0) = \sum h_i(0)g_i(0) \in I_{[\alpha]}\eta_d$, as desired.

(2) $\Longrightarrow$ (1): Let $0 \neq f(X) \in \operatorname{Ker}\pi$ and write $f(X) = a_nX^n + \cdots + a_1X + a$. Since $[K(\alpha){:}K] = d$, we have $n \geq d$. By the assumption that $a \in J = I_{[\alpha]}\eta_d$, it follows that $a = b\eta_d$ for some $b \in I_{[\alpha]}$. Put $g(X) = bX^d + (b\eta_1)X^{d-1} + \cdots + (b\eta_d)$. Note that $g(X) \in \operatorname{Ker}\pi$. As $f(0) = g(0)$, we get $f(X) - g(X) = X(h(X)) \in \operatorname{Ker}\pi$ for some $h(X) \in R[X]$. Since $R[\alpha]$ is an integral domain, $\operatorname{Ker}\pi$ is a prime ideal of $R[X]$, and hence $h(X) \in \operatorname{Ker}\pi$. Since $\deg h(X) \leq n-1$, we can prove $f(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$ by induction. Therefore α is an anti-integral element of degree d over R.

Under this preparation, we obtain the following result mentioned before.

Theorem 2. *The following statements are equivalent to each other.*

(1) *α is an anti-integral element of degree d over R,*

(2) *α is an anti-integral element of degree d over R_p for all $p \in Dp_1(R)$.*

Proof. (1) $\Longrightarrow$ (2): By assumption, we have the following exact sequence:

$$0 \longrightarrow I_{[\alpha]}\varphi_\alpha(X)R[X] \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

Take $p \in Dp_1(R)$. Tensoring $\text{-}\otimes_R R_p$, we have an exact sequence:

$$0 \longrightarrow I_{[\alpha]}\varphi_\alpha(X)R_p[X] \longrightarrow R_p[X] \longrightarrow R_p[\alpha] \longrightarrow 0.$$

This exact sequence implies that α is an anti-integral element of degree d over R_p.

(2) $\Longrightarrow$ (1): Consider the following canonical exact sequence:

$$0 \longrightarrow \operatorname{Ker}\pi \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

Let J denote the ideal generated by the set $\{g(0) \mid g(X) \in \operatorname{Ker}\pi\}$. We need to show that $I_{[\alpha]}\eta_d = J$ by Theorem 1. Since $I_{[\alpha]}\varphi_\alpha(X) \subseteq \operatorname{Ker}\pi$, we have $I_{[\alpha]}\eta_d \subseteq J$. We shall show the converse inclusion. Since α is an anti-integral element of degree d over R_p by assumption, we conclude that $(I_{[\alpha]}\eta_d)_p = J_p$ for all $p \in Dp_1(R)$ by Theorem 1. Thus we have $J \subseteq J_p = (I_{[\alpha]}\eta_d)_p$. Let $q \in \operatorname{Spec}(R)$ be a prime divisor of $I_{[\alpha]}\eta_d$. Since $I_{[\alpha]}\eta_d$ is a divisorial ideal, we see that $q \in Dp_1(R)$ ([4, Proposition 1.10]). Hence $J \subseteq \bigcap_{p \in Dp_1(R)}(I_{[\alpha]}\eta_d)_p = I_{[\alpha]}\eta_d$ ([4, Proposition 5.6]). This completes the proof.

References

[1] H. Matsumura: Commutative Algebra, Benjamin, New York (1970).

[2] S. Oda and K. Yoshida: Anti-integral extensions of Noetherian domains, Kobe J. Math. **5** (1988), 43–56.

[3] S. Oda, J. Sato and K. Yoshida: High degree anti-integral extensions of Noetherian domains, Osaka J. Math. **30** (1993), 119–135.

[4] K. Yoshida: On birational-integral extensions of rings and prime ideals of depth one, Japan J. Math. **8** (1982), 49–70.

J. SATO
OSAKA JUNIOR COLLEGE
HIRAO, MIHARA, MINAMI-KAWACHI, OSAKA 567, JAPAN

S. ODA
MATSUSAKA COMMERCIAL HIGH SCHOOL
TOYOHARA, MATSUSAKA, MIE 515–02, JAPAN

K. YOSHIDA
DEPARTMENT OF APPLIED MATHEMATICS
OKAYAMA UNIVERSITY OF SCIENCE
RIDAI, OKAYAMA 700, JAPAN

(*Received, September 16, 1994*)