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A GENERALIZATION OF THE BOTT-TAUBES RIGIDITY THEOREM

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1. Introduction. For a compact S^1-manifold M and S^1-vector bundles E, F over M, the S^1-index of an S^1-elliptic operator $D\colon \Gamma(E) \to \Gamma(F)$ is defined by (see [3])

$$\operatorname{ind} D = [\operatorname{Ker} D] - [\operatorname{Coker} D] \in R(S^1)$$

Note that we can decompose it into a finite sum

$$\operatorname{ind} D = \sum a_n L^n$$

where L^n denotes the representation of S^1 on $\mathbb{C}$ sending $e^{i\theta}$ to $e^{in\theta}$ and $a_n \in \mathbb{Z}$. We call the operator D rigid if $a_n = 0$ for all $n \neq 0$. Rigidity theorems of S^1-index are obtained in [4],[6],[7].

In [4, Proposition 12.2], the rigidity of $\tau_q'(M; E'(V/T))$ (see §2) is stated and the proof is left for the readers. In this paper we give the proof of this proposition. The construction of this paper is as follows. In §2 we give the precise statement of the rigidity theorem, in §3 we show that $\tau_q'(M; E'(V/T))$ can be considered as a meromorphic section of a flat line bundle over a torus, finally in §4 we show that it has no poles.

2. Statement of the theorem. Let G be a compact Lie group, X a G space and EG the total space of the universal G bundle. Denote $X \times EG/G$ by X_G where the quotient is taken relative to the product action. The equivariant cohomology $H_G^*(X; R)$ is defined by

$$H_G^*(X; R) \equiv H^*(X_G; R)$$

where R is a commutative ring with unit. For a G vector bundle V over X, V_G $(= V \times EG/G)$ is a vector bundle over X_G so that the characteristic classes of V_G are well defined in $H_G^*(X)$.

For each $1 \leq k \leq \infty$, consider $\mathbb{Z}_k$ $(= \mathbb{Z}/k\mathbb{Z}) \subset S^1$ where $\mathbb{Z}_\infty = S^1$. We have natural maps

$$\alpha(S^1; \mathbb{Z}_k) : X_{\mathbb{Z}_k} \longrightarrow X_{S^1}$$

which induces

$$\alpha(S^1;\mathbb{Z}_k)^* : H^*_{S^1} \longrightarrow H^*_{\mathbb{Z}_k}.$$

Note that if $\mathbb{Z}_k$ acts trivially on X, then $X_{\mathbb{Z}_k} = X \times B\mathbb{Z}_k$ and there is a map

$$\pi : X_{\mathbb{Z}_k} \longrightarrow X$$

which induces

$$\pi^* : H^*(X) \longrightarrow H^*_{\mathbb{Z}_k}(X).$$

Let M be a smooth oriented compact manifold, T its tangent bundle and V a real oriented vector bundle over M. Then we define the formal series in q,

$$R'_q(T) \equiv \sum_{n=1}^{\infty} R'_n q^{\frac{n}{2}} = \bigotimes_{n=0}^{\infty} \Lambda_{q^{n+\frac{1}{2}}} T \bigotimes_{m=1}^{\infty} S_{q^m} T$$

$$E'_q(V/T) \equiv \Big(\bigotimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}} V \Big) \Big(\bigotimes_{n=1}^{\infty} \Lambda_{q^{n-\frac{1}{2}}} T \Big)^{-1}$$

where

$$S_a(V) = \sum_{k=0}^{\infty} a^k S^k_{\mathbb{C}}(V_{\mathbb{C}}), \quad \text{and} \quad \Lambda_a(V) = \sum_{k=0}^{\infty} a^k \Lambda^k_{\mathbb{C}}(V_{\mathbb{C}})$$

and $S^k_{\mathbb{C}}$, $\Lambda^k_{\mathbb{C}}$ denote symmetric and exterior powers over $\mathbb{C}$, $V_{\mathbb{C}} = V \otimes \mathbb{C}$.

Recall that the S^1-index of an S^1-elliptic operator D can be specialized to individual elements of S^1. For $\lambda \in S^1$ we can define

$$\operatorname{ind}_\lambda D = \operatorname{trace}(\lambda|_{\operatorname{Ker} D}) - \operatorname{trace}(\lambda|_{\operatorname{Coker} D}).$$

We can state the theorem.

Theorem ([4, Proposition 12.1]). *Let M be a smooth oriented compact spin manifold of even dimension, on which S^1 acts by isometry, and V an even dimensional real oriented vector bundle over M with a compatible S^1 action and $w_2(V) = 0$. For each $1 < k \leq \infty$, let $i: M_k \hookrightarrow M$ be the inclusion of the fixed point set of $\mathbb{Z}_k \subset S^1 = S$ in M and so i induces $i_S: (M_k)_S \to M_S$. Assume the following two conditions are satisfied.*

(i) $\alpha(S^1;\mathbb{Z}_k)^* \circ i_S^* w_2(V_S - T_S) = 0 \in H^2_{\mathbb{Z}_k}(M_k;\mathbb{Z}_2)$

(ii) $\alpha(S^1;\mathbb{Z}_k)^* \circ i_S^* \frac{1}{2} p_1(V_S - T_S) = \pi^* \circ \alpha(S^1;\mathbb{Z}_1)^* \circ i_S^* \frac{1}{2} p_1(V_S - T_S) \in H^4_{\mathbb{Z}_k}(M_k;\mathbb{Z})$

Then

$$\tau'_q(M;E') \equiv \mathrm{ind}(\not{\partial} \otimes R'_q(T) \otimes E'_q(V/T))$$

is rigid, where $\not{\partial}: \Gamma(\Delta^+) \to \Gamma(\Delta^-)$ *is the Dirac operator (see* [5]*) on spin bundle* $\Delta = \Delta^+ \oplus \Delta^-$ *over* M.

Remark. Note that $w_2(V_S - T_S) = 0$, $p_1(V_S - T_S)/2 = 0$ imply (i) and (ii).

3. $\tau'_q(M;E')$ and meromorphic sections. First, we define a torus and a flat line bundle. For $\tau \in H = \{\textit{upper half plane}\} \subset \mathbb{C}$, we set

$$T_\tau \equiv \mathbb{C}/\mathbb{Z} \times \tau\cdot\mathbb{Z}$$

and define the flat line bundle L'_τ over T_τ by

$$L'_\tau \equiv \mathbb{C} \times \mathbb{C}/\mathbb{Z} \times \mathbb{Z} \to T_\tau$$

where $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{C} \times \mathbb{C}$ by sending $(k,l) \in \mathbb{Z} \times \mathbb{Z}$ and $(z,v) \in \mathbb{C} \times \mathbb{C}$ to $(z + k + l\cdot\tau, (-1)^{k+l}v)$.

We consider special sections of L'_τ. For this purpose, we introduce a function η_τ. For $\tau \in H$ and $z \in \mathbb{C}$, we define

$$\eta_\tau(z) \equiv \prod_{n=1}^{\infty}(1 + q^n q^{-\frac{1}{2}}\lambda)(1 + q^n q^{-\frac{1}{2}}\lambda^{-1})$$

where $q = e^{2\pi i\tau}$, $q^{1/2} = e^{\pi i\tau}$ and $\lambda = e^{2\pi iz}$. Since $\tau \in H$, $0 < |q| < 1$, and so η_τ converges for all $z \in \mathbb{C}$. Let E be a complex vector bundle over a space X. Then define

$$\hat{\eta}_\tau(E)(z) \equiv \bigotimes_{n=1}^{\infty}(\Lambda_{\lambda q^{n-\frac{1}{2}}}E \otimes \Lambda_{\lambda^{-1}q^{n-\frac{1}{2}}}E^*)$$

which is a formal power series in $q^{1/2}$ whose coefficients are vector bundles $\otimes$ finite Laurent series in $\lambda = e^{2\pi iz}$. If E is a line bundle

$$\hat{\eta}_\tau(E)(z) = \sum_{k=0}^{\infty}\eta_\tau{}^{(k)}(z)\frac{1}{(2\pi i)^k k!}(\log E)^k$$

where $\eta_\tau{}^{(k)} = (\frac{d}{dz})^k\eta_\tau$ and therefore, by the splitting principle, we can consider $\hat{\eta}(E)$ as an element of $K(X)\otimes_{\mathbb{Z}}\{\textit{meromorphic functions over } \mathbb{C}\}$

for general E. Similarly we define

$$\hat{\chi}_\tau(E)(z) \equiv \lambda^{\frac{1}{2}}(\Lambda_{-\lambda}E)^{-1} \bigotimes_{m=1}^{\infty} (\Lambda_{\lambda q^{m-\frac{1}{2}}} E \otimes \Lambda_{\lambda^{-1}q^{m-\frac{1}{2}}} E^*) \bigotimes_{m=1}^{\infty} (S_{\lambda q^m} E \otimes S_{\lambda^{-1}q^m} E^*)$$

where $\lambda^{1/2} = e^{\pi i z}$ for $z \in \mathbb{C}$. Then we can view $\hat{\chi}_\tau(E) \in K(X) \otimes_{\mathbb{Z}}$ $\{$*meromorphic sections over* T_τ *of* $(L'_\tau)^{\dim_{\mathbb{C}} E}\}$ (see [4, §12]).

We want to use the fixed point formula (see [1],[2]), therefore we consider the fixed point set of S^1 action. Let P be a connected component of the S^1-fixed point set. We decompose the tangent bundle into

$$T|_P = TP \oplus \bigoplus_i E_i^\sharp \tag{3.1}$$

where $E_i^\sharp$ is the underlying real bundle of a complex bundle $E_i \to P$, on which S^1 acts by $\lambda \to \lambda^{m_i}$ for some integer m_i and $|m_i| \neq |m_j|$ if $i \neq j$. We write

$$N_P \equiv \bigoplus_i E_i$$

Similarly

$$V|_P = V'_P \oplus \bigoplus_i F_i^\sharp$$

where S^1 acts trivially on V'_P and by $\lambda \to \lambda^{\nu_i}$ on F_i.

Lemma 3.1.

$$\tau'_q(M; E')(\lambda) = \sum_{\substack{\{P\}:S^1\text{-fixed} \\ \text{point sets}}} \mu'_P(\lambda)$$

$$\mu'_P(\lambda) = \tau'_q\left(P_{\det N_P}; \bigotimes_i \psi_{m_i}\hat{\chi}_\tau(E_i)(z) \otimes E'_q(V'_P/TP) \otimes \frac{\otimes_i \psi_{\nu_i}\hat{\eta}_\tau(F_i)}{\otimes_i \psi_{m_i}\hat{\eta}_\tau(E_i)}(z)\right)(1)$$

where for a complex line bundle L over a manifold X and a formal power series in q E_q with coefficients in $K(X)$, we define

$$\tau'_q(X_L; E_q)(\lambda) \equiv \operatorname{ind}_\lambda(d_S \otimes \Delta(TX; L)^{-1} \otimes R'_q(TX) \otimes E_q)$$

where $\Delta(TX; L)$ is a Spin$_C$*-bundle defined by L and d_S is the signature operator. And ψ_m denotes the operation of raising λ to the m-th power.*

Proof. By the fixed point formula, we can show the idenity (see [4, (12.11)])

$$\mathrm{ind}_\lambda\Big(\not\partial \otimes R'_q\Big) = \sum_{\{P\}} \tau'_q(P_{\det N_P};\ \bigotimes_i \psi_{m_i}\hat{\chi}_\tau(E_i)(z)\Big)$$

Now $\tau'_q(M;E')(\lambda) = \mathrm{ind}_\lambda(\not\partial \otimes R'_q \otimes E')$ by definition, therefore we have only to show

$$ch(E'|_P) = ch\left(E'_q(V'_P/TP) \otimes \frac{\otimes_i \psi_{\nu_i}\hat{\eta}_\tau(F_i)(z)}{\otimes_i \psi_{m_i}\hat{\eta}_\tau(E_i)(z)}\right)$$

but this can be shown by direct calculation.

Now we can interpret $\tau'_q(M;E)$ as a section.

Lemma 3.2. *$\tau'_q(M;E)(e^{2\pi i z})$ can be considered as a meromorphic section of $(L'_\tau)^\varepsilon$ over T_τ, where $\varepsilon = 0$ if the action of S^1 is even and $\varepsilon = 1$ if odd.*

Proof. By [4, Lemma 12.2], $\tau'_q(P_{\det N_P}; \psi_{m_i}\hat{\chi}_\tau(E_i)(z))(1)$ is a meromorphic section of $(L'_\tau)^\varepsilon$. We consider the $\hat{\eta}$-part. For $\xi \in \mathbb{C}$,

$$e^{2\pi i(\xi+1)} = e^{2\pi i \xi}, \quad e^{2\pi i(\xi+\tau)} = e^{2\pi i \tau}e^{2\pi i \xi} = q e^{2\pi i \xi}$$

so

$$\begin{aligned}
\eta_\tau(\xi+1) &= \eta_\tau(\xi) \\
\eta_\tau(\xi+\tau) &= \prod_{n=1}^{\infty}(1+q^n q^{-\frac{1}{2}} q\lambda)(1+q^n q^{-\frac{1}{2}} q^{-1}\lambda^{-1}) \\
&= \frac{1+q^{-\frac{1}{2}}\lambda^{-1}}{1+q^{\frac{1}{2}}\lambda}\eta_\tau(\xi) \\
&= q^{-\frac{1}{2}}\lambda^{-1}\eta_\tau(\xi).
\end{aligned}$$

Therefore

$$\begin{aligned}
\psi_m\eta_\tau(\xi+1) &= \eta_\tau(m\xi+m) = \eta_\tau(m\xi) = \psi_m\eta_\tau(\xi) \\
\psi_m\eta_\tau(\xi+\tau) &= \eta_\tau(m\xi+m\tau) \\
&= q^{-\frac{1}{2}}\lambda^{-m}q^{-(m-1)}\eta_\tau(m\xi+(m-1)\tau) \\
&= q^{-\frac{m}{2}}\lambda^{-m^2}q^{-\frac{m^2-m}{2}}\eta_\tau(m\xi) \\
&= \lambda^{-m^2}q^{-\frac{m^2}{2}}\psi_m\eta_\tau(\xi).
\end{aligned}$$

These imply

$$ch\psi_\nu\hat{\eta}_\tau(F)(z+1) = ch\psi_\nu\hat{\eta}_\tau(F)(z)$$
$$ch\psi_\nu\hat{\eta}_\tau(F)(z+\tau) = q^{-\frac{\nu^2}{2}}\lambda^{-\nu^2}e^{-\nu y}ch\psi_\nu\hat{\eta}_\tau(F)(z)$$

where $y = c_1(F)$. Write $y_i = c_1(F_i)$ and $x_i = c_1(E_i)$, then

$$(*)\qquad \frac{\prod q^{-\frac{\nu_i^2}{2}}\lambda^{-\nu_i^2}e^{-\nu_i y_i}}{\prod q^{-\frac{m_i^2}{2}}\lambda^{-m_i^2}e^{-m_i x_i}} = \lambda^{(\sum m_i^2 - \sum \nu_i^2)}q^{\frac{1}{2}(\sum m_i^2 - \sum \nu_i^2)}e^{(\sum m_i x_i - \sum \nu_i y_i)}$$

but one can show that $(\sum m_i^2 - \sum \nu_i^2)/2 = 0$, and $\sum m_i x_i - \sum \nu_i y_i = 0$ (see [4, (11.22), Lemma 11.3]).

Therefore $(*) = 1$, and $\tau_q'(P_{\det N_P}; \otimes\psi_{m_i}\hat{\chi}_\tau(E_i)(z) \otimes E_q'(V_P'/TP) \otimes [(\otimes\psi_{\nu_i}\hat{\eta}_\tau(F_i)(z))/(\otimes\psi_{m_i}\hat{\eta}_\tau(E_i)(z))])$ can be viewed as a meromorphic section of $(L_\tau')^\epsilon$. Therefore with Lemma 3.1, we complete the proof.

4. Proof of theorem; poles of $\tau_q'(M;E')$. Note that $\psi_\nu\eta_\tau$ has zeros at $e^{2\pi i/\nu}q^{1/2\nu}$ modulo $\exp[2\pi i(\mathbb{Z}\times\tau\cdot\mathbb{Z})]$, therefore $\tau_q'(M;E')$ possibly has poles at α^s where $\alpha = e^{2\pi i\tau/k}$ for some integers k and s. We may assume k is positive and k and s are relatively prime. We define

$$t_{\alpha^s}\tau_q'(M;E')(\lambda) \equiv \tau_q'(M;E')(\alpha^s\lambda)$$

and if we can identify $t_{\alpha^s}\tau_q'(M;E')$ with the S^1-index of an appropriate elliptic operator on some auxiliary manifold, since S^1-index has no poles on $\{|\lambda| = 1\}$, we conclude that $\tau_q'(M;E')$ has no poles.

Let M_k be the fixed point set of $\mathbb{Z}_k \subset S^1$, and for odd k we decompose the tangent bundle into

$$T|_{M_k} = TM_k \oplus T_1^\sharp \oplus \cdots \oplus T_{\frac{k-1}{2}}^\sharp \tag{4.1}$$

where each $T_r^\sharp$ is a real bundle which has a natural complex structure so that $\xi \in \mathbb{Z}_k$ acts as ξ^r. For even k, we have

$$T|_{M_k} = TM_k \oplus T_1^\sharp \oplus \cdots \oplus T_{\frac{k}{2}-1}^\sharp \oplus T_{\frac{k}{2}} \tag{4.2}$$

where $T_{k/2}$ is a real bundle on which $\xi \in \mathbb{Z}_k$ acts as (-1). Similarly we have

$$\begin{aligned} V|_{M_k} &= V_0 \oplus V_1^\sharp \oplus \cdots \oplus V_{\frac{k-1}{2}}^\sharp &&(k\text{: odd})\\ &= V_0 \oplus V_1^\sharp \oplus \cdots \oplus V_{\frac{k}{2}-1}^\sharp \oplus V_{\frac{k}{2}} &&(k\text{: even})\end{aligned}$$

M_k' denotes M_k, but oriented as follows. When k is odd, M_k has an induced orientation with respect to the decomposition (4.1). Call this orientation $+1$. Let P be a component of the S^1 fixed point set with $P \subseteq M_k$ and decompose $T|_P$ as (3.1). Choose the signs of m_j's so that each $m_j \not\equiv 0 \pmod k$ is congruent mod$k$ to some $r \in \{1, \cdots, (k-1)/2\}$. Choose the orientation of TP and choose the signs of those $m_j \equiv 0 \pmod k$ so that the induced orientation on $T|_P$ is correct one, then the induced orientation on $TM_k|_P$ will be the $+1$ orientation. For each m_j, define $(l_j, \omega_j) \in \mathbb{Z} \times \{0, \cdots, k-1\}$ by

$$s \cdot m_j = l_j \cdot k + \omega_j$$

and define

$$\varepsilon(P) \equiv \sum (\dim_{\mathbb{C}} E_j) \cdot l_j.$$

The orientation for M_k' is defined to be $(-1)^{\varepsilon(P)}$ on the component of M_k which contains P. This orientation is well defined under the spin assumption (see [4, §§8,9]). When k is even, an orientation is induced on $TM_k \oplus T_{k/2}$ by (4.2). Choose an orientation for TM_k and call it $+1$. Let P be a component of S^1 fixed point set with $P \subseteq M_k$ and choose the signs of those $m_j \not\equiv 0, k/2 \pmod k$ so that $m_j \pmod k \in \{1, \cdots, k/2 - 1\}$. Choose the signs for those $m_j \equiv 0, k/2 \pmod k$ and choose the orientation of TP to make the induced orientation on $TM_k \oplus T_{k/2}|_P$ correct. Introduce $\varepsilon_0 = 0, 1$ with $\varepsilon_0 = 0$ if the induced orientation on $TM_k|_P$ is correct, and $\varepsilon_0 = 1$ if incorrect.

Set

$$\varepsilon(P) \equiv \varepsilon_0 + \sum (\dim_{\mathbb{C}} E_j) \cdot l_j$$

and the orientation for M_k' is given by $(-1)^{\varepsilon(P)}$ as above.

Now we consider the transfer formula. Introduce

$$\omega(r) \equiv s \cdot r \pmod k \in \{0, \cdots, k-1\}.$$

Case 1; k is odd.

Lemma 4.1. *If the assumptions of Theorem hold, then the complex line bundle*

$$\nu_s \equiv \bigotimes_{r=1}^{\frac{k-1}{2}} \left[(\det V_r)^{\omega(r)} \otimes (\det T_r)^{-\omega(r)} \right]$$

admits a k-th root $\nu_s^{1/k}$ over M_k.

Proof. This is [4, Lemma 11.3].

Proposition 4.2.

$$t_{\alpha^s}\tau_q'(M;E')(\lambda) = \alpha^\sigma \tau_q'\Big(M_{k,L}'; \bigotimes_r \hat{\chi}_\tau(T_r)\Big(\frac{\omega(r)}{k}\cdot\tau\Big) \otimes \nu_s^{\frac{1}{k}} \otimes E_q'(V_0/T_0) \bigotimes_r \frac{\hat{\eta}_\tau(V_r)}{\hat{\eta}_\tau(T_r)}\Big(\frac{\omega(r)}{k}\cdot\tau\Big)\Big)(\lambda)$$

where

$$\sigma = \frac{1}{2k}\sum_r \omega(r)^2\{\dim_{\mathbb{C}} V_r - \dim_{\mathbb{C}} T_r\}$$
$$\Big(= \sum_i \Big(\frac{k}{2}l_i^2 + \omega_i l_i\Big) - \sum_i \Big(\frac{k}{2}l_i'^2 + \omega_i' l_i'\Big)\Big),$$

$(l_i', \omega_i') \in \mathbb{Z} \times \{0, \cdots, k-1\}$ is defined by $s\cdot\nu_i = l_i'\cdot k + \omega_i'$, and

$$L \equiv \bigotimes_{r=1}^{\frac{k-1}{2}} \det T_r.$$

Proof. By Lemma 3.1

$$\tau_q'(M;E')(\lambda) = \sum_{\{P\}} \mu'(P)(\lambda).$$

$$t_{\alpha^s}\mu'_P(\lambda) = \tau_q'\Big(P_{\det N_P}; \bigotimes_i \psi_{m_i}\hat{\chi}_\tau(E_i)\Big(z + \frac{s}{k}\tau\Big) \otimes E_q'(V_P'/TP) \bigotimes_i \frac{\psi_{\nu_i}\hat{\eta}_\tau(F_i)}{\psi_{m_i}\hat{\eta}_\tau(E_i)}\Big(z + \frac{s}{k}\tau\Big)\Big)(1)$$

and by the index theorem,

$$= 2^{\frac{\dim P}{2}}\Big(ch\Big[\Delta(TP;\det N_P)^{-1} \otimes R_q'(TP) \otimes \psi_{m_i}\hat{\chi}_\tau(E_i)\Big(z + \frac{s}{k}\tau\Big) \otimes E_q'(V_P'/TP) \otimes \frac{\psi_{\nu_i}\hat{\eta}_\tau(F_i)}{\psi_{m_i}\hat{\eta}_\tau(E_i)}\Big(z + \frac{s}{k}\tau\Big)\Big]\cdot\hat{\mathbb{L}}(P)\Big)[P]$$

$$= (-1)^{\sum l_i}\cdot\alpha^{\sum(\frac{k}{2}l_i^2 + \omega_i l_i) - \sum(\frac{k}{2}l_i'^2 + \omega_i' l_i')}\cdot\lambda^{\sum l_i m_i - \sum l_i'\nu_i'}\cdot 2^{\frac{\dim P}{2}}$$
$$\cdot\Big(e^{\sum l_i x_i - \sum l_i' y_i'} ch\Big[\Delta(TP;\det N_P)^{-1} \otimes R_q'(TP) \otimes \hat{\chi}_\tau(E_i)\Big(m_i z + \frac{\omega_i}{k}\tau\Big) \otimes E_q'(V_P'/TP) \otimes \frac{\hat{\eta}_\tau(F_i)(\nu_i z + \frac{\omega_i'}{k}\tau)}{\hat{\eta}_\tau(E_i)(m_i z + \frac{\omega_i}{k}\tau)}\Big]\cdot\hat{\mathbb{L}}(P)\Big)[P].$$

On the other hand, if M_k replaces M_k' in the right hand side $\times 1/\alpha^\sigma$, and apply the fixed point formula, the contribution from the S^1 fixed point set P is

$$\Big(ch\Big[\Delta(TM_k;L)^{-1}\otimes\hat{\chi}_\tau(T_r)\Big(\frac{\omega(r)}{k}\tau\Big)\otimes\nu_s^{\frac{1}{k}}\otimes E_q'(V_0/T_0)\otimes\frac{\hat{\eta}_\tau(V_r)}{\hat{\eta}_\tau(T_r)}\Big(\frac{\omega(r)}{k}\tau\Big)\Big|_P\Big]$$
$$\cdot\Big(ch_\lambda\delta\Big(\bigoplus_{r=1}^{\frac{k-1}{2}}T_r\Big)\Big/ch_\lambda\Lambda_{-1}\Big(\bigoplus_{r=1}^{\frac{k-1}{2}}T_r\otimes\mathbb{C}\Big)\Big)\cdot\hat{\mathbb{L}}(P)\Big)[P].$$

By the help of [4, (12.24)], we have only to show

$$\lambda^{\sum l_i m_i-\sum l_i'\nu_i}\cdot e^{\sum l_i x_i-\sum l_i' y_i}\cdot ch\Big(E_q'(V_P/TP)\otimes\frac{\hat{\eta}_\tau(F_i)(\nu_i z+\frac{\omega_i'}{k}\tau)}{\hat{\eta}_\tau(E_i)(m_i z+\frac{\omega_i}{k}\tau)}\Big)$$
$$=ch_\lambda\Big(\nu_s^{\frac{1}{k}}\otimes E_q'(V_0/T_0)\bigotimes_{r=1}^{\frac{k-1}{2}}\frac{\hat{\eta}_\tau(V_r)}{\hat{\eta}_\tau(T_r)}\Big(\frac{\omega(r)}{k}\tau\Big)\Big|_P\Big)$$

but this can be shown by direct calculation.

Case 2; k is even.

Lemma 4.3. *If the assumptions of Theorem hold, then*
(1) *The complex line bundle*

$$\nu_s\equiv\bigotimes_{r=1}^{\frac{k}{2}-1}\Big[(\det V_r)^{\omega(r)}\otimes(\det T_r)^{\omega(r)}\Big]$$

admits a $(k/2)$-th root $\nu_s^{2/k}$ over M_k.

(2) *The vector bundle $T_{k/2}\oplus V_{k/2}\to M_k$ has a* Spin_C*-structure defined by the line bundle $\nu_s^{2/k}$.*

Proof. See Lemma 11.4(1) and (11.38) of [4].

Proposition 4.4.

$$t_{\alpha^s}\tau_q'(M;E')(\lambda)=\alpha^\sigma\tau_q''\Big(M_{k,L}';\bigotimes_{r=1}^{\frac{k}{2}-1}\hat{\chi}_\tau(T_r)\Big(\frac{\omega(r)}{k}\tau\Big)\otimes R_q''(T_{\frac{k}{2}})$$
$$\otimes\,\nu_s^{\frac{2}{k}}\otimes\frac{\Delta(T_{\frac{k}{2}}\oplus V_{\frac{k}{2}};\nu_s^{\frac{2}{k}})}{\Delta(T_{\frac{k}{2}}\oplus V_{\frac{k}{2}})}\bigotimes_{n=1}^{\infty}\frac{\Lambda_{q^n}V_{\frac{k}{2}}}{\Lambda_{q^n}T_{\frac{k}{2}}}$$
$$\otimes\,E_q'(V_0/T_0)\bigotimes_{r=1}^{\frac{k}{2}-1}\frac{\hat{\eta}_\tau(V_r)}{\hat{\eta}_\tau(T_r)}\Big(\frac{\omega(r)}{k}\tau\Big)\Big)(\lambda)$$

where

$$\sigma = \frac{1}{2k}\sum_{r=1}^{\frac{k}{2}-1}\omega(r)^2\{\dim_{\mathbb{C}} V_r - \dim_{\mathbb{C}} T_r\} + \frac{1}{4k}\omega\left(\frac{k}{2}\right)^2\{\dim_{\mathbb{R}} V_{\frac{k}{2}} - \dim_{\mathbb{R}} T_{\frac{k}{2}}\}$$

$$\tau_q''(X_L;E)(\lambda) = \mathrm{ind}_\lambda\Big(d_S \otimes \Delta(TX \oplus T_{\frac{k}{2}}; L)^{-1} \otimes R_q'(TX) \otimes E_q\Big)$$

$$L = \bigotimes_{r=1}^{\frac{k}{2}-1} \det T_r, \quad \text{and}$$

$$R_q''(T_{\frac{k}{2}}) = \Delta(T_{\frac{k}{2}} \otimes \mathbb{C}) \bigotimes_{n=1}^{\infty} \Lambda_{q^n} T_{\frac{k}{2}} \bigotimes_{m=1}^{\infty} S_{q^{m-\frac{1}{2}}} T_{\frac{k}{2}}.$$

The proof of this proposition is quite similar to that of Proposition 4.2, so we omit the proof.

Thus $\tau_q'(M;E')$ has no poles on T_τ, so is rigid and we complete the proof of Theorem.

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