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THE ATIYAH-SINGER INDEX THEOREM FOR G-EQUIVARIANT REAL ELLIPTIC FAMILIES

Dedicated to the Memory of Professor Masahisa Adachi

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0. Introduction. The purpose of this paper is to give the index theorem for G-equivariant Real elliptic families. Our main result is Theorem 9.1. (See also Definition 2.1, Definition 3.3 and Definition 5.16.)

Let X be a closed (i.e. compact, without boundary) smooth manifold and let TX denote its tangent bundle. Let X be embedded in $\boldsymbol{R}^n$ and let N denote the normal bundle of X in $\boldsymbol{R}^n$. Then, the tangent bundle TN has a complex vector bundle structure over TX and the periodicity map $K(TX) \to K(TN)$ is defined. The topological index is defined to be the composition $K(TX) \to K(TN) \to K(T\boldsymbol{R}^n) = K(\boldsymbol{R}^{2n}) = K^{-2n}(\text{point}) = K(\text{point})$ where $K(TN) \to K(T\boldsymbol{R}^n)$ is the extension homomorphism of the open inclusion $TN \to T\boldsymbol{R}^n$. On the other hand, any element of $K(TX)$ is expressed by a principal symbol of an elliptic pseudo-differential operator on X and the analytical index $K(TX) \to \boldsymbol{Z} = K(\text{point})$ is defined to be the Fredholm index of this operator. The Atiyah-Singer index theorem asserts that the topological index coincides with the analytical index.

The index theorem is generalized in [4] to the case of fiber bundles $Z \to Y$. Namely, let Y be a compact Hausdorff space and Z a fiber bundle over Y with fiber X and structure group Diff(X) where Diff(X) denotes the topological group of diffeomorphisms of X endowed with the C^∞-topology. Let T_FZ denote the tangent bundle along the fibers of Z. (Namely, T_FZ is a fiber bundle over Y with fiber TX.) Then, the analytical index and the topological index are defined to be the homomorphisms $K(T_FZ) \to K(Y)$ where the analytical index is defined by families of elliptic operators with parameter space Y.

So, taking refinements of K, one may introduce the following types of the index theorem. (G denotes a compact Lie group.)

I. $K(TX) \longrightarrow K(\text{point})$
II. $KR(TX) \longrightarrow KR(\text{point})$
III. $K_G(TX) \longrightarrow K_G(\text{point})$
IV. $KR_G(TX) \longrightarrow KR_G(\text{point})$

V. $K(T_F Z) \longrightarrow K(Y)$
VI. $KR(T_F Z) \longrightarrow KR(Y)$
VII. $K_G(T_F Z) \longrightarrow K_G(Y)$
VIII. $KR_G(T_F Z) \longrightarrow KR_G(Y)$

The index theorem of type III is given in [3]. Since the forgetting map $KR_G(\text{point}) \to K_G(\text{point})$ is injective, the index theorem of type IV gives no more information than that of type III. The index theorem of type V is given in [4] and the index theorem of type VI is given in [5]. So, it remains to give index theorems of type VII and type VIII. In this paper, our purpose is to give the index theorem of type VIII. Then, the index theorem of type VII is given by forgetting all involutions.

The contents of sections are as follows.

In section 1, we recall the notion of the KR_G-theory, show that TN (where N denotes the normal bundle of a Real G-embedding of X) has a Real G-vector bundle structure over TX and define the !-homomorphism.

In section 2, under the definition of the families in [4], we introduce the notion of Real G-families. A Real G-family is a Real G-fiber bundle over the parameter space Y such that the bundle itself, the actions of G and the involution are "smooth along the fibers". Here we assume that Z satisfies certain smoothness conditions (cf. Definition 2.1) which are satisfied if Z itself is smooth. Then, G-equivariant Real elliptic families are defined to be Real G-sections of the bundle of elliptic operators over Y. For the detailed constructions of various bundles, [4] §1 should be refered to.

In section 3, we first construct an equivariant fiber-wise embedding of Z into a (finite dimensional) trivial vector bundle over Y. Once this is done, we can define the topological index $KR_G(T_F Z) \to KR_G(Y)$ in the same way as in [3], [4] and it, of course, becomes the natural refinement of the topological index in [3], [4].

In section 4, we give the definition of the index of G-equivariant Real elliptic families which is the natural refinement of those in [3], [4] (cf. Proposition 4.1).

In section 5, using the result in section 4, we define the analytical index and show that the definition is well-defined. Here we use the homotopy invariance of the analytical index, which is dealt with systematically by considering the families.

The process to prove the index Theorem 9.1 is essentially the same as in [3], [4] and owes to the commutativities of certain diagrams (cf. §9). In section 6, 7 and 8, we show the commutativities of those diagrams. These procedure depend upon the calculations of analytical indices defined in section 5.

In section 6, considering the construction of extension homomorphisms, we show that the analytical index satisfies the excision axiom, namely, commutes with the extension homomorphism.

In section 7, we show that the analytical index satisfies the normalization axiom, namely, gives the inverse of the Bott periodicity map. Here, using a simple property of the analytical index and the result in [3], we can avoid troublesome calculations.

In section 8, considering the construction of multiplications, we show that the analytical index satisfies the multiplicative axiom, namely, commutes with the Thom homomorphism.

In section 9, using results in sections 6, 7 and 8, we give the proof of the index theorem which asserts that the topological index coincides with the analytical index.

In this paper throughout, we will use, without proof, the results of Atiyah-Singer which are explicitly written in [3], [4] and [5]. So, if necessary, they (in particular, [4]) should be referred to.

1. !-homomorphisms in KR_G-theory. Let G be a compact Lie group with an involution τ and let G_R be the semidirect product $G \times_\tau Z_2$. Namely, $G_R = \{ga \mid g \in G, a \in Z_2\}$ and $(g\tau)(g'a) = g\tau(g')\tau a$ for the generator τ of Z_2, $g, g' \in G$ and $a \in Z_2$. Note that G_R is a compact Lie group.

Definition 1.1 Let E be a complex vector bundle over X. E is called a Real G-vector bundle if it satisfies the following conditions:

(1.1.1) E and X are G_R-spaces and the projection $E \to X$ is a G_R-map,
(1.1.2) $g: E \to E$ is a complex linear bundle map for any $g \in G \subset G_R$,
(1.1.3) $\tau: E \to E$ is an antilinear bundle map for the generator τ of $Z_2 \subset G_R$.

Throughout this paper, τ denotes the generator of $Z_2 \subset G_R$ or the involutive action of this generator. A Real G-vector bundle has to be distinguished from a G_R-vector bundle and a real G-vector bundle.

Definition 1.2. If X is a compact G_R-space, $KR_G(X)$ denotes as usual the Grothendieck group of the category of Real G-vector bundles over X. If X is a locally compact G_R-space, $KR_G(X)$ denotes the kernel of the restriction $KR_G(X^+) \to KR_G(+)$ where $X^+ = X \cup \{+\}$ denotes the one point compactification of X. $KR_G(X)$ is a commutative ring with the Whitney sum and the tensor product. It is not difficult to see that, if the involutions on X and G are trivial, $KR_G(X)$ is isomorphic to $KO_G(X)$.

Example 1.3. Let X be a smooth G_R-manifold and let TX be the tangent bundle of X. Then, TX is a G_R-manifold with the involution $\tau(x,v) = (\tau x, -\tau_* v)$ for $x \in X$ and $v \in T_x X$. If we give X a G_R-invariant riemannian metric, TX can be identified with the cotangent bundle T^*X which is a G_R-manifold with the involution $\tau(x,w) = (\tau x, -\tau^* w)$ for $x \in X$ and $w \in T_x^* X$. TX is always regarded as a G_R-space with this involution. This convention is essential in the following arguments. Note that, even if the involution on X is trivial, the involution on TX is not trivial. Note moreover that, though the exponential map with respect to a G_R-invariant metric commutes with τ, it does not commute with the above involution on TX.

Let E be a Real G-vector bundle over a locally compact G_R-space X. Then, by tensoring the exterior complex $\Lambda(E)$, the Thom homomorphism $KR_G(X) \to KR_G(E)$ is defined as usual. In [2], Atiyah gives the Bott periodicity theorem in KR_G-theory.

Theorem 1.4 ([2] Theorem (5.1)). *Let X be a compact G_R-space and W a Real G-module. (Namely, there exists a representation $\rho: G_R \to GL_R(W)$ such that $\rho|_G: G \to GL_C(W)$ and $\rho(\tau)$ is antilinear.) Then, the Thom homomorphism $KR_G(X) \to KR_G(X \times W)$ is an isomorphism.*

Now, we define the !-homomorphism in KR_G-theory. Let X be a closed (i.e. compact, without boundary) smooth G_R-manifold, Y a smooth G_R-manifold and $i: X \to Y$ a G_R-embedding. Let $N = \{p: N \to X\}$ be the normal bundle of X in Y which is identified with the unit open disk bundle with respect to a G_R-invariant riemannian metric on Y. Then, by the exponential map, N is regarded as an open G_R-submanifold of Y (with the involution on Y). Let $TX = \{q: TX \to X\}$ be the tangent bundle of X. Then, $q^*(N \otimes C)$ is a Real G-vector bundle over TX with the involution $((x,v), \alpha + \sqrt{-1}\beta) \to ((\tau x, -\tau_* v), \tau\alpha - \sqrt{-1}\tau\beta)$ for $x \in X$, $v \in T_x X$ and $\alpha, \beta \in N_x$. On the other hand, the tangent bundle TN is a neighborhood of TX in TY and is isomorphic to $q^*(N \otimes \boldsymbol{C})$ as a G_R-fiber bundle over TX. In fact, the isomorphism $\phi: TN \to q^*(N \otimes \boldsymbol{C})$ is given by $\phi((x,\omega),\xi+\eta) = ((x,p_*\xi),\omega+\sqrt{-1}\eta)$ for $x \in X$, $\omega \in N_x$ and $\xi+\eta \in T_\omega N$ where η is vertical and ξ is horizontal with respect to a G_R-invariant metric. The Reality of ϕ is proved as follows:

$$\begin{aligned}
\phi\cdot\tau((x,\omega),\xi+\eta) &= \phi((\tau x,\tau\omega),-\tau_*\xi-\tau_*\eta) && \text{(cf. Example 1.3)}\\
&= ((\tau x,-p_*\tau_*\xi),\tau\omega-\sqrt{-1}\tau\eta) && \begin{pmatrix}\tau \text{ preserves the orthogonal}\\ \text{decomposition and } \tau_*\eta \text{ is}\\ \text{identified with } \tau\eta.\end{pmatrix}\\
&= ((\tau x,-\tau_* p_*\xi),\tau\omega-\sqrt{-1}\tau\eta) && (p \text{ commutes with } \tau.)\\
&= \tau((x,p_*\xi),\omega+\sqrt{-1}\eta) = \tau\cdot\phi((x,\omega),\xi+\eta).
\end{aligned}$$

So, TN is G_R-diffeomorphic to $q^*(N\otimes\boldsymbol{C})$ and $p_*: TN \to TX$ has a Real G-vector bundle structure. Hence, the Thom homomorphism $KR_G(TX) \to KR_G(TN)$ is defined by tensoring the exterior complex $\Lambda(TN) = \Lambda q^*(N\otimes \boldsymbol{C})$. The additive homomorphism $i_!: KR_G(TX) \to KR_G(TY)$ is then defined to be the composition of the Thom homomorphism with the extension homomorphism $k_*: KR_G(TN) \to KR_G(TY)$ of the open inclusion $k: TN \to TY$.

2. The families. Let X be a closed smooth manifold and let Diff(X) denote the topological group of diffeomorphisms of X endowed with the C^∞-topology. Let Y be a compact Hausdorff G_R-space.

Definition 2.1. Let $Z = \{\pi: Z \to Y\}$ be a G_R-fiber bundle with fiber X and structure group Diff(X) (so that Z is a G_R-space and π is a G_R-map). We call Z an X-family over Y if the G_R-action on Z satisfies the following conditions:

(2.1.1) $g: Z_y\ (=\pi^{-1}(y)) \to Z_{gy}$ is a (smooth) diffeomorphism for any $g \in G_R$ and any $y \in Y$,

(2.1.2) Let $B \to Y$ be the principal Diff(X)-bundle associated to Z. Namely, for any $y \in Y$, B_y consists of admissible maps $X \to Z_y$. Then, G_R acts continuously on B on the left by $gu = g\cdot u: X \to Z_y \to Z_{gy}$ for any $g \in G_R$ and any $u \in B_y$.

Remark 2.2. Let $Y =$ point and $Z = X$. Then, the condition (2.1.1) says that, while X is a continuous G_R-space, the image of the homomorphism $G_R \to$ Homeo(X) (where Homeo(X) denotes the topological group of homeomorphisms of X endowed with the compact-open topology) is contained in Diff(X), and the condition (2.1.2) says that $G_R \to \mathrm{Diff}(X)$ is continuous with respect to the C^∞-topology.

Remark 2.3. Let Y be a smooth G_R-manifold (possibly with boundary) and let Z be a smooth G_R-fiber bundle over Y with fiber X. Then, it is clear that Z is an X-family. It is also clear that the restriction of a smooth G_R-fiber bundle to any closed G_R-subset of Y is a family.

Remark 2.4. According to [9] (or [7]), we recall the notion of the (Γ,α,G_R)-bundle. Let Γ be a compact Lie group and let $\alpha: G_R \to \mathrm{Aut}(\Gamma)$ be a homomorphism. We assume that the adjoint of α $\tilde{\alpha}: G_R \times \Gamma \to \Gamma$ is smooth. Let $\Gamma \times_\alpha G_R$ denote the semidirect product. A principal Γ-bundle $P \to Y$ is called a principal (Γ,α,G_R)-bundle if P is a left G_R-space, the projection is a G_R-map and the action of G_R and Γ are related as $g{\cdot}(x\gamma) = (gx){\cdot}\alpha(g)(\gamma)$ for any $x \in P$, any $g \in G_R$ and any $\gamma \in \Gamma$. Let X be a closed smooth $\Gamma \times_\alpha G_R$-manifold. Then, regarding X as a Γ-space, we can associate to P a fiber bundle $Z = P \times_\Gamma X$ with fiber X, which is called a (Γ,α,G_R)-bundle. It is obvious that the diagonal action of $\Gamma \times_\alpha G_R$ on $P \times X$ induces a G_R-action on Z and Z becomes a G_R-fiber bundle over Y. Then, it is easy to see that Z satisfies the conditions (2.1.1), (2.1.2) and Z is an X-family.

For the sake of the following arguments, we note a simple remark.

Remark 2.5. Let $f: Z \to \boldsymbol{R}$ be any function on an X-family Z such that the restriction $f_y = f|_{Z_y}: Z_y \to \boldsymbol{R}$ is smooth for any $y \in Y$ and all Z_y-derivatives of f_y are continuous in the y-direction. Then, the G_R-averaging of f has the same properties. Namely, when we set $\tilde{f}(g,y,x) = f(g{\cdot}(y,x))$ where $g \in G_R$, $y \in Y$ and $x \in Z_y$, it follows from the conditions of Definition 2.1 that x-derivatives of $\tilde{f}$ are continuous in g and y. Hence, when we set $f'(y,x) = \int_{G_R} \tilde{f}(g,y,x)\,dg$ where dg denotes the normalized Haar measure on G_R (so that $\int_{G_R} dg = 1$), $f'(y,x)$ is smooth in x and x-derivatives are continuous in y.

From Remark 2.5, it follows the following lemma.

Lemma 2.6. *Let $f: Z \to M$ be a continuous G_R-map from an X-family Z to a compact G_R-manifold M. Then, f can be approximated by* *a G_R-map $f': Z \to M$ such that the restriction $f'_y = f'|_{Z_y}$ is smooth for any $y \in Y$ and all Z_y-derivatives of f'_y are continuous in the y-direction.*

Proof. Let U be any topological space and let $f_0(y,x): U \times \boldsymbol{R}^n \to \boldsymbol{R}$ be any continuous function of compact support. Let $\phi(x): \boldsymbol{R}^n \to \boldsymbol{R}$ be a smooth function such that $\phi(x) \geq 0$, $\operatorname{supp}(\phi)$ is compact and $\int_{\boldsymbol{R}^n} \phi(x)\,dx = 1$, and we set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Then, $f_\varepsilon^\infty(y,x) = \int_{\boldsymbol{R}^n} \phi_\varepsilon(x - v) f_0(y,v)\,dv$ is smooth in x, x-derivatives of f_ε^∞ are continuous in y and f_ε^∞ converges to f uniformly as $\varepsilon \to 0$. Since Z is locally of the form $U \times \boldsymbol{R}^n$, by using the partitions of unity on X and Y, this result is extended to the case of Z. Namely, let M be G_R-equivariantly embedded in a finite dimensional real G_R-module V (cf. [6]) and let δ be any positive real valued function on V. Then, there exists $f^\infty: Z \to V$ such that the restriction $f_y^\infty = f^\infty|_{Z_y}$ is smooth for any $y \in Y$, Z_y-derivatives of f_y^∞ are continuous in the y-direction and f^∞ is a δ-approximation to f. Let $p: N \to M$ be a smooth G_R-tubular neighborhood of M in V. We may assume that the image of f^∞ is contained in N. Then, by Remark 2.5, the composition of p with the G_R-averaging of f^∞ is the required one.

Now, let $E \to X$ be a smooth complex vector bundle and let $\operatorname{Diff}(X,E)$ denote the topological group of diffeomorphisms of E which map fibers to fibers linearly endowed with the C^∞-topology (cf. [4], §1).

Definition 2.7. Let $\tilde{E} = \{p: \tilde{E} \to Z\}$ be a Real G-vector bundle over Z. We call $\tilde{E}$ an E-family over Y if it satisfies the following conditions:

(2.7.1) $\pi \cdot p: \tilde{E} \to Y$ is a fiber bundle with fiber E and structure group $\operatorname{Diff}(X,E)$,

(2.7.2) $g: \tilde{E}_y \to \tilde{E}_{gy}$ is a smooth isomorphism for any $g \in G_R$ and any $y \in Y$ where $\tilde{E}_y = (\pi \cdot p)^{-1}(y) = \tilde{E}|_{Z_y}$,

(2.7.3) The left G_R-action on the associated principal $\operatorname{Diff}(X,E)$-bundle over Y is continuous (cf. (2.1.2)).

Let $p: E_0 \to Z$ be any (continuous) Real G-vector bundle.

Remark 2.8. Note that $\pi \cdot p: E_0 \to Y$ always has a local triviality in the topological sense. In fact, let y_0 be any point in Y and let U be an open neighborhood of y_0 such that $\pi^{-1}(U) \simeq U \times \pi^{-1}(y_0)$. Then, $(\pi \cdot p)^{-1}(U) = E_0|_{\pi^{-1}(U)}$ is isomorphic to $U \times (\pi \cdot p)^{-1}(y_0) = U \times (E_0|_{\pi^{-1}(y_0)})$ on $\pi^{-1}(y_0)$. Since both X ($\simeq \pi^{-1}(y_0)$) and Y are compact Hausdorff, this isomorphism can be extended to an open neighborhood of $\pi^{-1}(y_0)$ in $\pi^{-1}(U)$ and there exists an open neighborhood U' of y_0 in Y such that $(\pi\cdot p)^{-1}(U')$ is isomorphic to $U' \times (\pi\cdot p)^{-1}(y_0)$.

For the smoothness conditions in Definition 2.7, we have to take an approximation of the classifying map of E_0 which satisfies certain smoothness conditions. Let W be a Real G-module and let $G_r(q,W)$ denote the Grassmann manifold of q-dimensional subspaces of W. $G_r(q,W)$ is a compact G_R-manifold which is a finite approximation to the classifying space of Real G-vector bundles. Let $f: Z \to G_r(q,W)$ be the classifying map of E_0. By Lemma 2.6, there exists a G_R-map $f': Z \to G_r(q,W)$ which is G_R-homotopic to f, smooth in the Z_y-direction for any $y \in Y$ and all Z_y-derivatives are continuous in the y-direction. Then, the pull back $\tilde{E}$ of the canonical smooth Real G-vector bundle ξ over $G_r(q,W)$ by f' is a family which is isomorphic to E_0 (cf. [4], p.123, Remark 2). Then, in fact, the local triviality of $\tilde{E}$ is given as follows. Let y_0 be any point in Y and let U be an open neighborhood of y_0 such that $\pi^{-1}(U) \simeq U \times \pi^{-1}(y_0)$. Then, an isomorphism

$$\psi : \tilde{E}|_{\pi^{-1}(U)} = (f'|_{\pi^{-1}(U)})^*\xi \longrightarrow U \times (\tilde{E}|_{\pi^{-1}(y_0)}) = U \times (f'|_{\pi^{-1}(Y_0)})^*\xi$$

is given by $\psi((y,x),v) = (y,(x,c(v)))$ where $(y,x) \in \pi^{-1}(U) \simeq U \times \pi^{-1}(y_0)$, $v \in \xi_{f'(y,x)}$ and $c: \xi_{f'(y,x)} \to \xi_{f'(x)}$ denotes the parallel translation along the unique geodesic (we assume that U is sufficiently small) from $f'(y,x)$ to $f'(x)$ with respect to a smooth riemannian metric on $G_r(q,W)$ and a smooth connection on ξ. Namely, the following proposition holds.

Proposition 2.9. *For any Real G-vector bundle E_0 over Z, there exists a family $\tilde{E}$ over Y which is isomorphic to E_0.*

Note that, if E_0 is a smooth Real G-vector bundle over a smooth G_R-fiber bundle Z, then E_0 itself is a family.

Let E and F be smooth complex vector bundles over X. Let $\tilde{E}$ (resp. $\tilde{F}$) be an E (resp. F)-family over Y. According to [4], §1, we recall the notion of families of sections and operators. Let $C^\infty(X,E)$ denote the Fréchet space of smooth sections $s: X \to E$ and let $H_s(X,E)$ denote the Sobolev space of distributional sections whose k-th derivatives are L^2-measurable for any $k \leq s$ (cf. [8]). Let $C^\infty(Z,\tilde{E})$ (resp. $H_s(Z,\tilde{E})$) denote the (infinite dimensional) vector bundle over Y with fiber $C^\infty(X,E)$ (resp. $H_s(X,E)$) which is associated to $\tilde{E}$. Namely, $C^\infty(X,\tilde{E}) = B(\tilde{E}) \times_{\mathrm{Diff}(X,E)}$

$C^\infty(X,E)$ where $B(\tilde{E})$ is the associated principal Diff(X,E)-bundle of $\tilde{E}$. Let $\mathcal{P}^m(X;E,F)$ denote the space of pseudo-differential operators $C^\infty(X,E) \to C^\infty(X,F)$ of order m and $\mathcal{P}^m_c(X;E,F)$ its completion in the Fréchet space $\prod_s Op^m_s(X;E,F)$ where $Op^m_s(X;E,F)$ denotes the Banach space of continuous operators $H_s(X,E) \to H_{s-m}(X,F)$ endowed with the operator norm. Let $\mathrm{Ell}^m(X;E,F)$ denote the subspace of $\mathcal{P}^m_c(X;E,F)$ which consists of elliptic operators of order m (cf. Remark 5.2). Let $\mathcal{P}^m(Z;\tilde{E},\tilde{F})$ (resp. $\mathcal{P}^m_c(Z;\tilde{E},\tilde{F})$, $\mathrm{Ell}^m(Z;\tilde{E},\tilde{F})$) denote the fiber bundle over Y with fiber $\mathcal{P}^m(X;E,F)$ (resp. $\mathcal{P}^m_c(X;E,F)$, $\mathrm{Ell}^m(X;E,F)$) which is associated to $\tilde{E}$ and $\tilde{F}$. (For details, see [4], §1.) Now, the G_R-action on the associated principal Diff(X,E)-bundle of $\tilde{E}$ defines Real G-vector bundle structures on $C^\infty(Z,\tilde{E})$ and $H_s(Z,\tilde{E})$ by $C^\infty(Z,\tilde{E})_{gy} = C^\infty(Z_{gy},\tilde{E}_{gy}) \ni g\jmath = g\cdot\jmath\cdot g^{-1}$ for any $g \in G_R$, $y \in Y$ and $\jmath \in C^\infty(Z,\tilde{E})_y$ (cf. (2.7.3)). Then, the G_R-action on $\mathcal{P}^m_c(Z;\tilde{E},\tilde{F})$ is defined by $\mathcal{P}^m_c(Z;\tilde{E},\tilde{F})_{gy} \ni gQ = g\cdot Q\cdot g^{-1}$:

$$C^\infty(Z,\tilde{E})_{gy} \longrightarrow C^\infty(Z,\tilde{E})_y \longrightarrow C^\infty(Z,\tilde{F})_y \longrightarrow C^\infty(Z,\tilde{E})_{gy}$$

for any $g \in G_R$, $y \in Y$ and $Q \in \mathcal{P}^m_c(Z;\tilde{E},\tilde{F})_y$. Moreover, $\mathcal{P}^m(Z;\tilde{E},\tilde{F})$ and $\mathrm{Ell}^m(Z;\tilde{E},\tilde{F})$ become G_R-subspaces of $\mathcal{P}^m_c(Z;\tilde{E},\tilde{F})$.

Definition 2.10. A continuous G_R-section $P = \{P_y\}_{y\in Y}: Y \to \mathrm{Ell}^m(Z;\tilde{E},\tilde{F})$ is called a G-equivariant Real elliptic family of order m and, throughout this paper, is denoted by *G.R.E.F.*.

Now, let T_FZ denote the tangent bundle along the fibers of Z. Namely, T_FZ is a fiber bundle over Y with fiber TX. A metric on Z is by definition a continuous euclidean metric ρ on T_FZ which is smooth along the fibers, namely, such that the restriction ρ_y of ρ to $(T_FZ)_y = T(Z_y)$ is a smooth riemannian metric on Z_y for any y and that all Z_y-derivatives of ρ_y are continuous in the y-direction. A metric on an E-family $\tilde{E}$ is by definition a positive definite continuous hermitian metric h on $\tilde{E}$ which is smooth along the fibers. By averaging ρ over G_R, we obtain a G_R-invariant metric on Z (cf. Remark 2.5). By averaging h over G, we obtain a hermitian metric h' and, by setting $h_0 = (h' + \overline{\tau^* h'})/2$, we obtain a hermitian metric h_0 such that the norm defined by h_0 is G_R-invariant. In this paper, metrics on Z and $\tilde{E}$ are assumed to be smooth along the fibers and G_R-invariant in the above sense. Then, the unit sphere bundle S_FZ, $S\tilde{E}$ are G_R-subsets of T_FZ, $\tilde{E}$, respectively, and T_FZ may be identified with the cotangent bundle along the fibers T_F^*Z. Note moreover that, if $P: Y \to \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F})$ is a G_R-section, then the adjoint $P^*: Y \to \mathcal{P}_c^m(Z; \tilde{F}, \tilde{E})$ with respect to metrics is also a G_R-section.

Example 2.11. Let Z be an X-family over Y with a metric. Let $\tilde{E} = \oplus_p \Lambda^{2p} T_F^* Z \otimes \boldsymbol{C}$, $\tilde{F} = \oplus_p \Lambda^{2p+1} T_F^* Z \otimes \boldsymbol{C}$, and let d_y be the exterior derivative on Z_y and d_y^* its adjoint. Then, the derivatives of the symbol of $d_y + d_y^*: C^\infty(Z, \tilde{E})_y \to C^\infty(Z, \tilde{F})_y$ are continuous in y (cf. [4], p.123~124) and $\{d_y + d_y^*\}_{y \in Y}$ define a *G.R.E.F.*: $Y \to \mathrm{Ell}^1(Z, \tilde{E}, \tilde{F})$.

3. The topological index. Let $Z = \{\pi: Z \to Y\}$ be an X-family over Y.

Proposition 3.1. *There exist a (finite dimensional) real G_R-module V and a G_R-map $i: Z \to Y \times V$ such that the composition of i with the projection $Y \times V \to Y$ coincides with π and i is an embedding along the fibers, namely, the restriction of i to Z_y is a smooth embedding $Z_y \to \{y\} \times V$ for any $y \in Y$.*

Proof. Let y_0 be any point in Y and let H be the isotropy subgroup of G_R at y_0. Let S be a slice at y_0 and let $T = G_R \times_H S$ be the tube. We may assume that Z is trivial over S and $Z|_S \simeq S \times X$. Since $\{y_0\} \times X$ is a closed H-manifold, there exist a real H-module W and an H-embedding $j = j(x): \{y_0\} \times X \to W$ (cf. [6]). We define $k = k(s, x): S \times X \to W$ to be the composition of the projection $S \times X \to \{y_0\} \times X$ with j. Moreover, we define an H-map $l = l(s, x): S \times X \to W$ by $l(s, x) = \int_H h \cdot k(h^{-1} \cdot (s, x))\, dh$ where dh denotes the normalized Haar measure on H. Then, $l(y_0, x) = j(x)$ and it follows from the same argument as in Remark 2.5 that the x-derivatives of $l(s, x)$ are continuous in s. Hence, there exists a slice $S' \subset S$ at y_0 such that $l(s, x): \{s\} \times X \to W$ is an embedding for any $s \in S'$. Now, it is known [6] that there exists a real G_R-module V such that V contains W as an H-submodule. Then, we may regard l as an H-map $Z|_{S'} \to V$. Let $T' = G_R \times_H S'$ be the tube. Since $Z|_{T'} = G_R \cdot (Z|_{S'})$, we can define a G_R-map $i_{T'}: Z|_{T'} \to V$ by $i_{T'}(gz) = g \cdot l(z)$ for any $g \in G_R$ and any $z \in Z|_{S'}$. Since l is an H-map, it is easy to see that $i_{T'}$ is well-defined and it is clear that the restriction of $i_{T'}$ to Z_y is an embedding for any $y \in T'$. Now, since Y is compact, Y is covered by finitely many T''s. Namely, there exist tubes $T_1, \ldots, T_N$ and G_R-maps $i_m: Z|_{T_m} \to V_m$ such that the restriction of i_m to Z_y is an embedding for any $y \in T_m$ $(1 \leq m \leq N)$. Let $\{f_m\}_{m=1}^N$ be a G_R-partition of unity subordinate to $\{T_m\}_{m=1}^N$. Then, the G_R-map $i: Z \to Y \times (V_1 \oplus \cdots \oplus V_N)$ given by

$$i(z) = (\pi(z), f_1(\pi(z))i_1(z) \oplus \cdots \oplus f_N(\pi(z))i_N(z))$$

is the required one.

Remark 3.2. When Z is smooth, there exists a G_R-embedding $\phi: Z \to V$ into a real G_R-module V and the G_R-map $i: Z \to Y \times V$ given by $i(z) = (\pi(z), \phi(z))$ is clearly an embedding along the fibers.

Now, let N be the normal bundle along the fibers of Z in $Y \times V$, namely, $N = \cup_{y \in Y} N_y$ where N_y is the normal bundle of Z_y in $\{y\} \times V$. It is easy to see that N becomes a G_R-fiber bundle over Y. Moreover, by taking a G_R-invariant metric on (the trivial V-family) $Y \times V$ and the exponential map in the fiber direction, N is regarded as in §1 as an open G_R-subset of $Y \times V$. Namely, there exists an open G_R-embedding along the fibers $N \to Y \times V$ and N can be regarded as an open manifold family over Y. Let $q: T_F Z \to Z$ be the projection. Then, the tangent bundle along the fibers $T_F N$ is isomorphic as in §1 to $q^*(N \otimes \boldsymbol{C})$ and the Thom homomorphism $KR_G(T_F Z) \to KR_G(T_F N)$ is defined by tensoring the exterior complex $\Lambda(T_F N)$. Moreover, $T_F N$ is an open G_R-subset of $Y \times TV = T_F(Y \times V)$ and the extension homomorphism $k_*: KR_G(T_F N) \to KR_G(Y \times TV)$ is defined. Then, $i_!: KR_G(T_F Z) \to KR_G(Y \times TV)$ is defined to be the composition of the Thom homomorphism with k_*.

Let $j: Y = Y \times \{o\} \to Y \times V$ be the canonical G_R-embedding where o denotes the origin of V. It is easy to see that TV $(\simeq V \otimes \boldsymbol{C})$ is a Real G-module (cf. Example 1.3) and

$$j_! : KR_G(T_F(Y \times \{o\})) = KR_G(Y) \longrightarrow KR_G(Y \times TV)$$

coincides with the Thom homomorphism. Hence, it follows from Theorem 1.4 that $j_!$ is an isomorphism.

Definition 3.3. The topological index $t\text{-ind}: KR_G(T_F Z) \to KR_G(Y)$ is defined to be the composition of $i_!$ with $j_!^{-1}$.

Remark 3.4. From the same argument as in [3], p.498, it can be proved that t-ind is independent of the choices of i and V.

4. The index of G-equivariant Real elliptic families. Let $\tilde{E}$ (resp. $\tilde{F}$) be the E (resp. F)-family over Y and $P = \{P_y\}_{y\in Y}: Y \to \mathrm{Ell}^m(Z;\tilde{E},\tilde{F})$ a *G.R.E.F.*.

Proposition 4.1. (1) *There exist a finite dimensional Real G-vector bundle L over Y and a continuous Real G-vector bundle homomorphism $T: L \to C^\infty(Z,\tilde{F})$ such that the map $Q_y: C^\infty(Z,\tilde{E})_y \oplus L_y \to C^\infty(Z,\tilde{F})_y$ given by $Q_y(u,v) = P_y(u) + T(v)$ is surjective for any $y \in Y$.*

(2) $\mathrm{Ker}Q = \cup_{y\in Y}\mathrm{kernel}(Q_y)$ *is a finite dimensional Real G-vector subbundle of $C^\infty(Z,\tilde{E}) \oplus L$ over Y.*

(3) $KR_G(Y) \ni [\mathrm{Ker}Q]-[L]$ *is independent of the choices of L and T.*

Proof. (1) It is obvious that (1) follows from the following lemma.

Lemma 4.2. *There exist a finite dimensional Real G-module W and a continuous Real G-vector bundle homomorphism $T: Y \times W \to C^\infty(Z,\tilde{F})$ such that the map $Q_y: C^\infty(Z,\tilde{E})_y \oplus W \to C^\infty(Z,\tilde{F})_y$ given by $Q_y(u,v) = P_y(u) + T(y,v)$ is surjective for any $y \in Y$.*

Proof of Lemma 4.2. Let h (resp. ρ) be a metric on $\tilde{F}$ (resp. Z) (cf. §2). Let $D: C^\infty(Z_y,\tilde{F}_y) \to C^\infty(Z_y,\tilde{F}_y \otimes T^*(Z_y))$ denote the covariant differentiation and let $\Delta = 1 + D^*D$. Then, $C^\infty(Z,\tilde{F})_y = C^\infty(Z_y,\tilde{F}_y)$ is a Fréchet space with the following seminorms $\{p^y_m\}_{m=1}^\infty$;

$$p^y_m(w) = \left\{\int_{Z_y} h_y\Big(\Delta^m w(x), w(x)\Big)\, dx_{\rho(y)}\right\}^{1/2}$$

for $w \in C^\infty(Z_y,\tilde{F}_y)$. Let Γ denote the set of continuous sections $s: Y \to C^\infty(Z,\tilde{F})$. Then, Γ is a Fréchet space with seminorms $p_m(s) = \sup_{y\in Y} p^y_m(s(y))$ for $s \in \Gamma$. Moreover, Γ is a (infinite dimensional) Real G-module with the continuous G_R-action $(gs)(y) = g{\cdot}s(g^{-1}{\cdot}y)$ for $g \in G_R$. Let Γ_a denote the union of the finite dimensional G_R-invariant subspaces of Γ. Then, it can be proved by the Peter-Weyl theorem that Γ_a is dense in Γ. In fact, if M is a finite dimensional G-invariant subspace of Γ, $M \oplus \tau M$ is a finite dimensional G_R-invariant subspace of Γ. Hence, the denseness of Γ_a in Γ follows from Theorem (2.5) in [10]. Now, for any $y \in Y$, P_y is elliptic and there exist finite elements $s^y_1, \ldots, s^y_{l(y)} \in \Gamma$ such that $s^y_1(y), \ldots, s^y_{l(y)}(y)$ span cokernel(P_y) where $C^\infty(Z_y,\tilde{F}_y) = \mathrm{image}(P_y) \oplus \mathrm{cokernel}(P_y)$. Since P_y is continuous in y, it follows from the standard argument (cf. [1], Appendix) that there exists an open neighborhood U_y of y such that

$s_1^y(y'), \ldots, s_{l(y)}^y(y')$ span cokernel$(P_{y'})$ for any $y' \in U_y$. Moreover, since Γ_a is dense in Γ and the evaluation map $\Gamma \to C^\infty(Z_y, \tilde{F}_y)$ is continuous, we may assume that $s_1^y, \ldots, s_{l(y)}^y \in \Gamma_a$. Suppose $Y = \cup_{i=1}^N U_{y_i}$ and let W be the finite dimensional Real G-submodule of Γ generated by $\left\{\{s_j^{y_i}\}_{j=1}^{l(y_i)}\right\}_{i=1}^N$. Then, $T: Y \times W \to C^\infty(Z, \tilde{F})$ given by $T(y, s) = s(y)$ is the required homomorphism. This completes the proof of Lemma 4.2.

(2) It follows from [4], Proposition (2.2) (or [1], Appendix) that kernel(Q_y) is finite dimensional for any $y \in Y$ and the dimension of kernel(Q_y) is (locally) constant in y. Note that [4] does not assert that kernel(Q_y) form a subbundle of $C^\infty(Z, \tilde{E}) \oplus L$. We show that kernel$(Q_y)$ form a subbundle Ker$Q$ of $C^\infty(Z, \tilde{E}) \oplus L$. Then, since it is obvious that Ker$Q$ is a G_R-subset of $C^\infty(Z, \tilde{E}) \oplus L$, Ker$Q$ becomes a Real G-vector subbundle of $C^\infty(Z, \tilde{E}) \oplus L$ with the restricted G_R-action. Now, for any fixed s and any $y \in Y$, let $P_y^s: H_s(Z, \tilde{E})_y \to H_{s-m}(Z, \tilde{F})_y$ denote the canonical extension of P_y and let $Q_y^s: H_s(Z, \tilde{E})_y \oplus L_y \to H_{s-m}(Z, \tilde{F})_y$ be given by $Q_y^s(u, v) = P_y^s(u) + T(v)$. Since P_y is elliptic, it follows from the regularity (cf. [4], Lemma (2.1)) that kernel$(Q_y^s) =$ kernel(Q_y). For any fixed $y_0 \in Y$, let U be an open neighborhood of y_0 and suppose that

$$H_s(Z, \tilde{E})|_U \simeq U \times H_s(X, E),$$
$$H_{s-m}(Z, \tilde{F})|_U \simeq U \times H_{s-m}(X, F) \quad \text{and}$$
$$L|_U \simeq U \times \boldsymbol{C}^q.$$

We set $H = H_s(X, E) \oplus \boldsymbol{C}^q$ and $H' = H_{s-m}(X, F)$. Let $\mathcal{L}(H, H')$ denote the Banach space of continuous operators $H \to H'$ endowed with the operator norm. Then, by the assumption, the map $U \to \mathcal{L}(H, H')$ given by $y \to Q_y^s$ is continuous, $Q_y^s: H \to H'$ is surjective and $Q_y^{s*}: H' \to H$ is injective for any y. Since, Q_y^s is a Fredholm operator, $Q_y^{s*}(H')$ is a closed subspace of H and $H =$ kernel$(Q_y^s) \oplus Q_y^{s*}(H')$. Hence it follows that $Q_y^s Q_y^{s*}: H' \to H'$ is bijective. Therefore it follows from the closed graph theorem that there exists $(Q_y^s Q_y^{s*})^{-1} \in \mathcal{L}(H', H')$. Moreover, the map $U \to \mathcal{L}(H', H')$ given by $y \to (Q_y^s Q_y^{s*})^{-1}$ is continuous. We set $q_y = 1 - Q_y^{s*}(Q_y^s Q_y^{s*})^{-1} Q_y^s$. Then, $q_y: H \to$ kernel(Q_y^s), $q_y|_{\text{kernel}(Q_y^s)} = 1$ and the map $U \to \mathcal{L}(H, H)$ given by $y \to q_y$ is continuous. Let $\phi_y: H =$ kernel$(Q_{y_0}^s) \oplus Q_{y_0}^{s*}(H') \to H$ be given by $\phi_y(u, v) = q_y(u) + v$. Then, the map $U \to \mathcal{L}(H, H)$ given by $y \to \phi_y$ is continuous and it is obvious that ϕ_{y_0} is an isomorphism. Hence, by the standard argument, there exists an open neighborhood U' of y_0 in U such that ϕ_y is an isomorphism for any $y \in U'$. Therefore, $q_y: \text{kernel}(Q^s_{y_0}) \to \text{kernel}(Q^s_y)$ is injective for any $y \in U'$. Since, $\text{kernel}(Q^s_y)$ is constant dimensional, $q_y: \text{kernel}(Q^s_{y_0}) \to \text{kernel}(Q^s_y)$ is an isomorphism for any $y \in U'$. Let $\Phi: U' \times H \to U' \times H$ be an isomorphism given by $\Phi(y,u) = (y, \phi_y(u))$. Then,

$$\Phi|_{U' \times \text{kernel}(Q^s_{y_0})} : U' \times \text{kernel}(Q^s_{y_0}) \longrightarrow \bigcup_{y \in U'} \text{kernel}(Q^s_y)$$

gives the required local triviality of $\text{Ker}Q$. From the arguments above, it follows that $\text{Ker}Q$ is a subbundle of $H_s(Z, \tilde{E}) \oplus L$ for any s. Hence, the inclusion $\text{Ker}Q \to C^\infty(Z, \tilde{E}) \oplus L$ is continuous and $\text{Ker}Q$ is a subbundle of $C^\infty(Z, \tilde{E}) \oplus L$.

(3) Let L' be another finite dimensional Real G-vector bundle over Y and $T': L' \to C^\infty(Z, \tilde{F})$ another homomorphism such that the map $Q'_y: C^\infty(Z, \tilde{E})_y \oplus L'_y \to C^\infty(Z, \tilde{F})_y$ given by $Q'_y(u, v') = P_y(u) + T'(v')$ is surjective for any $y \in Y$. For any $t \in I = [0,1]$, let $R_{(y,t)}: C^\infty(Z, \tilde{E})_y \oplus L_y \oplus L'_y \to C^\infty(Z, \tilde{F})_y$ be given by $R_{(y,t)}(u, v, v') = P_y(u) + (1-t)T(v) + tT'(v')$. Then, $R_{(y,t)}$ is surjective for any $(y,t) \in Y \times I$ and it follows from (2) that $\text{Ker}R = \cup_{(y,t) \in Y \times I} \text{kernel}(R_{(y,t)})$ is a finite dimensional Real G-vector bundle over $Y \times I$ where G_R acts on I trivially. Let $j_k: Y \to Y \times I$ be given by $j_k: Y = Y \times \{k\} \to Y \times I$ for $k = 0, 1$. Then, it is obvious that j_0 is G_R-homotopic to j_1 and hence that, $j_0^* = j_1^*: KR_G(Y \times I) \to KR_G(Y)$. Hence, it follows that

$$\begin{aligned} [\text{Ker}Q \oplus L'] - [L \oplus L'] &= j_0^*([\text{Ker}R] - [I \times (L \oplus L')]) \\ &= j_1^*([\text{Ker}R] - [I \times (L \oplus L')]) \\ &= [\text{Ker}Q' \oplus L] - [L \oplus L']. \end{aligned}$$

Therefore, it follows that $[\text{Ker}Q] - [L] = [\text{Ker}Q'] - [L']$. This completes the proof.

Definition 4.3. We define the index of a *G.R.E.F.* P by

$$\text{index}(P) = [\text{Ker}Q] - [L] \in KR_G(Y).$$

Remark 4.4. It is easy to see that the definition above of a *G.R.E.F.* is a refinement of that of an elliptic operator in [3] and that of an elliptic family in [4]. In fact, when Y = point and $P: C^\infty(X, E) \to C^\infty(X, F)$ is an single elliptic operator, we can set L = cokernel(P)

and T = inclusion. Then, it is clear that $\mathrm{Ker}Q = \mathrm{kernel}(P) \oplus 0$ and $[\mathrm{Ker}Q] - [L] = [\mathrm{kernel}(P)] - [\mathrm{cokernel}(P)]$.

5. The analytical index. Let Z be an X-family over Y. Let E and F be smooth complex vector bundles over X and $\tilde{E}$ (resp. $\tilde{F}$) the E (resp. F)-family over Y. Let $q: T_F Z \to Z$ denote the projection and, by an abuse of the notation, we also denote by q the restriction of q to $T_F Z - Z$ where Z is regarded as the zero-section of $T_F Z$. Using a metric, we identify $T_F Z$ with the cotangent bundle along the fibers $T_F^* Z$. Let $q_X: TX - X \to X$ denote the projection and let $\psi: q_X^* E \to q_X^* F$ be a homomorphism. ψ is called homogeneous of degree m if $\psi(x, \lambda\xi)(v) = \lambda^m \psi(x, \xi)(v)$ for any $\lambda > 0$, $x \in X$, $0 \neq \xi \in T_x X$ and $v \in E_x$. Note that a homogeneous homomorphism is determined by its restriction to the unit sphere bundle of TX. Let $Symb^m(X; E, F)$ (resp. $Symb_c^m(X; E, F)$) denote the space of smooth (resp. continuous) homomorphisms $\psi: q_X^* E \to q_X^* F$ which are homogeneous of degree m. Let $Is^m(X; E, F)$ denote the set of all $\psi \in Symb_c^m(X; E, F)$ for which $\psi(x, \xi): E_x \to F_x$ are isomorphisms for any $x \in X$ and any $0 \neq \xi \in T_x X$. As in §2, $Symb^m(Z; \tilde{E}, \tilde{F})$, $Symb_c^m(Z; \tilde{E}, \tilde{F})$ and $Is^m(Z; \tilde{E}, \tilde{F})$ are defined to be fiber bundles over Y with fiber $Symb^m(X; E, F)$, $Symb_c^m(X; E, F)$ and $Is^m(X; E, F)$, respectively (cf. [4], §1). Namely, for any $y \in Y$,

$$\begin{aligned} &Symb^m(Z; \tilde{E}, \tilde{F})_y \text{ (resp. } Symb_c^m(Z; \tilde{E}, \tilde{F})_y) \\ &\quad = \{\psi: q_y^* \tilde{E}_y \to q_y^* \tilde{F}_y; \psi \text{ is smooth (resp. continuous) homomorphism} \\ &\qquad\qquad \text{which is homogeneous of degree } m\} \end{aligned}$$

and

$$\begin{aligned} &Is^m(Z; \tilde{E}, \tilde{F})_y \\ &\quad = \{\psi \in Symb_c^m(Z; \tilde{E}, \tilde{F})_y;\ \psi(x, \xi): \tilde{E}_x \to \tilde{F}_x \text{ is an isomorphism} \\ &\qquad\qquad \text{for any } x \in Z_y \text{ and any } 0 \neq \xi \in T_x(Z_y)\}. \end{aligned}$$

On the other hand, the G_R-actions on $q^* \tilde{E}$ and $q^* \tilde{F}$ make $Symb^m(Z; \tilde{E}, \tilde{F})$, $Symb_c^m(Z; \tilde{E}, \tilde{F})$ and $Is^m(Z; \tilde{E}, \tilde{F})$ G_R-fiber bundles over Y.

Let $q_S: S_F Z \to Z$ denote the projection of the unit sphere bundle of $T_F Z$ and let $|\ |_{\tilde{F}}$ denote a norm on $\tilde{F}$ (cf. §2). For any $y \in Y$ and any $\psi \in Symb_c^m(Z; \tilde{E}, \tilde{F})_y$, we set $|\psi| = \sup\{|\psi(e)|_{\tilde{F}}; e \in (q_S)_Y^* S\tilde{E}_y\}$ where $S\tilde{E}_y$ is the unit sphere bundle of $\tilde{E}_y$ over Z_y. Moreover, for any continuous section $\phi: Y \to Symb_c^m(Z; \tilde{E}, \tilde{F})$ we set $\|\phi\| = \sup_{y \in Y} |\phi_y|$. Note that $Is^m(Z; \tilde{E}, \tilde{F})$ is an open G_R-subset of $Symb_c^m(Z; \tilde{E}, \tilde{F})$ and it is not difficult to see that the following proposition holds.

Proposition 5.1. *For any continuous section* $\phi: Y \to Is^m(Z; \tilde{E}, \tilde{F})$, *there exists* $\epsilon > 0$ *such that, if a continuous section* $\phi': Y \to Symb_c^m(Z; \tilde{E}, \tilde{F})$ *satisfies* $\|\phi - \phi'\| < \epsilon$, *then the image of* ϕ' *is contained in* $Is^m(Z; \tilde{E}, \tilde{F})$.

For any $y \in Y$ and any $Q \in \mathcal{P}^m(Z; \tilde{E}, \tilde{F})_y$, let $\sigma(Q) \in Symb^m(Z; \tilde{E}, \tilde{F})_y$ denote the principal symbol of Q. Namely, for any $x \in Z_y$, $0 \neq \xi \in T_x(Z_y)$ and $v \in \tilde{E}_x$,

$$\tilde{F}_x \ni \sigma(Q)(x, \xi)(v) = \lim_{\lambda \to \infty} \frac{e^{-\sqrt{-1}\lambda f} f_Q(e^{\sqrt{-1}\lambda f} \cdot s)(x)}{\lambda^m}$$

where f is a smooth function on Z_y such that $(df)_x = \xi$ and s is an element of $C^\infty(Z_y, \tilde{E}_y)$ such that $s(x) = v$. Then, a continuous fiber-preserving map $\sigma: \mathcal{P}^m(Z; \tilde{E}, \tilde{F}) \to Symb^m(Z; \tilde{E}, \tilde{F})$ is defined and is canonically extended to a continuous fiber-preserving map $\sigma: \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F}) \to Symb_c^m(Z; \tilde{E}, \tilde{F})$. By the convention in Example 1.3, it is not difficult to see that σ is a G_R-map.

Remark 5.2. Let $Q: C^\infty(X, E) \to C^\infty(X, F)$ be a pseudo-differential operator of order m. Q is elliptic if and only if $\sigma(Q) \in Is^m(X; E, F)$. In particular, a continuous section $P = \{P_y\}_{y \in Y}: Y \to \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F})$ defines an elliptic family if and only if $\sigma(P) = \{\sigma(P_y)\}_{y \in Y}$ is a continuous section $Y \to Is^m(Z; \tilde{E}, \tilde{F})$.

Proposition 5.3. *For any continuous* G_R*-section* $\phi: Y \to Symb_c^m(Z; \tilde{E}, \tilde{F})$ *and any* $\epsilon > 0$, *there exists a continuous* G_R*-section* $P: Y \to \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F})$ *such that* $\|\phi - \sigma(P)\| < \epsilon$.

Proof. It follows from [4], Proposition (1.6) that there exists a continuous section $Q: Y \to \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F})$ such that $\|\phi - \sigma(Q)\| < \epsilon$. Now, let dg denote the normalized Haar measure on G_R and we define a G_R-section $P: Y \to \mathcal{P}_c^m(Z; \tilde{E}, \tilde{F})$ by $P_y = \int_{G_R} (gQ)_y \, dg$ for any $y \in Y$ where $(gQ)_y = g \cdot Q_{g^{-1} \cdot y}$. Then, since $(g\phi)_y = \phi_y$ for any $y \in Y$ and any $g \in G_R$, it follows that

$$\begin{aligned} |\phi_y - \sigma(P_y)| &= \left| \int_{G_R} (g\phi)_y \, dg - \int_{G_R} \sigma(gQ)_y \, dg \right| \\ &= \left| \int_{G_R} \Big(g(\phi - \sigma(Q))\Big)_y dg \right| \leq \int_{G_R} \left| \Big(g(\phi - \sigma(Q))\Big)_y \right| dg \end{aligned}$$

$$= \int_{G_R} \left|\phi_{g^{-1}\cdot y} - \sigma(Q)_{g^{-1}\cdot y}\right| dg < \int_{G_R} \epsilon\, dg = \epsilon$$

for any $y \in Y$.

Definition 5.4. Let $\phi^0, \phi^1: Y \to Is^m(Z; \tilde{E}, \tilde{F})$ be continuous G_R-sections. Then, $\phi^0 \sim \phi^1$ iff there exists a continuous G_R-section $\hat{\phi}: Y \times I \to Is^m(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$ such that $\hat{\phi}|_{Y\times\{k\}} = \phi^k$ for $k = 0, 1$ where the G_R-action on I is trivial.

Remark 5.5. Let $\phi: Y \to Is^m(Z; \tilde{E}, \tilde{F})$ be any continuous G_R-section. Then, by Proposition 5.1, Proposition 5.3 and Remark 5.2, there exists a *G.R.E.F.* $P: Y \to \mathrm{Ell}^m(Z; \tilde{E}, \tilde{F})$ such that the image of $(1-t)\phi + t\sigma(P)$ is contained in $Is^m(Z; \tilde{E}, \tilde{F})$ for any $t \in I$, in particular, $\phi \sim \sigma(P)$.

Definition 5.6. Let $\phi: Y \to Is^m(Z; \tilde{E}, \tilde{F})$ be a continuous G_R-section. We define the index of ϕ by $\mathrm{index}(\phi) = \mathrm{index}(P) \in KR_G(Y)$ where $P: Y \to \mathrm{Ell}^m(Z; \tilde{E}, \tilde{F})$ is a *G.R.E.F.* such that $\phi \sim \sigma(P)$.

Proposition 5.7. *The definition of* index(ϕ) *is well-defined.*

Proof. Let $P': Y \to \mathrm{Ell}^m(Z; \tilde{E}, \tilde{F})$ be another *G.R.E.F.* such that $\phi \sim \sigma(P')$. Then, it is obvious that $\sigma(P) \sim \sigma(P')$, namely, there exists a continuous G_R-section $\hat{\phi}: Y \times I \to Is^m(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$ such that $\hat{\phi}|_{Y\times\{0\}} = \sigma(P)$ and $\hat{\phi}|_{Y\times\{1\}} = \sigma(P')$. On the other hand, it follows from Proposition 5.1 that there exists $\epsilon > 0$ such that, if a continuous G_R-section $\phi': Y \to Symb_c^m(Z; \tilde{E}, \tilde{F})$ satisfies $\|\sigma(P) - \phi'\| < \epsilon$ or $\|\sigma(P') - \phi'\| < \epsilon$, then the image of ϕ' is contained in $Is^m(Z; \tilde{E}, \tilde{F})$. Moreover, it follows from Proposition 5.3 that there exists a *G.R.E.F.* $Q: Y \times I \to \mathrm{Ell}^m(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$ such that $\|\hat{\phi} - \sigma(Q)\| < \epsilon$. Since it is obvious that $\|\sigma(P) - \sigma(Q|_{Y\times\{0\}})\| < \epsilon$ and $\|\sigma(P') - \sigma(Q|_{Y\times\{1\}})\| < \epsilon$, $(1-t)P + t(Q|_{Y\times\{0\}})$ and $(1-t)(Q|_{Y\times\{1\}}) + tP'$ define *G.R.E.F.*'s $Y \times I \to \mathrm{Ell}^m(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$. Hence, by connecting these *G.R.E.F.*'s, we obtain a *G.R.E.F.* $R: Y \times I \to \mathrm{Ell}^m(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$ such that $R|_{y\times\{0\}} = P$ and $R|_{y\times\{1\}} = P'$. Now, let $j_k: Y = Y \times \{k\} \to Y \times I$ be the inclusion for $k = 0, 1$. Since j_0 is G_R-homotopic to j_1, $\mathrm{index}(P) = j_0^*(\mathrm{index}(R)) = j_1^*(\mathrm{index}(R)) = \mathrm{index}(P')$.

Definition 5.8. Let $q: T_F Z - Z \to Z$ be the projection and let

$\psi: q^*\tilde{E} \to q^*\tilde{F}$ be a Real G-vector bundle homomorphism which gives an isomorphism outside the unit disk bundle $B_F^o Z$ of $T_F Z$, namely, outside a compact set which is contained in the interior of $B_F Z$. For any m, we define a G_R-section $\psi^m: Y \to Is^m(Z; \tilde{E}, \tilde{F})$ by $\psi_y^m(x,\xi) = |\xi|^m \psi_y(x, \xi/|\xi|)$ where $y \in Y$, $x \in Z_y$ and $0 \neq \xi \in T_x(Z_y)$.

Definition 5.9. Let ψ be as above. We define $\mathrm{index}(\psi) = \mathrm{index}(\psi^m) \in KR_G(Y)$.

Proposition 5.10. The definition of index(ψ) is independent of the choice of m.

Proof. Let $Q: Y \to \mathrm{Ell}^m(Z; \tilde{E}, \tilde{F})$ be a *G.R.E.F.* such that $\sigma(Q) \sim \psi^m$ and let $R: Y \to \mathrm{Ell}^k(Z; \tilde{E}, \tilde{F})$ be a *G.R.E.F.* such that $\sigma(R) \sim \psi^k$. If $k = m$, it follows from Proposition 5.7 that $\mathrm{index}(Q) = \mathrm{index}(R)$. So, we assume that $k > m$. By substituting $\phi(\xi)\,|\xi|^{k-m}(r(x,y)^{-1})_{ji}(dv_{p(y)}/dx)^{-1}$ for $K_{ij}(x,\xi,y)$ in the proof of [4], Proposition (2.4), we can construct a continuous section $P': Y \to \mathrm{Ell}^{k-m}(Z; \tilde{E}, \tilde{E})$ which has the following properties:

(5.10.1) P' is positive definite,

(5.10.2) $\sigma(P')$ is self-adjoint and positive definite outside the zero-section Z of $T_F Z$.

Let $P: Y \to \mathcal{P}_c^{k-m}(Z; \tilde{E}, \tilde{E})$ denote a continuous G_R-section given by $P = \int_{G_R}(gP')\,dg$. Then, by the G_R-invariance of metrics, it is not difficult to see that P also has the properties (5.10.1) and (5.10.2), in particular, P is a *G.R.E.F.*. Now, let $\phi: Y \to Is^{k-m}(Z; \tilde{E}, \tilde{E})$ be a continuous G_R-section given by $\phi_y(x,\xi) = |\xi|^{k-m} \cdot 1_{\tilde{E}_x} : \tilde{E}_x \to \tilde{E}_x$ for any $y \in Y$, $x \in Z_y$ and $0 \neq \xi \in T_x(Z_y)$. Then, since the eigenvalues of $\phi_y(x,\xi)$ and $\sigma(P)_y(x,\xi)$ are positive real and $\phi_y(x,\xi)$ is a scalar matrix, it is obvious that $(1-t)\sigma(P)_y(x,\xi) + t\phi_y(x,\xi)$ gives an isomorphism for any $t \in I$, hence, $\phi \sim \sigma(P)$. On the other hand, since it is obvious that $\psi^k = \psi^m \cdot \phi$, $\sigma(R) \sim \sigma(Q)\cdot\phi \sim \sigma(Q)\cdot\sigma(P) = \sigma(QP)$. Therefore, it follows from Proposition 5.7 that $\mathrm{index}(R) = \mathrm{index}(QP)$. So, it remains to show that $\mathrm{index}(QP) = \mathrm{index}(Q)$.

Lemma 5.11. *$P: C^\infty(Z, \tilde{E}) \to C^\infty(Z, \tilde{E})$ gives a bijective Real G-vector bundle homomorphism.*

Proof of Lemma 5.11. For any $y \in Y$, since P_y is positive definite, kernel$(P_y) = 0$. So, we show that P_y is surjective. Since $P_y: H_{k-m}(Z, \tilde{E})_y \to L^2(Z, \tilde{E})_y$ is elliptic, image(P_y) is a closed subspace of $L^2(Z, \tilde{E})_y$ and $L^2(Z, \tilde{E})_y$ = image$(P_y) \oplus$ kernel(P_y^*). Since P_y is positive definite, it follows that kernel(P_y^*) = 0 and, for any $v \in C^\infty(Z, \tilde{E})_y$, there exists $u \in H_{k-m}(Z, \tilde{E})_y$ such that $P_y(u) = v$. Then, it follows from the regularity of the elliptic equation (cf. [4], Lemma (2.1)) that $u \in C^\infty(Z, \tilde{E})_y$. This completes the proof of Lemma 5.11.

Let L be a finite dimensional Real G-vector bundle over Y and let $T: L \to C^\infty(Z, \tilde{F})$ be a Real G-vector bundle homomorphism such that the map $\Psi_y: C^\infty(Z, \tilde{E})_y \oplus L_y \to C^\infty(Z, \tilde{F})_y$ given by $\Psi_y(u, v) = Q_y(u) + T(v)$ is surjective for any $y \in Y$. Then, it follows from Lemma 5.11 that the map $\Phi_y: C^\infty(Z, \tilde{E})_y \oplus L_y \to C^\infty(Z, \tilde{F})_y$ given by $\Phi_y(u, v) = Q_y \cdot P_y(u) + T(v)$ is surjective for any $y \in Y$. Hence, index(QP) = [KerΦ] – [L]. On the other hand, the restriction of $P \oplus 1: C^\infty(Z, \tilde{E}) \oplus L \to C^\infty(Z, \tilde{E}) \oplus L$ gives an isomorphism Ker$\Phi \simeq$ KerΨ. Hence, index(QP) = [KerΨ] – [L] = index(Q). Proposition 5.10 has been proved.

Now, we define the analytical index. The well-definedness of the analytical index owes to the following lemmas.

Lemma 5.12. *Let $q: T_F Z \to Z$ be the projection. Let F (resp. F') be a continuous Real G-vector bundle over $T_F Z$ (resp. $T_F Z \times I$). Then, there exists a family $\tilde{E}$ such that $q^* \tilde{E}$ (resp. $q^* \tilde{E} \times I$) is isomorphic to F (resp. F').*

Proof. Let $i: Z \to T_F Z$ be the zero-section. Then, $i \cdot q$ is G_R-homotopic to the identity and F is isomorphic to $q^*(i^* F)$. On the other hand, by Proposition 2.9, there exists a family $\tilde{E}$ which is isomorphic to $i^* F$, and $q^* \tilde{E}$ is isomorphic to F. The result for F' follows from the fact that $T_F Z \times \{0\}$ is a G_R-deformation retract of $T_F Z \times I$.

Lemma 5.13. *Let $\tilde{E}$ and $\tilde{F}$ be families and let $\hat{\psi}: q^* \tilde{E} \times I \to q^* \tilde{F} \times I$ be a Real G-vector bundle homomorphism over $T_F Z \times I$ which gives an isomorphism outside $B_F^\circ Z \times I$. Then, $\psi_k = \hat{\psi}|_{T_F Z \times \{k\}}: q^* \tilde{E} \to q^* \tilde{F}$ gives an isomorphism outside $B_F^\circ Z$ for $k = 0, 1$. Then,* index(ψ_0) = index(ψ_1).

Proof. It is obvious that $\hat{\psi}^m|_{T_F Z \times \{k\}} = \psi_k^m$ for $k = 0, 1$, hence,

$\psi_0^m \sim \psi_1^m$.

Lemma 5.14. *Let $\psi: q^*\tilde{E} \to q^*\tilde{F}$ be a Real G-vector bundle isomorphism over T_FZ. Then,* $\text{index}(\psi) = 0$.

Proof. By Proposition 5.10, $\text{index}(\psi) = \text{index}(\psi^0)$ where $\psi_y^0(x,\xi) = \psi_y(x,\xi/|\xi|)$ for any $y \in Y$, $x \in Z_y$ and $0 \neq \xi \in T_x(Z_y)$. Let $\hat{\psi}: Y \times I \to Symb_c^0(Z \times I; \tilde{E} \times I, \tilde{F} \times I)$ be a continuous G_R-section given by $\hat{\psi}_{(y,t)}(x,\xi) = \psi_y(x, t\xi/|\xi|)$ for $t \in I$. Then, by the assumption, the image of $\hat{\psi}$ is contained in Is^0. Hence, $\hat{\psi}|_{Y\times\{0\}} \sim \hat{\psi}|_{Y\times\{1\}} = \psi^0$ and $\text{index}(\psi^0) = \text{index}(\hat{\psi}|_{Y\times\{0\}})$. We set $\hat{\psi}|_{Y\times\{0\}} = \Psi$. Then, $\Psi_y(x,\xi) = \psi_y(x,0)$. So, let $f: \tilde{E} \to \tilde{F}$ denote the isomorphism given by the restriction of ψ to the zero-section Z, and let $f': \tilde{E} \to \tilde{F}$ be an approximation of f such that the restriction $f'_y = f'|_{\tilde{E}_y}$ is smooth for any y and all Z_y-derivatives are continuous in the y-direction (cf. Lemma 2.6). Then, f' naturally defines $f'_*: C^\infty(Z,\tilde{E}) \to C^\infty(Z,\tilde{F})$ which is a G_R-section of $\mathcal{P}^0(Z;\tilde{E},\tilde{F})$ and $\sigma(f'_*) \sim \Psi$, hence $\text{index}(\psi) = \text{index}(f'_*)$. On the other hand, since there exists the inverse f'^{-1}, f'_* is clearly an isomorphism and $\text{index}(f'_*) = 0$.

Lemma 5.15. *Let $\psi_k: q^*\tilde{E}_k \to q^*\tilde{F}_k$ be a Real G-vector bundle homomorphism which gives an isomorphism outside $B_F^\circ Z$ ($k = 1,2$). Then, a homomorphism $\psi_1 \oplus \psi_2: q^*(\tilde{E}_1 \oplus \tilde{E}_2) \to q^*(\tilde{F}_1 \oplus \tilde{F}_2)$ is naturally defined and $\psi_1 \oplus \psi_2$ also gives an isomorphism outside $B_F^\circ Z$. Then,* $\text{index}(\psi_1 \oplus \psi_2) = \text{index}(\psi_1) + \text{index}(\psi_2)$.

Proof. Let $P_k: Y \to \text{Ell}^m(Z;\tilde{E}_k,\tilde{F}_k)$ be a *G.R.E.F.* such that $\sigma(P_k) \sim \psi_k^m$ ($k = 1,2$). Then, a *G.R.E.F.* $P_1 \oplus P_2: Y \to \text{Ell}^m(Z;\tilde{E}_1 \oplus \tilde{E}_2, \tilde{F}_1 \oplus \tilde{F}_2)$ is naturally defined and $\sigma(P_1 \oplus P_2) \sim (\psi_1 \oplus \psi_2)^m$, and it is easy to see that $\text{index}(P_1 \oplus P_2) = \text{index}(P_1) + \text{index}(P_2)$.

Now, let α be any element of $KR_G(T_FZ)$. Then, α can be expressed by a triple as in [1], §2.6 and [10], §3. Namely, by Lemma 5.12, there exist families $\tilde{E}$, $\tilde{F}$ and a Real G-vector bundle homomorphism $\psi: q^*\tilde{E} \to q^*\tilde{F}$ which gives an isomorphism outside a compact G_R-subset K of T_FZ such that $\{\psi: q^*\tilde{E} \to q^*\tilde{F}\}$ represents α. We may assume that K is contained in $B_F^\circ Z$.

Definition 5.16. The analytical index a-ind: $KR_G(T_FZ) \to KR_G(Y)$ is defined by $a\text{-ind}(\alpha) = \text{index}(\psi)$.

Remark 5.17. By Lemma 5.15, it is easy to see that a-ind is an additive homomorphism.

We show that the definition of a-ind is well-defined. Suppose that $\{\psi' : q^*\tilde{E}' \to q^*\tilde{F}'\}$ also represents α where ψ' gives an isomorphism outside a compact G_R-subset of T_FZ which is contained in the unit disk bundle of T_FZ. Note that the choice of a metric on Z may be changed and the former unit disk bundle need not coincide with the latter. Then, by the definition of the equivalence relation of triples and Lemma 5.12, there exist triples $\{\phi_1 : q^*\tilde{E}_1 \to q^*\tilde{F}_1\}$, $\{\phi_2 : q^*\tilde{E}_2 \to q^*\tilde{F}_2\}$, and a triple over $T_FZ \times I$ $(= T_F(Z\times I))$ $\{\hat{\psi} : q^*\tilde{E}_3\times I \to q^*\tilde{F}_3\times I\}$ which have the following properties:

(1) ϕ_1 and ϕ_2 give isomorphisms over T_FZ,

(2) $\hat{\psi}$ gives an isomorphism outside a compact G_R-subset of $T_FZ \times I$ which is contained in the unit disk bundle of $T_FZ \times I$ with respect to a metric on $Z \times I$ which connects the metric on $Z \times \{0\}$ with the metric on $Z \times \{1\}$,

(3) There exist isomorphisms f_1 and f_2 such that the following diagram is commutative,

$$\begin{array}{ccc} q^*\tilde{E}_3 & \xrightarrow{\hat{\psi}|_{T_FZ\times\{0\}}} & q^*\tilde{F}_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ q^*\tilde{E} \oplus q^*\tilde{E}_1 & \xrightarrow{\psi\oplus\phi_1} & q^*\tilde{F} \oplus q^*\tilde{F}_1 \end{array}$$

(4) There exist isomorphisms f_3 and f_4 such that the following diagram is commutative.

$$\begin{array}{ccc} q^*\tilde{E}_3 & \xrightarrow{\hat{\psi}|_{T_FZ\times\{1\}}} & q^*\tilde{F}_3 \\ {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow \\ q^*\tilde{E}' \oplus q^*\tilde{E}_2 & \xrightarrow{\psi'\oplus\phi_2} & q^*\tilde{F}' \oplus q^*\tilde{F}_2 \end{array}$$

Then, it follows from Lemma 5.13 that $\text{index}(\psi\oplus\phi_1) = \text{index}(\psi'\oplus\phi_2)$. Moreover, it follows from Lemma 5.14 and Lemma 5.15 that $\text{index}(\psi \oplus \phi_1) = \text{index}(\psi)$ and $\text{index}(\psi' \oplus \phi_2) = \text{index}(\psi')$, and the proof of the well-definedness of a-ind is completed.

6. The excision axiom. Let X^1 and X^2 be closed smooth manifolds and let Z^1 (resp. Z^2) be an X^1 (resp. X^2)-family over Y. Let N be the open manifold family over Y defined in §3. We assume that there exist open G_R-embeddings along the fibers $N \to Z^k$ which commute with the projection onto Y for $k = 1, 2$, namely, N_y is embedded as an open submanifold of Z_y^k for any y. Then, $T_F N$ is an open G_R-subset of $T_F Z^k$ and the extension homomorphisms $j_*^k : KR_G(T_F N) \to KR_G(T_F Z^k)$ are defined for $k = 1, 2$.

Proposition 6.1. *The following diagram is commutative.*

$$\begin{array}{ccccc} & & KR_G(T_F Z^1) & & \\ & {\scriptstyle j_*^1}\nearrow & & \searrow{\scriptstyle a\text{-ind}} & \\ KR_G(T_F N) & & & & KR_G(Y) \\ & {\scriptstyle j_*^2}\searrow & & \nearrow{\scriptstyle a\text{-ind}} & \\ & & KR_G(T_F Z^2) & & \end{array}$$

Proof. Let α be any element of $KR_G(T_F N)$. Let $\{\psi : q^*\tilde{E} \to q^*\tilde{F}\}$ be a triple which represents α where $q: T_F N \to N$ is the projection. Then, by the definition of KR_G-group of locally compact spaces, we may assume that there exists a compact G_R-subset K of $T_F N$ which has the following properties:

(6.1.1) ψ gives an isomorphism outside K,
(6.1.2) There exists a Real G-module V such that

$$q^*\tilde{E}|_{T_F N - K} \xrightarrow[\psi]{\cong} q^*\tilde{F}|_{T_F N - K} \xrightarrow{\cong} (T_F N - K) \times V \ \text{(trivial)}.$$

Let A be a compact G_R-subset of N such that A^o (= the interior of A) contains $\bar{K} = q(K)$. Let $\eta: N \to [0,1]$ be a G_R-invariant continuous function such that $\eta|_{N - A^o} \equiv 1$ and $\eta|_{\bar{K}} \equiv 0$. Let $\hat{\psi}: q^*\tilde{E} \times I \to q^*\tilde{F} \times I$ be a Real G-vector bundle homomorphism over $T_F N \times I$ given by $\hat{\psi}_{(y,t)}(x, \xi) = \psi_y(x, (1 - \eta(y,x) + t\eta(y,x))\xi)$ for any $(y,t) \in Y \times I$, $(y,x) \in N$ (i.e. $x \in N_y$) and $\xi \in T_x(N_y)$. Then, $\hat{\psi}$ gives an isomorphism outside $K \times I$. In fact, if $\hat{\psi}_{(y,t)}(x,\xi)$ is not an isomorphism, $(y, x, (1 - \eta(y,x) + t\eta(y,x))\xi) \in K$. Hence, it follows that $(y,x) \in \bar{K}$, $\eta(y,x) = 0$ and $(y,x,\xi) \in K$. Since it is obvious that $\hat{\psi}|_{T_F N \times \{1\}} = \psi$, α can be represented by $\{\chi : q^*\tilde{E} \to q^*\tilde{F}\}$ where $\chi = \hat{\psi}|_{T_F N \times \{0\}}$. It is obvious that $\chi_y(x,\xi)$ coincides with the isomorphism $\psi_y(x, 0): \tilde{E}_{(y,x)} \to \tilde{F}_{(y,x)}$ for any $(y,x) \in N - A^o$ (cf. (6.1.1)). Let B be a compact G_R-subset of N such that B^o contains A. It follows from (6.1.2) that there exists an isomorphism $\mu: \tilde{F}|_{B - A^o} \to (B - A^o) \times V$. We set $\nu = \mu \cdot f: \tilde{E}|_{B - A^o} \to (B - A^o) \times V$ where f denotes the restriction of χ (or ψ) to $N - A^o$. Then, by the clutching construction using μ and ν, we can construct an extension $\tilde{E}^k$ (resp. $\tilde{F}^k$) of $\tilde{E}$ (resp. $\tilde{F}$) to $Z^k - B$ ($k = 1,2$). Moreover, there exists an isomorphism $f^k: \tilde{E}^k|_{Z^k - A^o} \to \tilde{F}^k|_{Z^k-A^o}$ such that $f^k|_{B-A^o} = f$. Note that we may assume that $\tilde{E}^k$ and $\tilde{F}^k$ are families, f^k is smooth in the Z_y-direction for any y and all Z_y-derivatives of f^k are continuous in the y-direction. Let $f^k_*: C^\infty(Z^k - A^o, E^k) \to C^\infty(Z^k - A^o, F^k)$ denote the G_R-section of $\mathcal{P}^0(Z^k - A^o, \tilde{E}^k, \tilde{F}^k)$ which is naturally defined by f^k ($k = 1,2$). Then, since $\chi_y(x,\xi) = f^k(y,x)$ for any ξ and $(y,x) \in B - A^o$, $\sigma(f^k_*)$ coincides with χ on $q^{-1}(B - A^o)$. Hence, we can construct a homomorphism $\chi^k: q^{k*}\tilde{E}^k \to q^{k*}\tilde{F}^k$ over $T_F Z^k$ (where $q^k: T_F Z^k \to Z^k$ is the projection) which has the following properties ($k = 1,2$):

(6.1.3) χ^k (or $(\chi^k)^0$) coincides with $\sigma(f^k_*)$ on $T_F Z^k - q^{-1}(A^o)$,
(6.1.4) χ^k coincides with χ on $q^{-1}(B)$,
(6.1.5) χ^k gives an isomorphism outside the unit disk bundle of $T_F Z^k$.

Then, $j^k_*(\alpha)$ is represented by the triple $\{\chi^k: q^{k*}\tilde{E}^k \to q^{k*}\tilde{F}^k\}$.

Now, let $P^k: Y \to \mathrm{Ell}^0(Z^k; \tilde{E}^k, \tilde{F}^k)$ be a *G.R.E.F.* of order 0 such that $\|\sigma(P^k) - (\chi^k)^0\| < \epsilon$ for sufficiently small $\epsilon > 0$ ($k = 1,2$). Let $\gamma: Z^k \to [0,1]$ be a G_R-invariant function such that γ is smooth in the Z_y-direction for any y, all Z_y-derivatives of γ are continuous in the y-direction, $\gamma|_A \equiv 1$ and $\gamma|_{Z^k - B^o} \equiv 0$. Let $R^k: Y \to \mathcal{P}^0_c(Z^k; \tilde{E}^k, \tilde{F}^k)$ be a G_R-section given by $R^k_y(u) = (f^k_*)_y((1-\gamma^2)u) + \gamma(P^1 + P^2)_y(\gamma u)/2$ for any y and $u \in C^\infty(Z^k, \tilde{E}^k)_y$. Then, $\sigma(R^k) = \sigma(f^k_*)(1-\gamma^2) + \sigma(P^1)\gamma^2/2 + \sigma(P^2)\gamma^2/2$. Hence, it follows that

$$\begin{aligned}\|\sigma(R^k) - (\chi^k)^0\| &= \|\sigma(R^k) - (\chi^k)^0(1-\gamma^2) - (\chi^k)^0\gamma^2\| \\ &\le \|(\sigma(f^k_*) - (\chi^k)^0)(1-\gamma^2)\| \\ &\quad + \frac{1}{2}\|(\sigma(P^1) - (\chi^k)^0)\gamma^2\| + \frac{1}{2}\|(\sigma(P^2) - (\chi^k)^0)\gamma^2\|\end{aligned}$$

for $k = 1,2$. Since, $\mathrm{supp}(\gamma^2) \subset B$, it follows from (6.1.4) that $(\chi^1)^0\gamma^2 = (\chi^2)^0\gamma^2$. Hence, it follows that $\|(\sigma(P^i) - (\chi^k)^0)\gamma^2\| < \epsilon$ for $i,k = 1,2$. Moreover, it follows from (6.1.3) that $\sigma(f^k_*)(1-\gamma^2) = (\chi^k)^0(1-\gamma^2)$. Hence, it follows that $\|\sigma(R^k) - (\chi^k)^0\| < \epsilon$. Therefore, we may assume that the image of $\sigma(R^k)$ is contained in Is^0 and R^k is a *G.R.E.F.*, moreover, $a\text{-ind}(j^k_*(\alpha)) = \mathrm{index}(R^k)$ for $k = 1,2$.

Let $C^\infty_B(\tilde{E})$ (resp. $C^\infty_B(\tilde{F})$) denote the Fréchet subbundle of $C^\infty(Z^k, \tilde{E}^k)$ (resp. $C^\infty(Z^k, \tilde{F}^k)$) which consists of smooth sections whose supports are contained in B. Since R^k_y gives an isomorphism outside B, it is easy to see that cokernel(R_y^k) (= kernel(R_y^{k*})) is contained in $C_B^\infty(\tilde{F})_y$ for any y. Hence, in the proof of Lemma 4.2, we can set Γ to be the Fréchet space of continuous sections $s: Y \to C_B^\infty(\tilde{F})$. Therefore, it follows that there exist a finite dimensional Real G-vector bundle L over Y and a Real G-vector bundle homomorphism $T: L \to C_B^\infty(\tilde{F})$ such that the map $Q_y^k: C^\infty(Z^k, \tilde{E}^k)_y \oplus L_y \to C^\infty(Z^k, \tilde{F}^k)_y$ given by $Q_y^k(u,v) = R_y^k(u) + T(v)$ is surjective for any y and $k = 1,2$. Note that, since cokernel(R_y^1) = cokernel(R_y^2) for any y, L and T can be taken independently of $k = 1,2$. Let (u,v) be any element of kernel(Q_y^k). Then, from the definition of R_y^k, it follows that $-f^k((1-\gamma^2)u) = \gamma(P^1+P^2)(\gamma u)/2 + T(v)$. Since the support of the right term is contained in B and f^k gives an isomorphism outside A^o, it follows that the support of u is contained in B ($k = 1,2$). Hence, kernel(Q_y^k) is determined by the restriction of Q_y^k to $C_B^\infty(\tilde{E}) \oplus L$ and it follows that kernel(Q_y^1) = kernel(Q_y^2) for any y. Therefore, it follows that a-ind$(j_*^1(\alpha))$ = index(R^1) = $[\mathrm{Ker}\, Q^1] - [L] = [\mathrm{Ker}\, Q^2] - [L]$ = index(R^2) = a-ind$(j_*^2(\alpha))$.

7. The normalization axiom. Let V be the real G_R-module in §3. Let S denote the one point compactification of V. S is a compact G_R-manifold which is diffeomorphic to the sphere. Let $j: Y = Y \times \{o\} \to Y \times V$ denote the canonical inclusion and let $j_!: KR_G(Y) \to KR_G(Y \times TV)$ be the periodicity isomorphism (cf. §3). Let $k_*: KR_G(Y \times TV) \to KR_G(Y \times TS)$ denote the extension homomorphism of the open inclusion $Y \times TV = T_F(Y \times V) \to Y \times TS = T_F(Y \times S)$.

Proposition 7.1. *a-ind$(k_* j_!(\alpha)) = \alpha$ for any $\alpha \in KR_G(Y)$.*

Proof. Let $i: \{o\} \to V$ be the inclusion of the origin and let $l: V \to S$ be the open inclusion. Then, it follows from [3], p.501, (A.1) and (A.2) that $K_G(\text{point}) \ni a$-ind$(l_* i_!(\beta))$ = a-ind$((l \cdot i)_!(\beta)) = \beta$ for any $\beta \in K_G(\text{point})$. Hence, the following diagram is commutative

$$\begin{array}{ccccc} & & KR_G(TS) & & \\ & \swarrow^{a\text{-ind}} & \downarrow & \nwarrow^{l_* i_!} & \\ KR_G(\text{point}) & & K_G(TS) & & KR_G(\text{point}) \\ F\downarrow & \swarrow^{a\text{-ind}} & & \nwarrow^{l_* i_!} & \downarrow F \\ K_G(\text{point}) & & \xleftarrow[\text{identity}]{\qquad\qquad} & & K_G(\text{point}) \end{array}$$

where vertical arrows denote the forgetting maps. Then, since F is known to be injective, it follows that

(7.1.1) $KR_G(\text{point}) \ni a\text{-ind}(l_* i_!(\gamma)) = \gamma$ for any $\gamma \in KR_G(\text{point})$.

Let $ET: KR_G(Y) \otimes KR_G(TV) \to KR_G(Y \times TV)$ denote the external tensor product (where $\otimes$ denotes the tensor product as $KR_G(\text{point})$-modules) and let $\lambda_{TV} \in KR_G(TV)$ denote the Thom class of $TV \to$ point. Then, it is easy to see that $ET(\beta \otimes \lambda_{TV}) = j_!(\beta)$ for any $\beta \in KR_G(Y)$. Namely, the following diagram is commutative.

$$\begin{array}{ccc} KR_G(Y) \otimes KR_G(\text{point}) & \xrightarrow[1\otimes i_!]{\simeq} & KR_G(Y) \otimes KR_G(TV) \\ \| & & \downarrow ET \\ KR_G(Y) & \xrightarrow[j_!]{\simeq} & KR_G(Y \times TV) \end{array} \tag{7.1.2}$$

In particular, ET is also an isomorphism.

Now, let γ be any element of $KR_G(TS)$. Let E and F be Real G-vector bundles over S and let $Q: C^\infty(S, E) \to C^\infty(S, F)$ be a G-equivariant Real elliptic operator of order m such that $\sigma(Q)$ represents γ. Let M be any Real G-vector bundle over Y. Then, since $C^\infty(Y \times S, M \boxtimes E)$ is naturally isomorphic to $M \otimes C^\infty(S, E)$ as a Fréchet bundle over Y (where $\boxtimes$ denotes the external tensor product), there exists a *G.R.E.F.* $P = 1 \otimes Q: Y \to \mathrm{Ell}^m(Y \times S; M \boxtimes E, M \boxtimes F)$ such that $\sigma(P) = 1 \boxtimes \sigma(Q)$ represents $ET(M \otimes \gamma)$ where ET denotes the external tensor product $KR_G(Y) \otimes KR_G(TS) \to KR_G(Y \times TS)$. Since the map

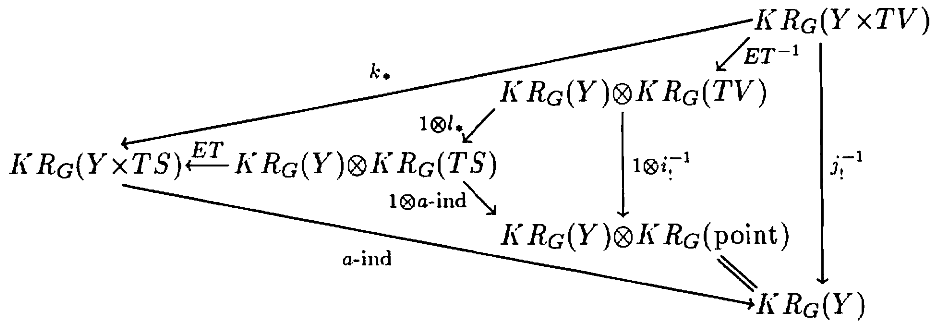
$$\begin{aligned} \Phi_y :\ & C^\infty(Y \times S, M \boxtimes E)_y \oplus (M_y \otimes \text{kernel}(Q^*)) \\ & \longrightarrow C^\infty(Y \times S, M \boxtimes F)_y \end{aligned}$$

given by $\Phi_y(u, v) = P_y(u) + v$ is clearly surjective and $\text{kernel}(\Phi_y) = \text{kernel}(P_y) = M_y \otimes \text{kernel}(Q)$ for any y, it follows that

$$\begin{aligned} a\text{-ind}(ET(M \otimes \gamma)) &= \text{index}(P) \\ &= M \otimes (\text{kernel}(Q) - \text{kernel}(Q^*)) \\ &= M \otimes a\text{-ind}(\gamma). \end{aligned} \tag{7.1.3}$$

Note that, in particular, a-ind is a $KR_G(\text{point})$-module homomorphism.

It follows from (7.1.1), (7.1.2) and (7.1.3) that the following diagram is commutative.



8. The multiplicative axiom. Let Z, V and N be as in §3. Let $O(N)$ denote the orthonormal frame bundle of N. $O(N)$ is a right $O(n)$ (where n is the fiber dimension of N) and left G_R-space. We define a family D over Y by $D = O(N)\times_{O(n)}S^n$ where S^n is the one point compactification of $\boldsymbol{R}^n$ with the trivial G_R-action. Then, there exists a G_R-embedding

$$\begin{aligned} i:\ Z = O(N)\times_{O(n)}\{o\} &\longrightarrow N = O(N)\times_{O(n)}\boldsymbol{R}^n \\ &\longrightarrow D = O(N)\times_{O(n)} S^n \end{aligned}$$

and $i_!: KR_G(T_FZ) \to KR_G(T_FD)$ is defined just as in §1 and §3.

Proposition 8.1. *The following diagram is commutative.*

$$\begin{array}{ccc} KR_G(T_FZ) & \xrightarrow{\quad i_! \quad} & KR_G(T_FD) \\ & {\scriptstyle a\text{-ind}}\searrow \quad \swarrow{\scriptstyle a\text{-ind}} & \\ & KR_G(Y) & \end{array}$$

We first recall the expression of $i_!$ by the local external product according to [3] and [4].

Let $T_F(D/Z)$ denote the Real G-vector bundle over D defined by $T_F(D/Z) = O(N)\times_{O(n)} TS^n$. Then, T_FD decomposes into horizontal and vertical parts with respect to a G_R-invariant metric on D as $T_FD = p^*T_FZ \oplus T_F(D/Z)$ where $p: D \to Z$ denotes the projection. Let $d: T_FD = p^*T_FZ \oplus T_F(D/Z) \to p^*T_FZ \times T_F(D/Z)$ denote the canonical inclusion, $\bar{p}: p^*T_FZ \to T_FZ$ the proper G_R-map given by the restriction of p_*, $r: O(N)\times TS^n \to TS^n$ the projection and $j: \{o\} \to S^n$ the canonical embedding. Let 1 denote the unit element of $KR_{G\times O(n)}(\text{point})$ where the involution of $O(n)$ is trivial. Then, $r^*j_!(1)$ $(= r^*j_!(1)/O(n))$ is an element of $KR_{G\times O(n)}(O(N)\times TS^n) = KR_G(T_F(D/Z))$. Let $\phi: KR_G(T_FZ) \to KR_G(T_FD)$ denote the homomorphism given by $\phi(\alpha) = d^*(\bar{p}^*\alpha \boxtimes r^*j_!(1))$ where $\boxtimes$ denotes the external tensor product on $p^*T_FZ \times T_F(D/Z)$. Then, ϕ coincides with $i_!$ (cf. [3], §4 and [4], §4).

Lemma 8.2. *There exists a $G\times O(n)$-equivariant Real elliptic operator of order* 1 $B: C^\infty(S^n, F_0) \to C^\infty(S^n, F_1)$ *(where F_0 and F_1 are smooth Real $G\times O(n)$-vector bundle over S^n) such that the following* (1) *and* (2) *are satisfied.*

(1) $\{\sigma(B): q_S^*F_0 \to q_s^*F_1\}$ *represents* $j_!(1)$ *where* $q_S: TS^n \to S^n$ *denotes the projection.*

(2) $\mathrm{kernel}(B^*) = 0$ *and* $\mathrm{kernel}(B)$ *is one dimensional trivial Real $G\times O(n)$-module.*

Proof. It follows from Proposition 7.1 that $a\text{-ind}(j_!(1)) = 1 \in KR_{G\times O(n)}(\mathrm{point})$. Hence, Lemma 8.2 follows from the same argument as in [4], Lemma (4.1).

Let F be a Real $G\times O(n)$-vector bundle over S^n and let $\tilde{E}$ be a Real G-vector bundle over Z. We define a Real G-vector bundle $\tilde{E}\boxtimes_l F$ over D by $\tilde{E}\boxtimes_l F = p^*\tilde{E}\otimes((\bar{r}^*F)/O(n))$ where $\bar{r}: O(N)\times S^n \to S^n$ denotes the projection and we call $\tilde{E}\boxtimes_l F$ the local external tensor product. Note that, if D is the trivial fiber bundle $Z\times S^n$, then the local external tensor product coincides with the ordinary external tensor product. Note moreover that, if $\tilde{E}$ is a family and F is smooth, then $\tilde{E}\boxtimes_l F$ is also a family.

Let $q_D: T_FD \to D$ denote the projection and let $i_D: D \to T_FD$ and $i_S: S^n \to TS^n$ denote zero-sections. Let $v: T_FD \to T_F(D/Z)$ denote the vertical projection and let $p_*: T_FD \to T_FZ$ be the differential of p. Then, it follows that

$$
\begin{aligned}
q_D^*(\tilde{E}\boxtimes_l F) &= q_D^*p^*\tilde{E}\otimes q_D^*((\bar{r}^*F)/O(n)) \\
&= (p_*)^*q^*\tilde{E}\otimes q_D^*((\bar{r}^*i_S^*q_S^*F)/O(n)) \\
&= (p_*)^*q^*\tilde{E}\otimes q_D^*i_D^*v^*((r^*q_S^*F)/O(n)) \\
&= (p_*)^*q^*\tilde{E}\otimes v^*((r^*q_S^*F)/O(n)) \\
&= d^*\{\bar{p}^*q^*\tilde{E}\boxtimes((r^*q_S^*F)/O(n))\}.
\end{aligned}
\tag{8.3}
$$

Now, since $\sigma(B)$ in Lemma 8.2 is $O(n)$-equivariant, a Real G-vector bundle homomorphism $\tilde{\sigma}(B): q_D^*(\tilde{E}\boxtimes_l F_0) \to q_D^*(\tilde{E}\boxtimes_l F_1)$ is defined by

$\tilde{\sigma}(B) = 1_{(p_*)^* q^* \tilde{E}} \otimes v^*((r^*\sigma(B))/O(n))$. Note that, if D is trivial, then $\tilde{\sigma}(B)$ coincides with the natural lift of $\sigma(B)$.

Let α be any element of $KR_G(T_F Z)$ which is represented by the triple $\{\psi: q^*\tilde{E}_0 \to q^*\tilde{E}_1\}$. Note that ψ is clearly G_R-homotopic to ψ^1 (cf. Definition 5.8) through homomorphisms which give isomorphisms outside a compact set. Hence, ψ is equivalent to ψ^1 (cf. §5) and we may assume that ψ is homogeneous of degree 1. Then, it follows from Lemma 8.2, (1) and the definition of ϕ that $\phi(\alpha)$ is represented by the triple $\{\Psi: q_D^*(\tilde{E}_0 \boxtimes_l F_0 \oplus \tilde{E}_1 \boxtimes_l F_1) \to q_D^*(\tilde{E}_1 \boxtimes_l F_0 \oplus \tilde{E}_0 \boxtimes_l F_1)\}$ where

$$\Psi = \begin{pmatrix} \tilde{\psi} & -\tilde{\sigma}(B)^* \\ \tilde{\sigma}(B) & \tilde{\psi}^* \end{pmatrix} \qquad (* \text{ denotes the adjoint})$$

and $\tilde{\psi} = (p_*)^*\psi \otimes 1$. Since, ϕ coincides with $i_!$, it suffices for the proof of Proposition 8.1 to show that $\mathrm{index}(\psi) = \mathrm{index}(\Psi)$.

Now, let $\{U_Y^i\}$ be a finite open covering of Y and let $\{U_X^j\}$ be a finite open covering of X. We may assume that Z is trivial over each U_Y^i and D is trivial over each $U^{ij} = U_Y^i \times U_X^j \subset Z$. Let $\{\mu_j^2\}$ be a smooth partition of unity subordinate to $\{U_X^j\}$ and let $\{\nu_i\}$ be a continuous partition of unity subordinate to $\{U_Y^i\}$. Let $A': Y \to \mathrm{Ell}^1(Z; \tilde{E}_0, \tilde{E}_1)$ be a *G.R.E.F.* such that $\sigma(A') \sim \psi$. We set $A_y^{ij} = \nu_i(y)(\mu_j A'_y \mu_j)$. Since $D \supset p^{-1}(U^{ij}) = U^{ij} \times S^n$, $(\tilde{E}\boxtimes_l F)|_{p^{-1}(U^{ij})} = (\tilde{E}|_{U^{ij}})\boxtimes F$ and the lifted operator $\tilde{A}_y^{ij} \in \mathcal{P}_c^1(D; \tilde{E}_0 \boxtimes_l F_0, \tilde{E}_1 \boxtimes_l F_1)_y$ is defined (cf. [3], (5.4)). We define a G_R-section $\tilde{A}: Y \to \mathcal{P}_c^1(D; \tilde{E}_0 \boxtimes_l F_0, \tilde{E}_1 \boxtimes_l F_0)$ by $\tilde{A}_y = \int_{G_R} g\cdot(\sum_{i,j} \tilde{A}_y^{ij})\, dg$. Note that $\tilde{A}$ is not in general an elliptic family. We define a G_R-section $A: Y \to \mathcal{P}_c^1(Z; \tilde{E}_0, \tilde{E}_1)$ by $A_y = \int_{G_R} g\cdot(\sum_{i,j} A_y^{ij})\, dg$. Then,

$$\begin{aligned} \sigma(A) &= \int_{G_R} g\cdot\Big(\sum_{i,j} \sigma(A^{ij})\Big)\, dg = \int_{G_R} g\cdot\Big(\sum_{i,j} \nu_i \mu_j^2 \sigma(A')\Big)\, dg \\ &= \int_{G_R} g\cdot\sigma(A')\, dg = \sigma(A'). \end{aligned}$$

Hence, A is a *G.R.E.F.* and $\sigma(A) \sim \psi$. Moreover,

$$\begin{aligned} \sigma(\tilde{A}) &= \int_{G_R} g\cdot\Big(\sum_{i,j} \sigma(\tilde{A}^{ij})\Big)\, dg \\ &= \int_{G_R} g\cdot\Big(\sum_{i,j} \tilde{\sigma}(A^{ij})\Big)\, dg \quad (\text{cf. [3], (5.4)}) \\ &= \Big(\widetilde{\int_{G_R} g\cdot\Big(\sum_{i,j} \sigma(A^{ij})\Big)\, dg}\Big) = \tilde{\sigma}(A) = (p_*)^*\sigma(A) \otimes 1. \end{aligned}$$

Since, B is $O(n)$-equivariant, by restricting the lifted operator of B on $O(N)_y \times S^n$, an operator $\tilde{B}_y$ over $O(N)_y \times_{O(n)} S^n = D_y$ is defined. Then, $\tilde{B}_y$ is the lifted operator of B on $p^{-1}(U^{ij}) \cap D_y$ and $\sigma(\tilde{B}) = \tilde{\sigma}(B)$. Now, we define a G_R-section $P: Y \to \mathcal{P}_c^1(D; \tilde{E}_0 \boxtimes_l F_0 \oplus \tilde{E}_1 \boxtimes_l F_1, \tilde{E}_1 \boxtimes_l F_0 \oplus \tilde{E}_0 \boxtimes_l F_1)$ by

$$P_y = \begin{pmatrix} \tilde{A}_y & -\tilde{B}_y^* \\ \tilde{B}_y & \tilde{A}_y^* \end{pmatrix}.$$

Lemma 8.4. (1) *P is a G.R.E.F..*
(2) $\sigma(P) \sim \Psi$. *In particular,* $\mathrm{index}(\Psi) = \mathrm{index}(P)$.

Proof. (1) Let (ξ, η) be any element of $T(D_y) = p^*T(Z_y) \oplus T(D_Y/Z_y)$ and $(u, v) \in q_D^*(\tilde{E}_0 \boxtimes_l F_0 \oplus \tilde{E}_1 \boxtimes_l F_1)_{(\xi,\eta)}$. We assume that

$$\begin{aligned} &\sigma(P)(\xi,\eta)(u \oplus v) \\ &= (\tilde{\sigma}(A)(\xi)(u) - \tilde{\sigma}(B)(\eta)^*(v)) \oplus (\tilde{\sigma}(B)(\eta)(u) + \tilde{\sigma}(A)(\xi)^*(v)) \\ &= 0. \end{aligned}$$

Then, since $\tilde{\sigma}(A)$ commutes with $\tilde{\sigma}(B)$, it follows that

$$\tilde{\sigma}(A)(\xi) \cdot \tilde{\sigma}(A)(\xi)^*(v) + \tilde{\sigma}(B)(\eta) \cdot \tilde{\sigma}(B)(\eta)^*(v) = 0.$$

Hence, it follows that $v = 0$ or both of $\tilde{\sigma}(A)(\xi)^*$ and $\tilde{\sigma}(B)(\eta)^*$ are degenerate, namely, both of $\sigma(A)(\xi)$ and $\sigma(B)(\eta)$ are degenerate. Therefore, it follows that $v = 0$ or $\xi = \eta = 0$, since A and B are elliptic. Moreover, it follows from the same argument that $u = 0$ or $\xi = \eta = 0$. Hence, it follows that $\sigma(P)(\xi, \eta)$ is non-degenerate outside the zero-section and P is elliptic.

(2) Since $\sigma(A) \sim \psi$, there exists a G_R-section $\chi: Y \times I \to Is^1(Z \times I; \tilde{E}_0 \times I, \tilde{E}_1 \times I)$ such that χ_0 $(= \chi|_{Y \times \{0\}}) = \sigma(A)$ and $\chi_1 = \psi$. Then, it follows from the same argument as in (1) that

$$\Phi_t = \begin{pmatrix} \tilde{\chi}_t(\xi) & -\tilde{\sigma}(B)(\eta)^* \\ \tilde{\sigma}(B)(\eta) & \tilde{\chi}_t(\xi)^* \end{pmatrix}$$

is non-degenerate for any $(\xi, \eta) \neq 0$ and any $t \in I$. Since, $\Phi_0 = \sigma(P)$ and $\Phi_1 = \Psi$, it follows that $\sigma(P) \sim \Psi$.

Since $\mathrm{index}(\psi) = \mathrm{index}(A)$, it suffices for the proof of Proposition 8.1 to show that $\mathrm{index}(A) = \mathrm{index}(P)$.

Now, let β be a generator of kernel(B) which is the one dimensional trivial Real $G \times O(n)$-module (cf. Lemma 8.2, (2)) and we define a homomorphism $f: C^\infty(Z,E) \to C^\infty(D, \tilde{E} \otimes_l F_0)$ by $f_y(u)(x,\theta) = u(x) \otimes \beta(\theta)$ for $u \in C^\infty(Z,\tilde{E})_y$ and $(x,\theta) \in Z_y \times S^n$ where $Z_y \times S^n$ is identified with D_y locally. Since $\beta \in C^\infty(S^n, F_0)$ is $G_R \times O(n)$-equivariant, it is easy to see that f is a well-defined injective Real G-vector bundle homomorphism and $\tilde{A}(f(u)) = f(A(u))$ (i.e. $\tilde{A}(u \otimes \beta) = A(u) \otimes \beta$). Let L be a finite dimensional Real G-vector bundle over Y and let $T: L \to C^\infty(Z, \tilde{E}_1)$ be a Real G-vector bundle homomorphism such that the map $Q_y: C^\infty(Z, \tilde{E}_0)_y \oplus L_y \to C^\infty(Z, \tilde{E}_1)_y$ given by $Q_y(u,v) = A_y(u) + T(v)$ is surjective for any y.

Lemma 8.5. (1) *The map* $R_y: C^\infty(D, \tilde{E}_0 \boxtimes_l F_0 \oplus \tilde{E}_1 \boxtimes_l F_1)_y \oplus L_y \to C^\infty(D, \tilde{E}_1 \boxtimes_l F_0 \oplus \tilde{E}_0 \boxtimes_l F_1)_y$ *given by* $R_y(u,v) = P_y(u) + (f \cdot T(v) \oplus 0)$ *is surjective for any* y.

(2) *Let* $h: C^\infty(Z, \tilde{E}_0) \oplus L \to C^\infty(D, \tilde{E}_0 \boxtimes_l F_0 \oplus \tilde{E}_1 \boxtimes_l F_1) \oplus L$ *be a Real* G*-vector bundle homomorphism given by* $h(u,v) = (f(u) \oplus 0, v)$*. Then, the restriction of* h *gives an isomorphism* $\mathrm{Ker}Q \simeq \mathrm{Ker}R$.

It follows from Lemma 8.5 that index(A) = index(P) and the proof of Proposition 8.1 is completed. In order to prove Lemma 8.5, we need the following simple lemma.

Lemma 8.6. (1) kernel$(\tilde{B}_y^*) = 0$ *for any* y.

(2) *For any* $\tilde{u} \in$ kernel$(\tilde{B}_y)$*, there exists an element* $u \in C^\infty(Z, \tilde{E})_y$ *such that* $\tilde{u} = u \otimes \beta$*, namely,* $\tilde{u} = f_y(u)$.

Proof. (1) Let $\tilde{u} = \sum_{i,j} u^{ij}(x,\theta) e_i(x) \otimes f_j(\theta)$ be any element of kernel$(\tilde{B}_y^*)$ where $\{e_i: Z_y \to \tilde{E}_y\}_i$ are local basis of $\tilde{E}_y$ and $\{f_j: S^n \to F_1\}_j$ are local basis of F_1. Then, it follows that $\tilde{B}_y^*(\tilde{u}) = \sum_i \{e_i(x) \otimes B^*(\sum_j u^{ij}(x,\theta) f_j(\theta))\} = 0$. Since kernel$(B^*) = 0$ (cf. Lemma 8.2), it follows that $\sum_j u^{ij}(x,\theta) f_j(\theta) = 0$ and $u^{ij}(x,\theta) = 0$ for any i,j.

(2) Let $\tilde{u} = \sum_{i,j} u^{ij}(x,\theta) e_i(x) \otimes f_j(\theta)$ be any element of kernel$(\tilde{B}_y)$. Then, it follows that $\tilde{B}_y(\tilde{u}) = \sum_i \{e_i(x) \otimes B(\sum_j u^{ij}(x,\theta) f_j(\theta))\} = 0$. Since kernel$(B)$ is generated by β, it follows that there exists a smooth function $w^i(x)$ such that $\sum_j u^{ij}(x,\theta) f_j(\theta) = w^i(x)\beta(\theta)$. Hence, it follows that $\tilde{u} = \{\sum_i w^i(x) e_i(x)\} \otimes \beta(\theta)$.

Proof of Lemma 8.5. (1) We first recall the construction of $\tilde{A}$. It is clear that $\tilde{A}^{ij}$ commutes with $\tilde{B}$. Moreover, since $\tilde{B}$ is G_R-equivariant, $\tilde{B}$

commutes with the averaging over G_R. Hence, it follows that $\tilde{A}$ commutes with $\tilde{B}$. Therefore, it follows that

$$\begin{aligned}
\text{cokernel}(P_y) &= \text{kernel}(P_y^*) = \text{kernel}(P_y P_y^*) \\
&= \text{kernel}\begin{pmatrix} \tilde{A}_y\tilde{A}_y^* + \tilde{B}_y^*\tilde{B}_y & 0 \\ 0 & \tilde{B}_y\tilde{B}_y^* + \tilde{A}_y^*\tilde{A}_y \end{pmatrix} \\
&= (\text{kernel}(\tilde{A}_y^*)) \cap (\text{kernel}(\tilde{B}_y)) \oplus (\text{kernel}(\tilde{B}_y^*)) \cap (\text{kernel}(\tilde{A}_y)).
\end{aligned}$$

Hence, it follows from Lemma 8.6 that, for any element $\tilde{u}$ of cokernel(P_y), there exists $u \in C^\infty(Z, \tilde{E}_1)_y$ such that $\tilde{u} = u \otimes \beta \oplus 0$. Now, it follows from the assumption that there exist $u' \in C^\infty(Z, \tilde{E}_0)_y$ and $v \in L_y$ such that $u = A_y(u') + T(v)$. Hence, it follows that

$$\begin{aligned}
\tilde{u} &= (A_y(u') \otimes \beta + T(v) \otimes \beta) \oplus 0 = (\tilde{A}_y(u' \otimes \beta) + f\cdot T(v)) \oplus 0 \\
&= P_y(u' \otimes \beta \oplus 0) + (f\cdot T(v) \oplus 0) = R_y(u' \otimes \beta \oplus 0, v).
\end{aligned}$$

Therefore, it follows that R_y is surjective.

(2) Let (u, v) be any element of kernel(Q_y). Then, it follows that

$$\begin{aligned}
R_y(h(u,v)) &= P_y(u \otimes \beta \oplus 0) + T(v) \otimes \beta \oplus 0 \\
&= (\tilde{A}_y(u \otimes \beta) \oplus 0) + (T(v) \otimes \beta \oplus 0) \\
&= (A_y(u) + T(v)) \otimes \beta \oplus 0 = 0.
\end{aligned}$$

Hence, it follows that $h(\text{Ker}Q)$ is contained in Ker$R$. On the other hand, let $(u_1 \oplus u_2, v)$ be any element of kernel(R_y). Then, it follows that

$$\begin{aligned}
&P_y(u_1 \oplus u_2) + (f\cdot T(v) \oplus 0) \\
&\quad = (\tilde{A}_y(u_1) - \tilde{B}_y^*(u_2) + T(v) \otimes \beta) \oplus (\tilde{B}_y(u_1) + \tilde{A}_y^*(u_2)) = 0.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\tilde{A}_y(\tilde{B}_y(u_1) + \tilde{A}_y^*(u_2)) - \tilde{B}_y(\tilde{A}_y(u_1) - \tilde{B}_y^*(u_2) + T(v) \otimes \beta) \\
&\quad = \tilde{A}_y\tilde{A}_y^*(u_2) + \tilde{B}_y\tilde{B}_y^*(u_2) = 0
\end{aligned}$$

and, hence, $\tilde{B}_y^*(u_2) = 0$. Therefore, it follows from Lemma 8.6 that $u_2 = 0$. Moreover, it follows that

$$\begin{aligned}
&\tilde{A}_y^*(\tilde{A}_y(u_1) - \tilde{B}_y^*(u_2) + T(v) \otimes \beta) + \tilde{B}_y^*(\tilde{B}_y(u_1) + \tilde{A}_y^*(u_2)) \\
&\quad = \tilde{A}_y^*\tilde{A}_y(u_1) + \tilde{B}_y^*\tilde{B}_y(u_1) + \tilde{A}_y^*(T(v) \otimes \beta) = 0.
\end{aligned}$$

Hence,

$$\tilde{B}_y^*\tilde{B}_y(\tilde{A}_y^*\tilde{A}_y(u_1) + \tilde{B}_y^*\tilde{B}_y(u_1) + \tilde{A}_y^*(T(v)\otimes\beta))$$
$$= \tilde{B}_y^*\tilde{A}_y^*\tilde{A}_y\tilde{B}_y(u_1) + \tilde{B}_y^*\tilde{B}_y\tilde{B}_y^*\tilde{B}_y(u_1) = 0$$

and, hence, $\tilde{B}_y^*\tilde{B}_y(u_1) = 0$. Therefore, it follows that $\tilde{B}_y(u_1) = 0$ and there exists an element u of $C^\infty(Z,\tilde{E}_0)_y$ such that $u_1 = u\otimes\beta$. Moreover, it follows that

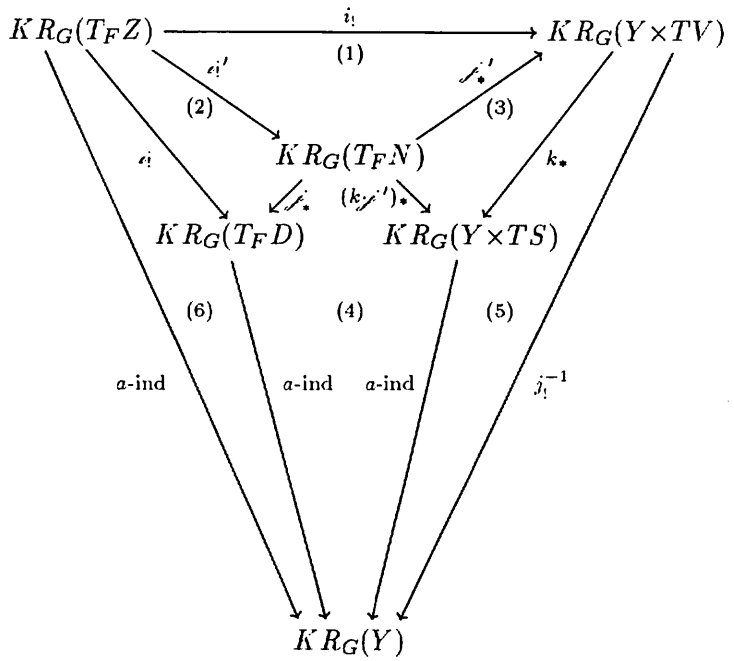
$$(A_y(u) + T(v))\otimes\beta = \tilde{A}_y(u_1) - \tilde{B}_y^*(u_2) + T(v)\otimes\beta = 0.$$

Hence, it follows that $(u,v)\in \mathrm{kernel}(Q_y)$ and $h\colon \mathrm{Ker}Q \to \mathrm{Ker}R$ is surjective. Since, f is injective, it follows that h gives an isomorphism.

9. The proof of the index theorem.

Theorem 9.1. a-ind $= t$-ind: $KR_G(T_FZ) \to KR_G(Y)$.

Proof. We consider the following diagram where the notation is as in §6, §7 and §8.



The commutativities of (1), (2) and (3) are obvious. The commutativities of (4), (5) and (6) follow from Proposition 6.1, Proposition 7.1

and Proposition 8.1, respectively. Hence, it follows that t-ind $= j_!^{-1} \cdot i_! =$ a-ind: $KR_G(T_F Z) \to KR_G(Y)$. This completes the proof.

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