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A NOTE ON THE K-THEORY OF CONSTRUCTIBLE SHEAVES OVER A CURVE

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Introduction. Let X be a smooth connected curve and let $Const(X)$ be the category of constructible sheaves over X. Since $Const(X)$ is an abelian category, we can define the K-group of $Const(X)$. The main result of this note is

Theorem.

$$K_*(Const(X)) \cong \bigoplus_{x\in X_1} K_*(Const(k(x))) \oplus K_*(\mathcal{R}(X))$$

where X_1 is the set of closed points of X and $\mathcal{R}(X)$ is a category related to the representations of absolute Galois group of its generic point (cf. Lemma 4).

In this note we use the Waldhausen's K-theory machine [4],[5]. Then the fibration theorem [4, Theorem 1.6.4] shows there exists a spectral sequence similar to Quillen spectral sequence of K-theory of coherent shaves [3, Theorem 5.4], and we calculate the E_1-term of this spectral sequence using the approximation theorem [4, Theorem 1.6.7].

1. The K-theory of constructible sheaves. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}^\bullet(\mathcal{A})$ be the category of its complexes. Let $co(\mathcal{C}^\bullet(\mathcal{A}))$ (resp. $quot(\mathcal{C}^\bullet(\mathcal{A}))$) consist of all degreewise monomorphisms (resp. epimorphisms). Let $w(\mathcal{C}^\bullet(\mathcal{A}))$ consist of all quasi-isomorphisms. Then $\mathcal{C}^\bullet(\mathcal{A})$ becomes a bi-Waldhausen category which satisfies the saturation and extension axioms, and the usual cylinder and cocylinder functors satisfy cylinder and cocylinder axioms [4],[5]. (Throughout this note, we denote a (bi-)Waldhausen category simply $\mathcal{C}$ or $w\mathcal{C}$ when the choice of $w(\mathcal{C})$ is particulaly important.)

Then we define the K-Theory of $\mathcal{A}$ by

$$\begin{aligned} K_i(\mathcal{A}) &= \pi_{i+1}(B_\bullet \mathcal{Q}(\mathcal{A})) = \pi_{i+1}(wS_\bullet\mathcal{C}^\bullet(\mathcal{A})) \\ &= \pi_{i+1}(wS_\bullet\mathcal{C}^\bullet(\mathcal{A})^{op}). \quad \text{(See [3],[4],[5].)} \end{aligned}$$

Now let X be a quasi-compact, quasi-separated scheme and R be a Noetheian ring. We denote simply $Const(X)$ the category of constructible sheaves of abelian groups or R-modules. Note that $Const(X)$ is an abelian category [6, IX]. Therefore we have the K-theory of constructible shraves $K_*(Const(X))$.

For any morphism $f: X \to Y$, $f^*: Const(Y) \to Const(X)$ is an exact functor, hence it induces a homomorphism of K-groups which will be denoted by

$$f^*:\ K_*(Const(Y)) \longrightarrow K_*(Const(X)).$$

In this way K_* becomes a contravariant functor from schemes to abelian groups.

We now consider the situation: $i: Z \to X$ is a closed subscheme and $j: U \to X$ its complement. Let $v(C^\bullet(Const(X)))$ be the subcategory of $C^\bullet(Const(X))$ whose morphisms are f so that j^*f is a quasi-isomorphism of $C^\bullet(Const(U))$. Then $v(C^\bullet(Const(X)))$ defines another structure of bi-Waldhausen category, denoted by $vC^\bullet(Const(X))$, satisfying the (*co-*) cylinder, saturation, and extension axioms. Therefore by the localization theorem, we have a homotopy fiber sequence:

$$wS_\bullet C^\bullet(Const(X))^v \longrightarrow wS_\bullet C^\bullet(Const(X)) \longrightarrow vS_\bullet C^\bullet(Const(X)).$$

Since the funtors $i_*: Const(Z) \to Const(X)$ and $j^*: Const(X) \to Const(U)$ are exact, they define exact functors of Waldhausen categories:

$$i_*:\ wC^\bullet(Const(Z)) \longrightarrow wC^\bullet(Const(X))^v,$$
$$j^*:\ vC^\bullet(Const(X)) \longrightarrow wC^\bullet(Const(U)).$$

Lemma 1. *The following morphisms are homotopy equivalences:*

$$i_*:\ wS_\bullet C^\bullet(Const(Z)) \longrightarrow wS_\bullet C^\bullet(Const(X))^v,$$
$$j^*:\ vS_\bullet C^\bullet(Const(X)) \longrightarrow wS_\bullet C^\bullet(Const(U)).$$

Proof. For i_*: Let $F^\bullet \in C^\bullet(Const(X))^v$, $G^\bullet \in C^\bullet(Const(Z))$, and $\varphi: F^\bullet \to i_*G^\bullet$. Put $G'^\bullet = T^\vee(i^*F^\bullet \to G^\bullet)$, where $T^\vee$ is the usual cocylinder functor, φ' the composition $F^\bullet \to i_*i^*F^\bullet \to i_*G'^\bullet$, and α the canonical projection $G'^\bullet \to G^\bullet$. Then α is a fibration and $\varphi = \alpha\circ\varphi'$. Since $j_!$ is exact and $j^*F^\bullet$ is acyclic, the exact sequence

$$0 \longrightarrow j_!j^*F^\bullet \longrightarrow F^\bullet \longrightarrow i_*i^*F^\bullet \longrightarrow 0$$

shows $F^\bullet \to i_* i^* F^\bullet$ is a qusi-isomorphism, hence so is φ'. Hence i_* satisfies the (dual of) approximation axioms. For j^*: Let $F^\bullet \in \mathcal{C}^\bullet(Const(X))$, $E^\bullet \in \mathcal{C}^\bullet(Const(U))$, and ψ: $E^\bullet \to j^* F^\bullet$. Put $F'^\bullet = T^\vee(j_! E^\bullet \to F^\bullet)$, ψ' the composition $E^\bullet \to j^* j_! E^\bullet \to j^* F'^\bullet$, and β the canonical projection. Then these data show j^* satisfies the approximation axioms.

Theorem 2. *We have an isomorphism:*

$$K_*(Const(X)) \cong K_*(Const(Z)) \oplus K_*(Const(U)).$$

Proof. By the above lemma, we have a long exact sequence:

$$\begin{aligned} \cdots \longrightarrow K_i(Const(Z)) \longrightarrow K_i(Const(X)) \longrightarrow K_i(Const(U)) \longrightarrow \\ \longrightarrow K_{i-1}(Const(Z)) \longrightarrow \cdots , \end{aligned}$$

and the functors $j_!$ and i^* give a splitting of this exact sequence.

Let $w_p(\mathcal{C}^\bullet(Const(X)))$ be the subcategory of $\mathcal{C}^\bullet(Const(X))$ whose morphisms are f so that the support of $H^*(T^\vee(f))$ is of codimension $\geq p$. Then we have homotopy fiber sequences:

$$\begin{aligned} wS_\bullet \mathcal{C}^\bullet(Const(X))^{w_{p-1}} &\longrightarrow wS_\bullet \mathcal{C}^\bullet(Const(X))^{w_p} \\ &\longrightarrow w_{p+1} S_\bullet \mathcal{C}^\bullet(Const(X))^{w_p}. \end{aligned}$$

The usual arguments show

Theorem 3. *There is a spectral sequence*

$$E_1^{p,q} = \pi_{-p-q-1}(w_{p+1} S_\bullet \mathcal{C}^\bullet(Const(X))^{w_p}) \Longrightarrow K_{-p-q}(Const(X)),$$

which is convergent when X has finite (Krull) dimension.

In the next section, we calculate this spectral sequence in the case of curves.

2. Proof of main theorem. Let X be a normal connected irreducible scheme of dimension one, and let $g: \eta \hookrightarrow X$ be its generic point, and X_1 be the set of closed points. Write $G_\eta = \mathrm{Gal}(k(\bar{\eta})/k(\eta))$ and $G_x = \mathrm{Gal}(k(\bar{x})/k(x))$ where $\bar{\eta}$ and $\bar{x}$ are chosen so that $k(\bar{\eta}) = k(\eta)^{sep}$, $k(\bar{x}) = k(x)^{sep}$. For each $x \in X_1$, choose an embedding $\mathcal{O}_{X,\bar{x}} \to k(\bar{\eta})$.

Then we have a filtration $G_\eta \supseteq D_x \supseteq I_x \supseteq \{1\}$, and an isomorphism $D_x/I_x \cong G_x$. Using these notations, we have an equivalence of categories

$$Const(X) \cong \left\{ \begin{array}{l} (M_\eta, (M_x, \phi_x)_{x \in X_1}); \\ \quad M_\eta\text{: a finite } G_\eta\text{-Module,} \\ \quad M_x\text{: a finite } G_x\text{-Module,} \\ \quad \phi_x : M_x \to M_\eta^{I_x}\text{: a } G_x\text{-homomorphism which} \\ \quad\quad \text{satisfies there exists a non-empty open sub-} \\ \quad\quad \text{set } U \text{ of } X \text{ s.t. } M_x \to M_\eta^{I_x} \hookrightarrow M_\eta \text{ is an} \\ \quad\quad \text{isomorphism for } x \in U. \end{array} \right\},$$

where "finite" means a finite group if we consider sheaves of abelian groups, or a R-module with finite representation if we consider sheaves of R-modules. In terms of this identification, a morphism $M^\bullet \to N^\bullet$ is in $w_1(C^\bullet(Const(X)))$ if and only if $M^\bullet_\eta \to N^\bullet_\eta$ is a quasi-isomorphism.

Lemma 4. *We have the following isomorphisms:*

$$\pi_{i+1}(wS_\bullet C^\bullet(Const(X)^{w_1}) \cong \bigoplus_{x \in X_1} K_i(Const(k(x))),$$
$$\pi_{i+1}(w_1 S_\bullet C^\bullet(Const(X)) \cong K_i(\mathcal{R}(X)),$$

where $\mathcal{R}(X)$ is the full subcategory of finite G_η-modules whose objects are M such that $M^{I_x} \to M$ are isomorphisms for $x \in U$ for some non-empty open subset U of X.

Proof. Let $Const(X_1)$ be the full subcategory of $Const(X)$ consisting of objects whose supports are of dimension zero. Clearly,

$$K_i(Const(X_1)) \cong \bigoplus_{x \in X_1} K_i(Const(k(x))).$$

Consider the functor:

$$\Phi : C^\bullet(Const(X_1)) \longrightarrow C^\bullet(Const(X))^{w_1}$$

defined by $\Phi A^\bullet = (0, (A^\bullet_x, 0))$. Let $B^\bullet \in C^\bullet(Const(X))^{w_1}$, $A^\bullet \in C^\bullet(Const(X_1))$, and $f: B^\bullet \to \Phi A^\bullet$. Choose a non-empty open subset U^i of X so that $A^i_x = 0$ and $B^i_x \to B^i_\eta$ are isomorphisms for $x \in U^i$. Define the complex $C^\bullet_x$ by

$$C^i_x = \begin{cases} B^i_x & \text{if } x \in \overline{U^i} \cup (\overline{U^{i-1}} \cap \overline{U^{i+1}}); \\ \mathrm{Im}[d: B^{i-1}_x \to B^i_x] & \text{if } x \in \overline{U^{i-1}} \cap U^i \cap U^{i+1}; \\ \mathrm{Coim}[d: B^{i-1}_x \to B^i_x] & \text{if } x \in U^{i-1} \cap U^i \cap \overline{U^{i+1}}; \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, the canonical maps $B_x^\bullet \to C_x^\bullet$ are quasi-isomorphisms, $(C_x^\bullet)_{x\in X_1}$ defines an object of $\mathcal{C}^\bullet(Const(X_1))$, denoted by $\mathcal{C}^\bullet$, and the composition $B^\bullet \to \Phi C^\bullet \to \Phi A^\bullet$ is equal to f. Put $B'^\bullet = T^\vee(C^\bullet \to A^\bullet)$, then we obtain a fibration $B'^\bullet \to A^\bullet$ and weak equivalence $B^\bullet \to \Phi C^\bullet \to \Phi B'^\bullet$. Hence we have an isomorphism

$$\pi_{i+1}(wS_\bullet\mathcal{C}^\bullet(Const(X))^{w_1}) \cong \bigoplus_{x\in X_1} K_i(Const(k(x))).$$

Define functors $g^*: Const(X) \to \mathcal{R}(X)$ and $g_!: \mathcal{R}(X) \to Const(X)$ by $g^*M = M_\eta$ and $g_!M_\eta = (M_\eta, M_\eta^{I_x}, incl.))$. Then the similar argument as lemma 1 shows

$$\pi_{i+1}(w_1S_\bullet\mathcal{C}_\bullet(Const(X))) \cong K_i(\mathcal{R}(X)).$$

Using this lemma, we obtain

Theorem 5. *Let X be a normal connected irreducible scheme of dimension one. Then there is an isomorphism:*

$$K_i(Const(X)) \cong \bigoplus_{x\in X_1} K_i(Const(k(x))) \oplus K_i(\mathcal{R}(X)).$$

The following corollary results from Theorem 2 and 5.

Corollary 6. *Let $X \hookrightarrow \bar{X}$ be an open immersion of curves where $\bar{X}$ is smooth connected irreducible. Then*

$$K_i(Const(X)) \cong \bigoplus_{x\in X_1} K_i(Const(k(x))) \oplus K_i(\mathcal{R}(\bar{X})).$$

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