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ON SOME PRODUCTS OF β-ELEMENTS IN THE HOMOTOPY OF THE MOORE SPECTRUM II

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1. Introduction. The present paper is a continuation of [26] with the same title. Throughout this paper, p denotes a prime number greater than 3.

Consider the sets $\mathcal{B}'$ and $\mathcal{B}$ of integers given by

$$\mathcal{B}' = \{(s,n,j) \in Z_+^3 \mid s,j > 0,\ p \nmid s \text{ and } j \leq a_n, \text{ and } j \leq p^n \text{ if } s = 1\},$$
$$\mathcal{B} = \{(s,n,j,i) \in Z_+^4 \mid s,j > 0,\ p \nmid s \text{ and } p^i | j \leq a_{n-i}, \text{ and } j \leq p^n \text{ if } s = 1\},$$

where Z_+ denotes the set of all non-negative integers and a_n the integer defined by $a_0 = 1$ and $a_n = p^n + p^{n-1} - 1$ if $n > 0$. Let S^0 and M denote the p-localized sphere and the mod p Moore spectrum, respectively. In their paper [9], Miller, Ravenel and Wilson introduced the β-elements in the E_2-terms $E_2^{*,*}(S^0)$ and $E_2^{*,*}(M)$ of the Adams-Novikov spectral sequences for computing the homotopy groups $\pi_*(S^0)$ and $\pi_*(M)$, respectively. Besides, they showed implicitly that $\beta'_{sp^n/j} \in \pi_*(L_2M)$ (resp. $\beta_{sp^n/j,i} \in \pi_*(L_2S^0)$) if $(s,n,j) \in \mathcal{B}'$ (resp. $(s,n,j,i) \in \mathcal{B}$). Here L_2 denotes the Bousfield localization functor with respect to the v_2-telescope $v_2^{-1}BP$ of the Brown-Peterson spectrum BP at p (*cf.* [25]). The object of these papers is to seek out non-trivial products of β-elements in the stable homotopy groups $\pi_*(M)$. Here note that there is a choice of defining β-elements since β-elements are the generators of the E_2-terms $E_2^{*,*}(S^0)$ and $E_2^{*,*}(M)$, and that ours are slightly different from theirs (see §3). This may be related to constructing homotopy β-elements in $\pi_*(S^0)$.

Virtually, we find non-trivial products in the E_2-term $E_2^{*,*}(L_2M)$ of the Adams-Novikov spectral sequence for computing $\pi_*(L_2M)$. Note that the E_2-term $E_2^{s,t}(L_2M)$ is isomorphic to the homotopy group $\pi_{t-s}(L_2M)$. In fact, the E_2-terms $E_2^{s,*}(L_2M)$ of the Adams-Novikov spectral sequence for L_2M are null if $s > 4$ and $p > 3$ (*cf.* [18]), and so the spectral sequence collapses and arises no extension problem because of the sparseness of the spectral sequence. Thus our non-trivial products in the homotopy groups $\pi_*(M)$ are obtained by pulling back those in $\pi_*(L_2M)$ under the induced map $\eta_*: \pi_*(M) \to \pi_*(L_2M)$ of the localization map η, if the products are in the image of η_*.

To state our results, consider a map $c: Z_+ \to Z_+$ sending n to $c(n)$ defined by

$$n = c(n)p^l - \frac{p^l - 1}{p - 1}$$

with $p \nmid c(n) + 1$ for some $l \geq 0$. We also use the integer $A(m)$ (*cf.* [22]) defined by

$$\begin{aligned} A(tp^n) &= (p+1)\frac{p^n - 1}{p - 1} + 2 & &\text{if } p \nmid t(t+1), \\ &= (p+1)\Big(p^{n+1} - p^n + \frac{p^n - 1}{p - 1}\Big) + 2 & &\text{if } p^2 | t+1. \end{aligned}$$

As a sequel to Theorem A of Part I [26], we have the following

Theorem AII. *Let $(s, n+r, p^r a_{n-i}, i)$ be an element of $\mathcal{B}$ with $n > i \geq 0$. Then, in $\pi_*(L_2M)$,*

$$\beta'_t \beta_{sp^{n+r}/p^r a_{n-i}, i+1} \neq 0$$

if $p \nmid t$ for even $r \geq 2$, or if $p | c$ and $p^2 \nmid c + p$ for odd $r \geq 1$, where $c = c(t + sp^{n+r} - p^{n+r-i-1} + (p^r + 1)/(p+1))$.

The following is a sequel to Theorem B of Part I.

Theorem BII. *Let (s, n, j) be an element of $\mathcal{B}'$. Then the following are the relations in $\pi_*(L_2M)$.*

1) *$\beta'_{up/k}\beta'_{sp^n/j} \neq 0$ if $p^2 | u+1$ and $p^n + p^{n-1} - p^i < k + j \leq p^n + p^{n-1} - p^i + p^2 + 1$ for $(u, 1, k) \in \mathcal{B}'$ and even i with $0 < i < n$.*

2) *$\beta'_{tp^m - sp^n + p^{n-1} - k(i)}\beta'_{sp^n/j} \neq 0$ if $p \nmid t(t+1)$ or $p^2 | t+1$, and if $p^n + p^{n-1} - p^i \leq j \leq p^n + p^{n-1} - p^i + A(tp^m) + 1$ for $(tp^m - sp^n + p^{n-1} - k(i), 0, 1) \in \mathcal{B}'$ and even i with $0 < i < n$.*

3) *$\beta'_{up^r/k}\beta'_{sp^n/j} \neq 0$ if $r > 0$ and $p^n + p^{n-1} - p^i \leq k + j < p^n + p^{n-1} - p^i + p$ for $(u, r, k) \in \mathcal{B}'$ and odd i with $0 < i < n$.*

It is well known that β'_t survives to a homotopy element of $\pi_*(M)$ for each $t > 0$ (*cf.* [16]). Besides, $\beta_{sp^{n+r}/p^{r+1}, n}$ is a homotopy element of $\pi_*(S^0)$ for each $(s, n+r, p^{r+1}, n)$ in $\mathcal{B}$ according to Lin's results [3] and [7]. Then setting $i = n - 1$ in Theorem AII we obtain

Corollary. *In the homotopy groups $\pi_*(M)$ of the Moore spectrum, we have the non-trivial elements*

$$\beta'_t\beta_{sp^{n+r}/p^{r+1},n} \neq 0 \quad \textit{if } p\nmid st \textit{ and } r \textit{ is even} \geq n-1 \geq 0,$$

and

$$\beta'_t\beta_{sp^{n+r}/p^{r+1},n} \neq 0 \quad \textit{if } p\nmid s,\ p|t+1,\ p^2\nmid t+1 \textit{ and } r \textit{ is odd} \geq n-1 \geq 0.$$

Here the case $r = 0$ follows from [22]. We note that Corollary states a part of the results obtained from Theorem AII, which means that we can say more about the product $\beta'_t\beta_{sp^{n+r}/p^{r+1},n}$ from Theorem AII. For example, it is non-trivial if $p\nmid s$, $p^2|t+2$, and $p^3\nmid t+p^2+2$ for odd $r \geq n-1 > 0$.

2. Notations and facts known. In this paper we virtually compute in a cobar complex. We prepare some notations and results. Let (A, Γ) be a Hopf algebroid with Γ A-flat. Then we can do the homological algebra in the category of Γ-comodules (*cf.* [16]). The Ext-groups are defined to be the cohomology of an injective resolution. In our computation an injective resolution is replaced by a relatively injective resolution, and a typical example of relatively injective resolution is the cobar resolution. Therefore an Ext-group $\mathrm{Ext}_\Gamma(A, M)$ for a comodule M is a homology group of a cobar complex $\Omega^*_\Gamma M$, which is given by:

$$\Omega^s_\Gamma M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (s \text{ factors})$$

with differential $d_s \colon \Omega^s_\Gamma M \to \Omega^{s+1}_\Gamma M$ given by

$$\begin{aligned} &d_s(m \otimes \gamma_1 \otimes \cdots \otimes \gamma_s) \\ &\quad = \psi(m) \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \\ &\qquad + \sum_{i=1}^{s} (-1)^i m \otimes \gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_s \\ &\qquad - (-1)^s m \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes 1. \end{aligned}$$

Here ψ is the structure map of the comodule M.

Consider the Hopf algebroid

$$(BP_*, BP_*(BP)) = (Z_{(p)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots])$$

associated to the Brown-Peterson ring spectrum BP (*cf.* [1]). The structure maps of it are well known (*cf.* [1],[9],[16]), and we will cite them where they are used. We define another Hopf algebroid

$$(E, E_*E) = (Z_{(p)}[v_1, v_2, v_2^{-1}], E[t_1, t_2, \cdots] \otimes_{BP_*} E),$$

whose structure is induced from that of $(BP_*, BP_*(BP))$. Here the action of BP_* on E is given by setting $v_n \cdot 1 = v_n$ if $n \leq 2$ and $v_n \cdot 1 = 0$ otherwise for $1 \in E$. This Hopf algebroid coincides with the one associated to the Johnson-Wilson spectrum $E(2)$ such that

$$\pi_*(E(2)) = E.$$

Let (A, Γ) denote one of the above Hopf algebroids. Recall [9] the Γ-comodules N_j^i and M_j^i, which are defined inductively by $N_n^0 = A/(p, v_1, \cdots, v_{n-1})$, $M_j^i = v_{i+j}^{-1} N_j^i$ and the exact sequence

$$(2.1) \qquad 0 \longrightarrow N_j^i \longrightarrow M_j^i \longrightarrow N_j^{i+1} \longrightarrow 0.$$

The comodule structures of them are induced from the right unit map $\eta_R: A \to \Gamma$ and we denote them by the same notation η_R. Here note that this construction works as well for $(A, \Gamma) = (E, E_*E)$ as $(BP_*, BP_*(BP))$. Besides, $M_j^i = 0$ if $i + j > 2$ and $= N_j^i$ if $i + j = 2$, for the case $\Gamma = E_*E$.

In this paper we consider them only for $i + j \leq 2$ and have

$$N_0^0 = A, \qquad N_1^0 = A/(p), \qquad N_0^1 = A/(p^\infty),$$
$$N_2^0 = A/(p, v_1), \quad N_1^1 = A/(p, v_1^\infty), \quad N_0^2 = A/(p^\infty, v_1^\infty) \quad \text{and}$$
$$M_j^i = v_{i+j}^{-1} N_j^i.$$

We end this section with recalling [8] an isomorphism

$$\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M_{2-i}^i) = \mathrm{Ext}^*_{E_*E}(E_*, M_{2-i}^i \otimes_{BP_*} E_*)$$

under the canonical map $\lambda: (BP_*, BP_*(BP)) \to (E, E_*E)$, for the $BP_*(BP)$-comodule M_{2-i}^i with $0 \leq i \leq 2$.

Hereafter, we use the abbreviation $H^*- = \mathrm{Ext}^*_\Gamma(A, -)$ for $(A, \Gamma) = (BP_*, BP_*(BP))$ or $= (E, E_*E)$. It is convenient for defining β-elements in both Ext-groups, though it may be confusing.

3. Computation of β-elements. Let S^0 and M be the p-local sphere spectrum and the mod p Moore spectrum, respectively, and L_2 the $v_2^{-1}BP$-localization functor (*cf.* [25]). Then the Ext-groups $H^*N_1^0$

and $H^*N_0^0$ for $BP_*(BP)$ (resp. E_*E) are the E_2-terms of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(M)$ and $\pi_*(S^0)$ (resp. $\pi_*(L_2M)$ and $\pi_*(L_2S^0)$), respectively (*cf.* [25]). The E_2-term $H^tN_1^0$ for $\pi_*(L_2M)$ is trivial if $t > 4$. In fact, $H^tM_1^0 = 0$ if $t > 1$ by [9], $H^tM_1^1 = 0$ if $t > 3$ by [18], and $H^{t-1}M_1^0 \to H^{t-1}M_1^1 \to H^tN_1^0 \to H^tM_1^0$ is exact by (2.1). Therefore the sparseness of the spectral sequence yields an isomorphism

$$H^*N_1^0 = H^*A/(p) \cong \pi_*(L_2M).$$

By this, an element of the E_2-term can be identified with a homotopy element for the case $A = E$.

In order to define the β-elements, consider the boundary homomorphisms

$$(3.1)\qquad \begin{aligned} \delta_1&\colon H^1N_0^1 \longrightarrow H^2N_0^0,\\ \delta_0&\colon H^0N_0^2 \longrightarrow H^1N_0^1 \quad\text{and}\\ \delta_0'&\colon H^0N_1^1 \longrightarrow H^1N_1^0\end{aligned}$$

associated to the short exact sequences of (2.1).

The map $\lambda\colon BP_* \to E$ is extended to the one $\lambda\colon v_2^{-1}BP_* \to E$ and we will use the same notation x for both the elements $x \in v_2^{-1}BP_*$ and $\lambda(x) \in E$.

In [9], Miller, Ravenel and Wilson introduced elements $x_n \in v_2^{-1}BP_*$ defined by

$$(3.2)\qquad \begin{aligned} x_0 &= v_2,\\ x_1 &= v_2^p - v_1^pv_2^{-1}v_3\\ x_2 &= x_1^p - v_1^{p^2-1}v_2^{p^2-p+1} - v_1^{p^2+p-1}v_2^{p^2-2p}v_3\\ x_n &= x_{n-1}^p - 2v_1^{a_n-p}v_2^{p^n-p^{n-1}+1} \quad \text{for } n \geq 3,\end{aligned}$$

where $a_n = p^n + p^{n-1} - 1$, and showed that

$$(3.3)\qquad d_0(x_n) = \varepsilon_n v_1^{a_n}v_2^{p^n-p^{n-1}}t_1 \quad\text{in } \Omega_\Gamma^1 v_2^{-1}A/(p, v_1^{1+a_n})$$

for $n > 0$ and $\varepsilon_n = \min\{n, 2\}$. Consider subsets of $\mathcal{B}'$ and $\mathcal{B}$ in Introduction given by

$$\begin{aligned}\mathcal{B}'(i) &= \{(s,n,j) \in Z_+^3 \mid s,j > 0,\ p \nmid s \text{ and } j \leq p^ia_{n-i},\\ &\qquad\qquad \text{and } j \leq p^n \text{ if } s = 1\},\\ \mathcal{B}^{(r)} &= \{(s,n+r,j,i) \in Z_+^4 \mid s,j > 0,\ p \nmid s \text{ and } p^i | j \leq p^ra_{n-i},\\ &\qquad\qquad \text{and } j \leq p^{n+r} \text{ if } s = 1\}.\end{aligned}$$

Then,

$$\mathcal{B}' = \bigcup_{i\geq 0} \mathcal{B}'(i) \quad \text{and} \quad \mathcal{B} = \bigcup_{r\geq 0} \mathcal{B}^{(r)}.$$

Note that x_n^s belongs to $\Omega_\Gamma^0 A/(p, v_1^j)$ (resp. $\Omega_\Gamma^0 A/(p^i, v_1^j)$) if $(s,n,j) \in \mathcal{B}'$ (resp. $(s,n,j,i) \in \mathcal{B}$) (*cf.* [9]), and that there is a monomorphism $A/(p,v_1^j) \hookrightarrow N_1^1$ (resp. $A/(p^i,v_1^j) \hookrightarrow N_0^2$) sending x to x/v_1^j (resp. $x/p^i v_1^j$). These with (3.3) imply that

$$x_n^s/v_1^j \in H^0 N_1^1 \quad \text{for } (s,n,j) \in \mathcal{B}', \text{ and}$$
$$x_n^s/p^{i+1}v_1^j \in H^0 N_0^2 \quad \text{for } (s,n,j,i) \in \mathcal{B},$$

and further that

$$x_{n-i}^{sp^i}/v_1^j \in H^0 N_1^1 \quad \text{for } (s,n,j) \in \mathcal{B}'(i), \text{ and}$$
$$x_{n-i}^{sp^{r+i}}/p^{i+1}v_1^j \in H^0 N_0^2 \quad \text{for } (s,n+r,j,i) \in \mathcal{B}^{(r)}.$$

Using these elements, we define the β-elements by

$$\begin{aligned} \beta'_{sp^n/j} &= \delta'_0(x_{n-i}^{sp^i}/v_1^j) \in H^1 A/(p) \\ &\qquad \text{for } (s,n,j) \in \mathcal{B}'(i) - \mathcal{B}'(i+1), \text{ and} \\ \beta_{sp^{n+r}/j,i+1} &= \delta_1\delta_0(x_{n-i}^{sp^{r+i}}/p^{i+1}v_1^j) \in H^2 A \\ &\qquad \text{for } (s,n+r,j,i) \in \mathcal{B}^{(r)} - \mathcal{B}^{(r+1)} \end{aligned} \tag{3.4}$$

in the E_2-terms of Adams-Novikov spectral sequences computing $\pi_*(M)$ and $\pi_*(S^0)$, respectively, for $A = BP_*$, and $\pi_*(L_2M)$ and $\pi_*(L_2S^0)$ for $A = E$. Here we notice that β-elements in [9] are defined by using x_n instead of $x_{n-i}^{p^i}$ as is done here. The subscripts of β-elements are given as follows:

$$\beta_{a/b,c} = \delta_1\delta_0((v_2^a + v_1 x)/p^c v_1^b)$$

for some $x \in BP_*$ such that $(v_2^a + v_1x)/p^cv_1^b \in H^0N_0^2$. Thus our β's are good to be considered. We abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$, $\beta_{sp^n/1}$ to β_{sp^n} and $\beta'_{sp^n/1}$ to β'_{sp^n} as is our custom.

From here on, in this section, we set $(A,\Gamma) = (E, E_*E)$.

In this paper, we use the same notation for both a homology class and its representing cycle.

Define an integer $k(n)$ for each positive integer n by

$$k(n) = \frac{p^n - (-1)^n}{p+1}.$$

Note that $k(2n+1) = pk(2n)+1$. We have $v_2t_1^{p^2} \equiv v_2^p t_1 \mod (p, v_1)$ in Γ, where $\eta_R(v_i) = 1 \otimes v_i \cdot 1 = 0$ for $i > 2$ by the action of BP_* on E. Therefore, inductively we obtain

$$t_1^{p^i} \equiv \begin{cases} v_2^{k(i)}t_1 & \mod (p, v_1) \quad \text{for even } i, \\ v_2^{k(i)-1}t_1^p & \mod (p, v_1) \quad \text{for odd } i, \end{cases} \tag{3.5}$$

in E_*E.

Lemma 3.6. *Consider integers s, n, j and i such that $(s,n,j) \in \mathcal{B}'(i)$. Then in the cobar complex $\Omega^1_{E_*E}E/(p)$,*

$$\beta'_{sp^n/j} \equiv \varepsilon_{n-i}v_1^{p^n+p^{n-1}-p^i-j}v_2^{e(s,n;i)}t_1 \qquad \mod (v_1^{p^n+p^{n-1}-p^i-j+1})$$

for even i with $0 \le i < n$ and

$$\beta'_{sp^n/j} \equiv \varepsilon_{n-i}v_1^{p^n+p^{n-1}-p^i-j}v_2^{e(s,n;i)-1}t_1^p \mod (v_1^{p^n+p^{n-1}-p^i-j+1})$$

for odd i with $0 < i < n$.

Here $\varepsilon_k = \min\{2,k\}$ for $k > 0$ and

$$e(s,n;i) = sp^n - p^{n-1} + k(i).$$

Proof. By the definition of δ', $\delta'(x_{n-i}^{sp^i}/v_1^j)$ is obtained by $v_1^{-j}d_0(x_{n-i}^{sp^i})$. Noticing that $d_0(x) = \eta_R(x) - x$, we obtain that

$$d_0(x_{n-i}^{sp^i}) \equiv s\varepsilon_{n-i}v_1^{p^ia_{n-i}}v_2^{sp^n-p^{n-1}}t_1^{p^i}$$

$\mod (p, v_1^{p^ia_{n-i}+p^i})$ by (3.3) and the binomial theorem. Now apply (3.5), and we have the desired result.

The Ext-group $H^2E/(p,v_1)$ was computed by Ravenel [15] to be a $F_p[v_2, v_2^{-1}]$-vector space spanned by the basis $\{h_0 \otimes \zeta_2, h_1 \otimes \zeta_2, g_0, g_1\}$. Here $\zeta_2 \in E_*E$, $g_0, g_1 \in E_*E \bigotimes_E E_*E$ are given by

$$\begin{aligned} \zeta_2 &= v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}), \\ g_0 &= v_2^{-p}(t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p) \quad \text{and} \\ g_1 &= v_2^{-p^2-1}(t_2^p \otimes t_1^{p^3} + t_1^p \otimes t_2^{p^2}). \end{aligned}$$

In the following, ζ denotes a cycle that is congruent to $\zeta_2 \bmod (p, v_1)$. Such a cycle is known to exist in E_*E/J for any ideal $J = (p^i, v_1^j)$ by the results [19] on ζ. We introduce an element b_0 of $E_*E \otimes_E E_*E$ defined by

$$b_0 = \frac{1}{p}\sum_{k=0}^{p-1}\binom{p}{k} t_1^k \otimes t_1^{p-k}.$$

This is denoted by $-T$ in [22] and [18]. Then [22;(3.2.5)] says that

$$b_0^{p^i} \equiv \begin{cases} v_2^{pk(i)} b_0 & \bmod (p, v_1) \quad \text{for even } i, \\ v_2^{pk(i)-p} b_0^p & \bmod (p, v_1) \quad \text{for odd } i. \end{cases}$$

The diagonal map $\Delta: E_*E \to E_*E \otimes_E E_*E$ sends t_3 to $t_3 \otimes 1 + 1 \otimes t_3 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p - v_2 b_0^p$. This gives the homologous relations

$$g_0 \equiv v_2^{-p+1} b_0^p \quad \text{and} \quad g_1 \equiv v_2^{-1} b_0$$

both $\bmod (p, v_1)$ in the cobar complex $\Omega^2_{E_*E}E$. Putting these together gives other homologous relations:

$$b_0^{p^i} \equiv \begin{cases} v_2^{k(i+1)} g_1 \bmod (p, v_1) & \text{for even } i, \\ v_2^{k(i+1)} g_0 \bmod (p, v_1) & \text{for odd } i. \end{cases} \tag{3.7}$$

Here $pk(i) + (-1)^i = k(i+1)$.

Lemma 3.8. *Let s, n, r, j and i be integers such that $p \nmid s > 0$, $r > 0$, $n > i \geq 0$, $p^i | j$, $1 \leq j \leq p^r a_{n-i}$ and $r \geq i$. Then we have*

$$\beta_{sp^{n+r}/j,i+1} \equiv \begin{cases} -\varepsilon_{n-i} s v_1^{p^r a_{n-i}-j} v_2^{e(s,n+r;i,r)} g_0 \bmod (p, v_1^{p^r a_{n-i}-j+1}) \\ \hfill \textit{for even } r, \\ -\varepsilon_{n-i} s v_1^{p^r a_{n-i}-j} v_2^{e(s,n+r;i,r)} g_1 \bmod (p, v_1^{p^r a_{n-i}-j+1}) \\ \hfill \textit{for odd } r. \end{cases}$$

Here the integers are defined by:

$$a_n = p^n + p^{n-1} - 1 \quad \text{and} \quad e(s, n; i, r) = sp^n - p^{n-i-1} + k(r).$$

Proof. First we compute $\delta_0(x_{n-i}^{sp^{r+i}}/p^{i+1}v_1^j) \in H^1 N_0^1$. Since $p^i | j$ and $d_0(v_1) = pt_1$, we see that $d_0(v_1^{-j} x_{n-i}^{sp^{r+i}}) \equiv v_1^{-j} d_0(x_{n-i}^{sp^{r+i}}) \bmod (p^{i+1})$ in $\Omega^1_{E_*E} v_1^{-1} E$, and moreover we have

$$d_0(x_{n-i}^{sp^{r+i}}) \equiv \varepsilon_{n-i} sp^i v_1^{p^r a_{n-i}} v_2^{sp^{n+r} - p^{n+r-i-1}} t_1^{p^r}$$

mod $(p^{i+1}, v_1^{p^r a_{n-i}+p^r})$ in $\Omega^1_{E_*E}E$ by (3.3) and the binomial theorem. Therefore the definition of δ_0 shows that

$$\delta_0(x_{n-i}^{sp^{r+i}}/p^{i+1}v_1^j) = \varepsilon_{n-i}sv_1^{p^r a_{n-i}-j}v_2^{sp^{n+r}-p^{n+r-i-1}}t_1^{p^r}/p + y$$

for some y divisible by a higher power of v_1 than that shown.

Note that $d_1(t_1^p) = -pb_0$, and we see that

$$\delta_1\delta_0(x_{n-i}^{sp^{r+i}}/p^{i+1}v_1^j) \equiv -\varepsilon_{n-i}sv_1^{p^r a_{n-i}-j}v_2^{sp^{n+r}-p^{n+r-i-1}}b_0^{p^{r-1}}$$

mod $(p, v_1^{p^r a_{n-i}-j+1})$ by the definition of the boundary homomorphism. Now use (3.7) to get the results.

4. Proofs of Theorems in §1. In this section (A, Γ) denotes again one of the Hopf algebroids $(BP_*, BP_*(BP))$ and (E, E_*E). The proofs of the theorems are based on the structure of $H^*M_1^1$. To state the results, we prepare some notations. First we consider subsets of the set Z of integers:

$$\Lambda(0) = \{s \mid p \nmid s(s+1)\}, \quad \Lambda(2) = \{tp^2 - 1 \mid t \in Z\} \quad \text{and}$$
$$\Lambda_0 = \{sp^n \mid n \geq 0,\ s \in \Lambda(0) \cup \Lambda(2)\}.$$

Next we define integers a_n and $A(m)$ for integers n and m by

$$a_n = p^n + p^{n-1} - 1; \text{ and}$$
$$A(sp^n) = \begin{cases} (p+1)\dfrac{p^n-1}{p-1} + 2 & \text{if } s \in \Lambda(0), \\ (p+1)\Big(p^{n+1} - p^n + \dfrac{p^n - 1}{p-1}\Big) + 2 & \text{if } s \in \Lambda(2). \end{cases}$$

Furthermore, we denote by $F_p[v_1]\{x/v_1^\infty\}$, the $F_p[v_1]$-module isomorphic to $F_p[v_1, v_1^{-1}]/F_p[v_1]$ generated by x/v_1^j's for $j > 0$ as a F_p-vector space. We also denote by $F_p[v_1]\langle x/v_1^a\rangle$, the $F_p[v_1]$-module generated by x/v_1^a that is isomorphic to $F_p[v_1]/(v_1^a)$.

By [22], we then have

(4.1) *$H^1M_1^1$ is the direct sum of $F_p[v_1]\{t_1/v_1^\infty\}$, $F_p[v_1]\{\zeta/v_1^\infty\}$ and*

$$\begin{array}{ll} F_p[v_1]\langle v_2^m t_1/v_1^{A(m)}\rangle & \text{for } m \in \Lambda_0, \\ F_p[v_1]\langle v_2^{tp-1}t_1^p/v_1^{p-1}\rangle & \text{for } t \in Z, \text{ and} \\ F_p[v_1]\langle x_n^s\zeta/v_1^{a_n}\rangle & \text{for } n \geq 0,\ s \in Z - pZ. \end{array}$$

By [18], we have

(4.2) *$H^2M_1^1$ is the direct sum of $F_p[v_1]\{t_1\otimes\zeta/v_1^\infty\}$ and*

$$\begin{array}{ll} F_p[v_1]\langle v_2^m t_1\otimes\zeta/v_1^{A(m)}\rangle & \text{for } m\in\Lambda_0,\\ F_p[v_1]\langle v_2^{tp-1}t_1^p\otimes\zeta/v_1^{p-1}\rangle & \text{for } t\in Z,\\ F_p[v_1]\langle v_2^s g_0/v_1\rangle & \text{for } s+1\in Z-pZ, \text{ and}\\ F_p[v_1]\langle v_2^{sp^n-(p^{n-1}-1)/(p-1)}g_1/v_1^{a_n}\rangle & \text{for } n>0,\ s+1\in Z-pZ. \end{array}$$

We denote the notation $sp^n/j,i$ or sp^n/j (not a fraction) by a capital letter K, so that β-elements are denoted by β_K and β'_K. The β-elements are, as is seen in the previous section, defined for both BP_* and E, and β_K (resp. β'_K) for E coincides with the λ-image of β_K (resp. β'_K) for BP_*, where $\lambda:(BP_*,BP_*(BP))\to(E,E_*E)$ is the canonical map of Hopf algebroids.

Lemma 4.3. *Let (s,n,j) and (u,r,k) be elements of $\mathcal{B}'$ and a and b integers ≥ 0, and put $K=sp^n/j$.*

1) *Suppose that $\beta'_K\equiv tv_1^a v_2^b t_1 \mod(v_1^{a+1})$ in $\Omega_\Gamma^1 A/(p)$ for some unit t. Then,*

$$\beta'_{up^r/k}\beta'_K\neq 0\in H^2A/(p)$$

if $0<k-a\leq A(up^r+b)$ and $up^r+b\in\Lambda_0$.

2) *Suppose that $\beta'_K\equiv tv_1^a v_2^b t_1^p \mod(v_1^{a+1})$ in $\Omega_\Gamma^1 A/(p)$ for some unit t. Then,*

$$\beta'_{up^r/k}\beta'_K\neq 0\in H^2A/(p)$$

if $0<k-a<p$ and $p|up^r+b+1$.

Proof. Consider the diagram

$$\begin{array}{ccccc} H^1M_1^0 & \longrightarrow & H^1N_1^1 & \xrightarrow{\delta_1} & H^2A/(p)\\ & & \downarrow\lambda_* & & \\ & & H^1M_1^1, & & \end{array}$$

in which the upper sequence is the exact sequence associated to the short exact one (2.1). Note that λ_* is the identity if $A=E$. Since $H^*A/(p)$ acts on $H^*N_1^*$ and $H^*M_1^*$, we see that

$$\beta'_{up^r/k}\beta'_K=\delta_0(x_r^u/v_1^k)\beta'_K=\delta_1(x_r^u\beta'_K/v_1^k).$$

The structure of $H^1M_1^0$ is given in [15] to be $F_p[v_1, v_1^{-1}]\{t_1\}$. Thus in our case δ_1 maps $x_r^u\beta'_K/v_1^k$ monomorphically. Therefore it suffices to show that $\lambda_*(x_r^u\beta'_K/v_1^k) \neq 0$. Using the hypothesis, we see that

$$\lambda_*(x_r^u\beta'_K/v_1^k) = \begin{cases} sv_2^{up^r+b}t_1/v_1^{k-a} & \text{for 1)}, \\ sv_2^{up^r+b}t_1^p/v_1^{k-a} & \text{for 2)}. \end{cases}$$

Compare now the powers of v_1 and v_2 with those of (4.1), and we obtain the lemma.

We define integers $c(n)$ and $l(n)$ for each non-negative integer n so that

$$n = c(n)p^{l(n)} - \frac{p^{l(n)}-1}{p-1}, \quad l(n) \geq 0 \text{ and } p \not| \, c(n)+1.$$

Note that $c(n)$ and $l(n)$ are uniquely determined for each n. In a similar fashion to the above lemma, we prove the following

Lemma 4.4. *Let (s,n,j,i) and (u,r,k) be elements of $\mathcal{B}$ and $\mathcal{B}'$, respectively, and a and b non-negative integers. Put $K = sp^n/j, i+1$.*

1) *Suppose $\beta_K = sv_1^av_2^bg_0 \mod (p, v_1^{a+1})$ in $\Omega^2_{E_*E}E$. Then,*

$$\beta'_{up^r/k}\beta_K \neq 0 \in H^3A/(p)$$

if $k = a+1$ and $p \not| \, up^r + b + 1$.

2) *Suppose $\beta_K = sv_1^av_2^bg_1 \mod (p, v_1^{a+1})$ in $\Omega^2_{E_*E}E$. Then,*

$$\beta'_{up^r/k}\beta_K \neq 0 \in H^3A/(p)$$

if $0 < k-a \leq a_{l+1}$, $p|c$ and $p^2 \not| \, c+p$ for $l = l(up^r+b)$ and $c = c(up^r+b)$.

Proof. Since $H^2M_1^0 = 0$ by [15], $\delta_2\colon H^2N_1^1 \to H^3A/(p)$ is a monomorphism. So we will find a condition that $\lambda_*(x_r^u\beta_K/v_1^k) = x_r^u\beta_K/v_1^k \neq 0$ in $H^2M_1^1$ for the localization map $\lambda_*\colon H^2N_1^1 \to H^2M_1^1$. This will give the lemma, since all Ext-groups here are H^*A-modules. The hypothesis shows

$$\lambda_*(x_r^u\beta_K/v_1^k) = \begin{cases} sv_2^{up^r+b}g_0/v_1^{k-a} & \text{for 1)}, \\ sv_2^{up^r+b}g_1/v_1^{k-a} & \text{for 2)}. \end{cases}$$

Now the lemma follows from (4.2).

Proof of Theorem AII. First consider the case that $\lambda_*(\beta_K) = \beta_K$ has a factor g_0, where $K = sp^{n+r}/j, i+1$. Lemma 3.8 says that r is even $\geq i$ and

$$(4.5) \quad a = p^r a_{n-i} - j \text{ and } b = sp^{n+r} - p^{n+r-i-1} + k(r)$$

for a and b in Lemma 4.4. Thus we have conditions:

$$(4.6) \quad k = p^r a_{n-i} - j + 1 \text{ and } p \nmid tp^m + sp^{n+r} - p^{n+r-i-1} + k(r) + 1$$

for a non-trivial product $\beta'_{tp^m/k}\beta_K$. In our case, $m = 0$, $k = 1$ and $j = p^r a_{n-i}$, and hence the second condition of (4.6) is now rewritten to be $p \nmid t$, since $k(r) \equiv -1 \mod p$ for even r.

Consider now an odd integer $r \geq i$. Then, Lemma 3.8 also shows (4.5) for this case, and Lemma 4.4 2) similarly yields the desired condition.

Proof of Theorem BII. Suppose that $(u, r, k), (s, n, j) \in \mathcal{B}'$ and consider an integer i such that $0 < i < n$.

First suppose that i is even greater than 0, and so $n > i \geq 2$. Then integers a and b in Lemma 4.3 are

$$p^n + p^{n-1} - p^i - j \text{ and } e(s, n; i) = sp^n - p^{n-1} + k(i),$$

respectively, by Lemma 3.6. By this, we have the condition

$$(4.7) \quad \begin{array}{l} 0 < k - p^n - p^{n-1} + p^i + j \leq A(up^r + sp^n - p^{n-1} + k(i)) \quad \text{and} \\ up^r + sp^n - p^{n-1} + k(i) \in \Lambda_0, \end{array}$$

which certifies the non-trivial product $\beta'_{up^r/k}\beta'_{sp^n/j}$.

For the case $r = 1$, suppose $u = u'p^2 - 1$ for some u' and $p^n + p^{n-1} - p^i < k + j \leq p^n + p^{n-1} - p^i + p^2 + 1$. Then $up^r + b = up + sp^n - p^{n-1} + k(i) \in \Lambda_0$ and $A(up^r + b) = p^2 + 1$. Therefore this case satisfies (4.7) and we have 1).

If $r = 0$, then put $u + sp^n - p^{n-1} + k(i) = tp^m$ with $t \in \Lambda(0) \cup \Lambda(2)$, and we have the second.

We turn now to the case that i is odd, which indicates $n > i \geq 1$. Then the integers a and b in the hypothesis of Lemma 4.3 are

$$a = p^n + p^{n-1} - p^i - j \text{ and } b = sp^n - p^{n-1} + pk(i-1)$$

by Lemma 3.6, and the condition is

$$0 < k - p^n - p^{n-1} + p^i + j < p \text{ and } p | up^r + sp^n - p^{n-1} + pk(i-1),$$

which is valid if $p^n + p^{n-1} - p^i < k + j < p^n + p^{n-1} - p^i + p$ and $r > 0$. Thus we complete the proof.

We note that the conditions (4.6) and (4.7) do not yield any more non-trivial relations than we have obtained. In fact, in the proof of Theorem AII, if $m > 0$, then the second condition of (4.6) indicates $r = 0$, since $k(r) \equiv -1 \mod p$ if r is even and ≥ 2. This contradicts to the condition $r \geq i > 0$. In the proof of Theorem BII, if $r \geq 2$, then $up^r + sp^n - p^{n-1} + k(i) \equiv p - 1 \mod p^2$, since i is even. This contradicts to the condition that it belongs to Λ_0. In a similar fashion, we see that the relations for odd r of Theorem AII are also the best results that we get from Lemma 4.4 2).

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