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THE ELEMENT $\bar{\kappa}$ IS NOT IN THE IMAGE OF THE S^1-TRANSFER

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

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Introduction. The present note may be regarded as a continuation of [13]. The notation and results of [13] are used freely here. Let $g_n: CP^{n-1} \to S^{-1}$ be the S^1-transfer map and set $B_k = \mathrm{Im}\{g_{n*}: \pi^S_{k-1}(CP^{n-1}) \to \pi^S_k(S^0)$ for $k \leq 2n-1\}$. Let p be a prime number. We denote by $\pi_k(X;p)$ the p-primary components of $\pi_k(X)$. The purpose of this note is to prove the following

Theorem. i) $\pi^S_{19}(CP;2) \cong Z_{64} \oplus Z_4$.
ii) $\pi^S_{20}(CP;2) \cong Z_2 \oplus Z_2 \oplus Z_2$.
iii) $B_{20} = 2\pi^S_{20}(S^0)$ *and* $B_{21} = \pi^S_{21}(S^0)$.

The result i) was already obtained by Mosher in [10]. The result iii) shows that the element $\bar{\kappa} \in \pi^S_{20}(S^0)$ given in [6] is not in the image of the S^1-transfer.

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1. Some homotopy groups of Q_n. We denote by $\widetilde{\eta\kappa} \in \langle i, \eta, \eta\kappa\rangle$ a coextension of $\eta\kappa$. By making use of $(6)_n = (6)_{n,19}$ of [13] for $n = 1$, we have $\pi^S_{19}(CP^2) = Z_{480}\{\zeta_2\rho\} \oplus Z_4\{\widetilde{\eta\kappa}\}$, where $2\widetilde{\eta\kappa} \in -i\langle\eta, \eta\kappa, 2\iota\rangle \ni i\nu\kappa$ mod 0 and $240\zeta_2\rho \in -i\langle\eta, 2\iota, 240\rho\rangle \ni i\bar{\mu}$ mod 0.

In $(6)_2$, we have $\gamma_2\sigma^2 = 2i\nu\sigma^2 = 0$, $\gamma_2\kappa = 2i\nu\kappa = 0$ and $\gamma_2\alpha_1\beta_1 = 0$. So we have $\pi^S_{19}(CP^3) = Z_{480}\{i\zeta_2\rho\} \oplus Z_4\{i\widetilde{\eta\kappa}\} \oplus Z_3\{\tilde{\alpha}_1\beta\}$, where $\tilde{\alpha}_1 \in \langle i, \gamma_2, \alpha_1\rangle$ is a coextension of α_1.

In $(6)_3$ and in the 2-primary components, we have $\gamma_3\zeta \in -i\langle\gamma_2, \eta, \zeta\rangle \supset \pm i\nu\langle 2\iota, \eta, \zeta\rangle \ni 0 \bmod i_*\pi^S_7(CP^2;2)\circ\zeta = \{i\bar{\nu}\zeta\} = 0$ ([11]). For $\bar{\nu}\zeta \in -i\langle\eta, \nu, \zeta\rangle \ni 0 \bmod i\eta\rho = 0$. Let $\tilde{\zeta} \in \langle i, \gamma_3, \zeta\rangle$ be a coextension of ζ. Then we have $8\tilde{\zeta} \in -i\langle\gamma_3, \zeta, 8\iota\rangle$ and $2\langle\gamma_3, \zeta, 8\iota\rangle \subset \langle 2\gamma_3, \zeta, 8\iota\rangle = \langle\pm i\zeta_2\nu, \zeta, 8\iota\rangle \supset \pm i\zeta_2\langle\nu, \zeta, 8\iota\rangle \ni \pm 4i\zeta_2\rho$ ([4]) $\bmod 8\pi^S_{19}(CP^3) = \{8i\zeta_2\rho\}$. So we have $\langle\gamma_3, \zeta, 8\iota\rangle \ni 2i\zeta_2\rho \bmod \{4i\zeta_2\rho, i\nu\kappa\}$. Applying $g_4: CP^3 \to S^{-1}$ to this relation, we have $g_4(i\zeta_2\rho) = g_3\zeta_2\rho \in \pi_5^S(S^0)\circ\rho = 0$, $g_4(i\nu\kappa) = \nu^2\kappa = 4\bar{\kappa}$ and $g_4\langle\gamma_3,\zeta,8\iota\rangle = -\langle g_4,\gamma_3,\zeta\rangle\circ 8\iota \subset 8\pi_{20}^S(S^0) = 0$ ([6]). Therefore we have $\langle\gamma_3,\zeta,8\iota\rangle \ni 2i\zeta_2\rho \bmod 4i\zeta_2\rho$ and $8\tilde{\zeta} = \pm 2i\zeta_2\rho$.

In the 3-primary components, $8\gamma_3\alpha_3' = \pm 4i\zeta_2\underset{\smile}{\alpha_1}\alpha_3' = 0$, where α_3' is a generator of $\pi_{11}^S(S^0) \cong Z_9$ ([14]). So, for $\widetilde{\alpha_3'} \in \langle i,\gamma_3,\alpha_3'\rangle$, we have $9\widetilde{\alpha_3'} \in -i\langle\gamma_3,\alpha_3',9\iota\rangle$. $8\langle\gamma_3,\alpha_3',9\iota\rangle \subset 4\langle\pm i\zeta_2\alpha_1,\alpha_3,3\iota\rangle \ni \pm i\zeta_2\alpha_4 \bmod 3\pi_{19}^S(CP^3;3) = 0$ by (13.8) and (13.11) of [14]. Therefore $\widetilde{\alpha_3'}$ is of order 27. We denote by α_1' and α_2 generators of $\pi_{11}^S(S^0;7) \cong Z_7$ and $\pi_{15}^S(S^0;5) \cong Z_5$, respectively.

By summarizing the above argument, we have the following

Proposition 1.1. *$\pi_{19}^S(CP^4) = Z_{128}\{\tilde{\zeta}\} \oplus Z_2\{i\zeta_2\rho \pm 4\tilde{\zeta}\} \oplus Z_4\{i\widetilde{\eta\kappa}\} \oplus Z_{27}\{\widetilde{\alpha_3'}\} \oplus Z_5\{i\zeta_2\alpha_2\} \oplus Z_7\{\widetilde{\alpha_1'}\}$, where $\widetilde{\alpha_1'} \in \langle i,\gamma_3,\alpha_1'\rangle$ and $64\tilde{\zeta} = i\bar{\mu}$.*

Hereafter we shall work in the 2-primary components, unless otherwise stated.

First we determine $\pi_k^S(Q_n)$ for $k = 19$ and 20 and $n \leq 5$.

Lemma 1.2. i) *$\pi_{19}^S(Q_2) = Z_2\{i\eta *\} \oplus Z_2\{i\eta\rho\}$, where $\omega_2\mu = i\eta\rho$.*

ii) *$\pi_{20}^S(Q_2) = Z_2\{i\nu\kappa\} \oplus Z_2\{i\eta\eta *\} \oplus Z_2\{i\eta^2\rho\} \oplus Z_2\{i\bar{\mu}\}$, where $\omega_2\eta\mu = i\eta^2\rho$.*

iii) *$\pi_{19}^S(Q_3) = Z_2\{\tilde{\sigma}\eta\} \oplus Z_2\{i\eta *\}$.*

iv) *$\pi_{19}^S(Q_4) \cong \pi_{19}^S(Q_3)$, where $\omega_4\eta \equiv i\tilde{\sigma}\eta \bmod i\eta *$ and $\pi_{19}^S(Q_5) \cong Z \oplus Z_2\{i\eta *\}$.*

v) *$\pi_{20}^S(Q_3) = Z_4\{\widetilde{\eta\varepsilon}\} \oplus Z_2\{\tilde{\sigma}\eta^2\} \oplus Z_2\{i\eta\eta *\} \oplus Z_2\{i\bar{\mu}\}$, where $\widetilde{\eta\varepsilon} \in \langle i,\omega_2,\eta\varepsilon\rangle$ and $2\widetilde{\eta\varepsilon} = i\nu\kappa$.*

vi) *$\pi_{20}^S(Q_4) \cong \pi_{20}^S(Q_3)$ and $\pi_{20}^S(Q_5) = Z_4\{i\widetilde{\eta\varepsilon}\} \oplus Z_2\{i\eta\eta *\} \oplus Z_2\{i\bar{\mu}\}$.*

Proof. Obviously $i_*: \pi_k^S(S^3) \to \pi_k^S(Q_2)$ is an isomorphism for $k = 19$ and 20. By Lemma 4.2 of [13], we have $\omega_2\mu \in \omega_2\langle\eta,8\sigma,2\iota\rangle \subset \langle\omega_2\eta,8\sigma,2\iota\rangle = \langle i\varepsilon,8\sigma,2\iota\rangle \supset i\langle\varepsilon,8\sigma,2\iota\rangle \ni i\eta\rho \bmod \omega_2\eta\pi_8^S(S^0) + 2\pi_{19}^S(Q_2) = 0$. This leads us to i) and ii). We recall the element $\tilde{\sigma} \in \langle i,\omega_2,\sigma\rangle \subset \pi_{18}^S(Q_3)$ ([13]). By Lemmas 4.1 and 4.2 of [13], we have $\omega_2\varepsilon = i\eta\kappa$. So, i) implies iii). The first half of iv) is trivial. By making use of Proposition 2.4 and Example 2.2 of [7], we have $\omega_4''\eta = i\sigma\eta$, where ω_4'' is ω_4 followed by the projection $Q_4 \to Q_3^4 = Q_4/Q_2$. So we have $\omega_4\eta \equiv i\tilde{\sigma}\eta \bmod i\eta *$. This leads us to iv). Since $\omega_2\eta\varepsilon = i\varepsilon^2 = 0$, we have a coextension $\widetilde{\eta\varepsilon} \in \langle i,\omega_2,\eta\varepsilon\rangle \subset \pi_{20}^S(Q_3)$ of $\eta\varepsilon$. Here $2\widetilde{\eta\varepsilon} \in -i'\langle\omega_2,\eta\varepsilon,2\iota\rangle \supset i'\langle i\varepsilon,\varepsilon,2\iota\rangle \supset$

$i\langle\varepsilon,\varepsilon,2\iota\rangle \ni i\nu\kappa \bmod 2i_*\pi_{20}^S(Q_2) = 0$. For $\langle\varepsilon,\varepsilon,2\iota\rangle = \langle\varepsilon,\langle\nu^2,2\iota,\eta\rangle,2\iota\rangle \equiv \langle\langle\varepsilon,\nu^2,2\iota\rangle,\eta,2\iota\rangle + \langle\varepsilon,\nu^2,\langle 2\iota,\eta,2\iota\rangle\rangle = \langle\eta\kappa,\eta,2\iota\rangle + \langle\varepsilon,\nu^2,\eta^2\rangle \ni \nu\kappa + 0 \bmod \varepsilon\pi_9^S(S^0) + \pi_{15}^S(S^0)\circ\eta^2 = \{\eta^2\rho\}$. So, ii) implies v). Since $\omega_3\nu^2 = 0$ by Lemma 1.2 of [13], we have the first half of vi). By iv), we have $\omega_4\eta^2 \equiv i\bar{\sigma}\eta^2 \bmod i\eta\eta*$ and hence the second half of vi). This completes the proof.

We recall the canonical map $h_n: CP^{2n+1} \to HP^n$ [13] and we set $\gamma_n' = \gamma_n(H): S^{4n+3} \to HP^n$ and $p_n' = p_n(H): HP^n \to S^{4n}$. By (5) of [13], we have an exact sequence

$$(*)_n \qquad \pi_{20}^S(Q_n) \xrightarrow{t_{n*}} \pi_{19}^S(CP^{2n-1}) \xrightarrow{h_{n-1*}} \pi_{19}^S(HP^{n-1}) \xrightarrow{f_{n-1*}} \pi_{19}^S(Q_n) \longrightarrow \cdots .$$

Lemma 1.3. *i)* $t_3\widetilde{\eta\varepsilon} \equiv \pm i\widetilde{\eta\kappa} + \gamma_5\varepsilon \bmod i\bar{\mu}$ *and* $t_{3*}\pi_{20}^S(Q_3) = Z_4\{i\widetilde{\eta\kappa} + \gamma_5\varepsilon\} \oplus Z_2\{\gamma_5\eta\sigma\} \oplus Z_2\{i\bar{\mu}\}$.

ii) $t_{k*}\pi_{20}^S(Q_k) = Z_4\{i\widetilde{\eta\kappa}\} \oplus Z_2\{i\bar{\mu}\}$ *for* $k = 4$ *and* 5.

Proof. We recall the element $\tilde{\sigma}'' = t_3\tilde{\sigma}$. By Lemma 1.2 and $(*)_2$, we have the relations $t_3\bar{\sigma}\eta^2 = \tilde{\sigma}''\eta^2$, $t_3(i\eta\eta*) = 0$, $t_3(i\eta^2\rho) = 0$ and $t_3(i\bar{\mu}) = i\bar{\mu}$. By Proposition 4.18 of [13], $\tilde{\sigma}''\eta^2 \equiv \gamma_5\sigma\eta \bmod i\eta\eta* = 0$. So we have $t_{3*}\pi_{20}^S(Q_3) = \{t_3\widetilde{\eta\varepsilon}, \gamma_5\eta\sigma, i\bar{\mu}\}$. Since $h_2(i\widetilde{\eta\kappa}) = ip_2\widetilde{\eta\kappa} = i\eta\kappa$ and $h_2\gamma_5\varepsilon = \gamma_2'\varepsilon \in -i\langle\nu,2\nu,\varepsilon\rangle \ni i\eta\kappa \bmod 0$, we have $i\widetilde{\eta\kappa} + \gamma_5\varepsilon \in t_{3*}\pi_{20}^S(Q_3)$ by $(*)_3$. Applying $p_5: CP^5 \to S^{10}$ to this relation, we obtain $i\widetilde{\eta\kappa} + \gamma_5\varepsilon \in \{t_3\widetilde{\eta\varepsilon}, i\bar{\mu}\}$. Hence, by the relation $2t_3\widetilde{\eta\varepsilon} = i\nu\kappa = 2i\widetilde{\eta\kappa}$, we have i). By i) and Lemma 1.2, we have ii). This completes the proof.

In $(6)_4$, we have the relations $\gamma_4\mu = it_2\omega_2\mu = i\eta\rho = 0$ and $\gamma_4\eta\mu = 0$ by Lemma 1.2. Let $\tilde{\mu} \in \langle i,\gamma_4,\mu\rangle$ be a coextension of μ. Then we have the following

Proposition 1.4. $\pi_{19}^S(CP^5) = Z_{128}\{i\tilde{\zeta}\} \oplus Z_4\{\tilde{\mu} \pm 2i\tilde{\zeta}\} \oplus Z_4\{i\widetilde{\eta\kappa}\} \oplus Z_2\{\gamma_5\eta\sigma\} \oplus Z_2\{\gamma_5\varepsilon\}$.

Proof. We have $2\tilde{\mu} \in -i\langle\gamma_4,\mu,2\iota\rangle \supset i\langle t_2\omega_2,\mu,2\iota\rangle$ and $\langle t_2\omega_2,\mu,2\iota\rangle \subset \langle t_2,i\eta\rho,2\iota\rangle \supset \langle t_2,i\eta,2\iota\rangle\circ\rho \supset \langle i,\eta,2\iota\rangle\circ\rho \ni i\zeta_2\rho \bmod t_{2*}\pi_{20}^S(Q_2) + 2\pi_{19}^S(CP^3) = 2\pi_{19}^S(CP^3)$ by Lemma 1.2. So we have $2\tilde{\mu} \equiv i\zeta_2\rho \bmod 2i_*\pi_{19}^S(CP^3)$. Therefore, by a suitable choice of $\tilde{\mu}$, we have $2\tilde{\mu} = i\zeta_2\rho$

and $2(\tilde{\mu} \pm 2i\tilde{\zeta}) = i\zeta_2\rho \pm 4i\tilde{\zeta}$. Hence, by Proposition 1.1, we have the proposition. This completes the proof.

2. Determination of $\pi_{19}^S(HP^n;2)$. To determine the group extension of $\pi_{19}^S(CP^6)$, we shall examine $\pi_{19}^S(HP^n)$ for $n \leq 5$. We set $\tilde{\zeta}' = h_2 i\tilde{\zeta}$. Then we have $p_2'\tilde{\zeta}' = \zeta$ and $8\tilde{\zeta}' = \pm 2h_2 i\zeta_2\rho = \pm 4i\rho$. So we have $\pi_{19}^S(HP^2) = Z_{64}\{\tilde{\zeta}'\} \oplus Z_4\{i\rho \pm 2\tilde{\zeta}'\} \oplus Z_2\{i\eta\kappa\}$. We set $HP_k^n = HP^n/HP^{k-1}$. Since $HP_2^3 = S^8 \bigcup_{2\nu_8} e^{12} = \Sigma^5 Q_2$ and $\langle 2\nu, \sigma, 16\iota\rangle \ni 2x\zeta$ for an odd integer x ([14]), we have $\pi_{19}^S(HP_2^3) = Z_{64}\{\tilde{\sigma}\} \oplus Z_2\{xi\zeta + 8\tilde{\sigma}\}$, where $\tilde{\sigma} = \Sigma^\infty\tilde{\sigma}_{13}$ for $\tilde{\sigma}_{13} \in \{i, 2\nu_{10}, \sigma_{13}\}$. We also have $\pi_{20}^S(HP_2^3) = Z_2\{\tilde{\sigma}\eta\} \oplus Z_2\{\tilde{\varepsilon}\}$, where $\tilde{\varepsilon} \in \langle i, 2\nu, \varepsilon\rangle$.

We set $\bar{\nu}_n' = \Sigma^{n-5}\bar{\nu}_5'$ for $\bar{\nu}_5' \in \{\nu_5, 2\nu_8, p''\}$, where $p'' = \Sigma^{-1}p'$: $\Sigma^{-1}HP_2^3 \to S^{11}$ is the collapsing map. We recall that $\Sigma HP^3 = S^5 \bigcup_{\bar{\nu}_5'} C(HP_2^3)$. Since $\bar{\nu}_7'\tilde{\sigma}_{13} \in \{\nu_7, 2\nu_{10}, \sigma_{13}\} \subset \pi_{21}(S^7) = \{\sigma'\sigma_{14}, \kappa_7\}$ ([14]), we have $\bar{\nu}'\tilde{\sigma} = 0$ or κ. On the other hand, $\bar{\nu}'\tilde{\sigma} \in \langle \nu, 2\nu, \sigma\rangle \ni 0 \bmod \sigma^2$. So we have $\bar{\nu}'\tilde{\sigma} = 0$ and we can define an element $\tilde{\sigma}' \in \langle i, \bar{\nu}', \tilde{\sigma}\rangle \subset \pi_{19}^S(HP^3)$ as a coextension of $\tilde{\sigma}$.

Lemma 2.1. $\pi_{19}^S(HP^3) = Z_{64}\{\tilde{\sigma}'\} \oplus Z_{64}\{i\tilde{\zeta}'\}$.

Proof. Let $M^n = S^{n-1} \bigcup_{2\iota_{n-1}} e^n$ be a Moore space. Since $2\nu_5 = \tilde{\eta}_5\tilde{\eta}_6$ and $\tilde{\eta}_6 \circ 2\nu_8 \in i\{2\iota_6, \eta_6, 2\iota_7\}\circ\nu_8 = i\eta_6^2\nu_8 = 0$, we can take $\bar{\nu}_5'$ so that $2\bar{\nu}_5' = \tilde{\eta}_5\beta + a\sigma'''p'$. Here $\beta = \bar{\tilde{\eta}}_6 \in \{\tilde{\eta}_6, 2\nu_8, p''\} \subset [HP_2^3, M^7]$ is an extension of $\tilde{\eta}_6$ and $a = 0$ or 1. Now we have $64\tilde{\sigma}' \in -i\langle\bar{\nu}', \tilde{\sigma}, 64\iota\rangle$ and $\beta\tilde{\sigma} \in \langle\tilde{\eta}, 2\nu, \sigma\rangle \subset i_*\pi_{14}^S(S^1) = 0$. So we have $2\langle\bar{\nu}', \tilde{\sigma}, 64\iota\rangle \subset \langle\tilde{\eta}\beta, \tilde{\sigma}, 64\iota\rangle + \langle 8a\sigma p', \tilde{\sigma}, 64\iota\rangle \supset \tilde{\eta}\langle\beta, \tilde{\sigma}, 64\iota\rangle \bmod \tilde{\eta}\beta\pi_{20}^S(HP_2^3) + 64\pi_{15}^S(S^0) = 0$. For $\langle 8a\sigma p', \tilde{\sigma}, 64\iota\rangle \subset \langle 8a\sigma, \sigma, 64\iota\rangle \subset \langle 8a\sigma, 2\sigma, 32\iota\rangle \supset 32a\langle\sigma, 2\sigma, 8\iota\rangle = 0 \bmod 0$ and $\tilde{\eta}\beta\pi_{20}^S(HP_2^3) = 2(\bar{\nu}' + 4a\sigma p')\circ\pi_{20}^S(HP_2^3) = 0$. We have $\langle\beta, \tilde{\sigma}, 64\iota\rangle \subset \pi_{15}^S(M^2) = \{i\sigma^2, i\kappa\}$ and so $\tilde{\eta}\langle\beta, \tilde{\sigma}, 64\iota\rangle \ni 0 \bmod \{\eta\kappa\} + \tilde{\eta}\beta\pi_{20}^S(HP_2^3) = \{\eta\kappa\}$. Since $\eta\kappa$ is not divisible by 2, we have $2\langle\bar{\nu}', \tilde{\sigma}, 64\iota\rangle \ni 0 \bmod 0$ and $128\tilde{\sigma}' = 0$. Now we consider an exact sequence associated with $\bar{\nu}'$:

$$\pi_{20}^S(HP_2^3) \xrightarrow{\bar{\nu}_*'} \pi_{19}^S(S^4) \xrightarrow{i_*} \pi_{19}^S(HP^3) \xrightarrow{p_*} \pi_{19}^S(HP_2^3) \xrightarrow{\bar{\nu}_*'} \cdots.$$

We have $\bar{\nu}'\tilde{\sigma}\eta = 0$, $\bar{\nu}'\tilde{\varepsilon} \in \langle\nu, 2\nu, \varepsilon\rangle \ni \eta\kappa \bmod 0$ and $p(xi\tilde{\zeta}' + 8\tilde{\sigma}') = xi\zeta + 8\tilde{\sigma}$. Since $128\tilde{\sigma}' = 0$, we have $16(xi\tilde{\zeta}' + 8\tilde{\sigma}') = 16xi\tilde{\zeta}' = \pm 8xi\rho$. Thus we deduce that $xi\tilde{\zeta}' + 8\tilde{\sigma}'$ is of order 64, $2(xi\tilde{\zeta}' + 8\tilde{\sigma}') = yi\rho$ for an odd integer y and $\tilde{\sigma}'$ is of order 64. This completes the proof.

By Proposition 4.18 of [13], we have $\sharp(\gamma_5\sigma) = 2$. Here $\sharp\alpha$ stands for the order of α. Let $\widetilde{2\sigma} \in \langle i, \gamma_5, 2\sigma\rangle \subset \pi_{19}^S(CP^6)$ be a coextension of 2σ.

Lemma 2.2. *$h_{2*}\pi_{19}^S(CP^5) = \pi_{19}^S(HP^2)$ and $h_3 i\widetilde{2\sigma} = 2\tilde{\sigma}'$ for a suitable choice of $\widetilde{2\sigma}$.*

Proof. By Proposition 4.18 of [13], we have $t_3\tilde{\sigma}\eta = \tilde{\sigma}''\eta \neq 0$ and $t_3 i\eta * = i\eta * \neq 0$ in $\pi_{18}^S(CP^5)$. So, by Lemma 1.2, $t_{3*}: \pi_{19}^S(Q_3) \to \pi_{18}^S(CP^5)$ is monomorphic, and hence we have the first half by $(*)_3$. Since $p_3' h_3 i\widetilde{2\sigma} = 2\sigma = p_3'(2\tilde{\sigma}')$, we have $h_3 i\widetilde{2\sigma} \equiv 2\tilde{\sigma}' \mod i_*'\pi_{19}^S(HP^2) = i_*' h_{2*}\pi_{19}^S(CP^5) = h_{3*} i_* \pi_{19}^S(CP^5)$. So there exists an element $\alpha \in \pi_{19}^S(CP^5)$ such that $h_3(i(\widetilde{2\sigma} + i\alpha)) = 2\tilde{\sigma}'$. So, by a suitable choice of $\widetilde{2\sigma}$, we have the second half. This completes the proof.

Lemma 2.3. i) *$2\gamma_4' = ai\tilde{\zeta}' + 2bi\tilde{\sigma}'$ for odd integers a and b.*

ii) *The element $\gamma_3'\nu$ is of order 4 and $\pi_{19}^S(HP^4) = Z_{128}\{\gamma_4'\} \oplus Z_{64}\{i\tilde{\sigma}'\}$.*

iii) *$\pi_{19}^S(HP^5) = Z_{64}\{i\tilde{\sigma}'\}$.*

Proof. We recall that γ_4' is of order 128 ([12]) and that $p_4'\gamma_4' = 4\nu$. So, by Lemma 2.1, we have $2\gamma_4' = ai\tilde{\zeta}' + bi\tilde{\sigma}'$, where a or b is odd. Therefore we have $2\gamma_4'' = ai\zeta + bi\tilde{\sigma}$, where γ_4'' is γ_4' followed by the projection $HP^4 \to HP_2^4$. Since γ_4'' is of order 64 ([1]), we have $32\gamma_4'' = 16bi\tilde{\sigma}$ and $b = 2b'$ for an odd integer b'. Hence a is odd. This leads us to i).

Obviously we have $\pi_{18}^S(HP^2) = Z_2\{i\sigma^2\} \oplus Z_2\{i\kappa\} \oplus Z_2\{\tilde{\eta}''\mu\}$, where $\tilde{\eta}'' \in \langle i, \nu, \eta\rangle \subset \pi_9^S(HP^2)$. Since $\gamma_2' \in \langle i, \nu, 2\nu\rangle$, $\gamma_2'\nu \in -i\langle \nu, 2\nu, \nu\rangle = 0$ and $\gamma_3' \in \langle i, \gamma_2', \nu\rangle$, we have $2\gamma_3'\nu \in -i\langle \gamma_2', \nu, 2\nu\rangle \equiv -i\langle \nu, 2\nu, \nu, 2\nu\rangle \ni i\kappa \mod i\sigma^2$ ([4], [6]). Here $\langle \nu, 2\nu, \nu, 2\nu\rangle$ is a tertial composition ([4]). So we have the first half of ii). i) and Lemma 2.1 imply the second half of ii), which implies directly iii). This completes the proof.

Remark. The result iii) of Lemma 2.3 does not coincide with that of Theorem 5.3 of [3] which asserts that $\pi_{19}^S(HP) \cong Z_{32} \oplus Z_2$.

Proposition 2.4. *$\pi_{19}^S(CP^7) \cong \pi_{19}^S(CP^6) = Z_{128}\{i\tilde{\zeta}\} \oplus Z_{32}\{\widetilde{2\sigma}'\} \oplus Z_4\{i\overline{\eta\kappa}\}$, where $\widetilde{2\sigma}' = \widetilde{2\sigma} + 2bi\tilde{\zeta}$ with $b = 0$ or 1 and $8\widetilde{2\sigma}' \equiv i(\bar{\mu} \pm 2i\tilde{\zeta}) \mod \{32i\tilde{\zeta}, \overline{\eta\kappa}\}$.*

Proof. In $(6)_6$, we have $\gamma_6\nu^2 = it_3\omega_3\nu^2 = 0$ by Lemma 1.2 of [13].

So we have the first isomorphism. In $(*)_4$, by Lemmas 1.3 and 2.2, we have $32\widetilde{2\sigma} = ai\widetilde{\eta\kappa} + bi\bar{\mu}$ for integers a and b. Applying g_7 to this relation, we obtain $0 = 2a\bar{\kappa}$. So we have $a \equiv 0 \bmod 4$, and hence we have that $32\widetilde{2\sigma} = bi\bar{\mu} = 64bi\tilde{\zeta}$. By Lemmas 2.2 and 2.3, we have $h_3 i(16(\widetilde{2\sigma}+2bi\tilde{\zeta})) = 32(\tilde{\sigma}' + bi\tilde{\zeta}') \neq 0$. Therefore $\widetilde{2\sigma} + 2bi\tilde{\zeta}$ is of order 32.

We have $8\widetilde{2\sigma} \in -i\langle\gamma_5, 2\sigma, 8\iota\rangle$ and $\langle\gamma_5, 2\sigma, 8\iota\rangle \ni \bar{\mu} \bmod i_*\pi_{19}^S(CP^4)$. So, by $(6)_4$ and Proposition 1.4, we have the rest of the proposition. This completes the proof.

3. Determination of $\pi_{19}^S(CP;2)$. By Proposition 4.19 of [13], $\gamma_7\nu$ is of order 4. Let $\widetilde{4\nu} \in \langle i, \gamma_7, 4\nu\rangle \subset \pi_{19}^S(CP^8)$ be a coextension of 4ν.

Lemma 3.1. i) $2h_4 i\widetilde{4\nu} \equiv 2\gamma_4' \bmod 2i_*\pi_{19}^S(HP^3)$.

ii) $2\widetilde{4\nu} \equiv xi\tilde{\zeta} \bmod \{i\widetilde{2\sigma}', i\widetilde{\eta\kappa}\}$ *and* $\widetilde{4\nu}$ *is of order* 256, *where* x *is an odd integer.*

iii) *Let* y *and* c *be odd integers and* d, d' *be integers. Then* $h_4 i\widetilde{4\nu} \equiv \gamma_4' + yi\tilde{\sigma}' + di\tilde{\zeta}', 2(i\widetilde{4\nu} - c\gamma_9 - d'i\tilde{\zeta}) \equiv yi\widetilde{2\sigma}' \bmod i\widetilde{\eta\kappa}$ *and* $\sharp(i\widetilde{4\nu} - c\gamma_9 - d'i\tilde{\zeta}) = 64$.

Proof. By Lemma 2.3 and from the definition, we have $\gamma_4' \in \langle i, \gamma_3', 4\nu\rangle$. So we have $2\widetilde{4\nu} \in -i\langle\gamma_7, 4\nu, 2\iota\rangle$ and $2h_4 i\widetilde{4\nu} \in -i\langle\gamma_3', 4\nu, 2\iota\rangle = \langle i, \gamma_3', 4\nu\rangle \circ 2\iota \ni 2\gamma_4' \bmod 2i_*\pi_{19}^S(HP^3)$. Hence we have i). By i), Lemmas 2.1, 2.2 and 2.3, $2h_4 i\widetilde{4\nu} \equiv h_4(ai\tilde{\zeta} + bi\widetilde{2\sigma}) \bmod \{h_4 i\widetilde{2\sigma}, 2h_4 i\tilde{\zeta}\}$ for odd integers a and b. So, by $(*)_4$, Lemma 1.3 and Proposition 1.4, we have $2i\widetilde{4\nu} \equiv ai\tilde{\zeta} \bmod \{i\widetilde{2\sigma}, 2i\tilde{\zeta}\} + \{i\widetilde{\eta\kappa}\}$. Since $i_*: \pi_{19}^S(CP^8) \to \pi_{19}^S(CP^9)$ is monomorphic, we have ii).

We have $p_4' h_4 i\widetilde{4\nu} = 4\nu = p_4'\gamma_4'$. So, by Lemma 2.1, we can set $h_4 i\widetilde{4\nu} = \gamma_4' + di\tilde{\zeta}' + yi\tilde{\sigma}'$ for some integers y and d. Assume that $y = 2y'$. Then, by Lemmas 1.3, 2.2 and by$(*)_5$, we have $i\widetilde{4\nu} \equiv \gamma_9 + di\tilde{\zeta} + y'i\widetilde{2\sigma} \bmod t_{5*}\pi_{20}^S(Q_5) = \{i\widetilde{\eta\kappa}, i\bar{\mu}\}$. Applying $p_9: CP^9 \to S^{18}$ to this relation, we have $\eta = 0$. This is a contradiction and we have the first relation of iii). We recall $i\bar{\mu} = 16i\zeta_2\rho = 64\tilde{\zeta}$. Since $64\gamma_8 = 16i\rho$ and $128\gamma_9 = \zeta_9\eta$ ([13]), we have $128\gamma_9 \in \langle i', \gamma_8, 128\iota\rangle \circ \eta \subset \langle i', 16i\rho, 2\iota\rangle \circ \eta = -i'\langle 16i\rho, 2\iota, \eta\rangle \supset -i\langle 16\rho, 2\iota, \eta\rangle \ni i\bar{\mu} \bmod i_*'\pi_{18}^S(CP^8)\circ\eta$. By Proposition 4.19 of [13], the indeterminancy consists of $i\widetilde{\sigma^2\eta} = 0$, because $\widetilde{\sigma^2\eta} \in -i\langle\eta, \sigma^2, \eta\rangle \bmod i_*\{\eta\eta*, \eta^2\rho\} = 0$ and the bracket contains an element of $\langle\sigma^2, \eta, 2\eta\rangle = \langle\sigma^2, \eta, 0\rangle = 0$ ([14]). So we have $i\bar{\mu} = 128\gamma_9$. We set $d' = d - yb$, where b is the integer in Proposition 2.4. By the first relation of iii), Proposition 2.4 and $(*)_5$ lead us to the rest of iii). This completes the proof.

We set $\widetilde{4\nu}' = \widetilde{4\nu} - d'i\tilde{\zeta}$. By Proposition 2.4, Lemma 3.1 and by $(6)_n$ for $n = 7$, 8 and 9, we have the following

Proposition 3.2. i) $\pi_{19}^S(CP^8) = Z_{256}\{\widetilde{4\nu}'\} \oplus Z_{32}\{i\widetilde{2\sigma}'\} \oplus Z_4\{i\widetilde{\eta\kappa}\}$.
ii) $\pi_{19}^S(CP^9) = Z_{256}\{\gamma_9\} \oplus Z_{64}\{i\widetilde{4\nu}' - c\gamma_9\} \oplus Z_4\{i\widetilde{\eta\kappa}\}$.
iii) $\pi_{19}^S(CP^{10}) = Z_{64}\{i\widetilde{4\nu}'\} \oplus Z_4\{i\widetilde{\eta\kappa}\}$, *where* $2i\widetilde{4\nu}' \equiv yi\widetilde{2\sigma}' \bmod i\widetilde{\eta\kappa}$.

4. Determination of $\pi_{20}^S(CP;2)$. First we recall that $\pi_{14}^S(Q_2) = Z_{64}\{\tilde{\sigma}\} \oplus Z_2\{xi\zeta + 8\tilde{\sigma}\}$ with x odd and $\pi_{14}^S(Q_3) = Z_{64}\{i\tilde{\sigma}\} \oplus Z_8\{\widetilde{2\nu}''\}$, where $\tilde{\sigma} \in \langle i, 2\nu, \sigma\rangle$, $\widetilde{2\nu}'' = \widetilde{2\nu}' - 2i\tilde{\sigma}$ and $\widetilde{2\nu}' \in \langle i, \overline{2\nu}, \widetilde{2\nu}\rangle$. We shall determine $\pi_{21}^S(Q_2)$. Let $\tilde{\kappa} \in \langle i, 2\nu, \kappa\rangle$ be a coextension of κ. Then we have $2\tilde{\kappa} \in -i\langle 2\nu, \kappa, 2\iota\rangle \supset i\nu\langle 2\iota, \kappa, 2\iota\rangle \ni i\nu\eta\kappa = 0 \bmod 2i\nu*$. So, by a suitable choice of $\tilde{\kappa}$, we have that $\tilde{\kappa}$ is of order 2. We have $2\tilde{\sigma}\sigma \in -i\langle 2\nu, \sigma, 2\sigma\rangle \ni -2i\nu* \bmod 0$. So we have the following

Lemma 4.1. $2\tilde{\sigma}\sigma = -2i\nu*$ *and* $\pi_{21}^S(Q_2) = Z_8\{\tilde{\sigma}\sigma\} \oplus Z_2\{i\nu* + \tilde{\sigma}\sigma\} \oplus Z_2\{\tilde{\kappa}\} \oplus Z_2\{i\eta\bar{\mu}\}$.

By making use of $(6)_n = (6)_{n,20}$ of [13] for $n = 1$, we have $\pi_{20}^S(CP^2) = Z_4\{i\nu*\}$. We recall that $\tilde{\sigma}' = t_2\tilde{\sigma}$ and set $\tilde{\kappa}' = t_2\tilde{\kappa}$. By making use of $(6)_2$ or $(*)_2$, we have the following

Proposition 4.2. $2\tilde{\sigma}'\sigma = 2i\nu*$ *and* $\pi_{20}^S(CP^3) = Z_4\{\tilde{\sigma}'\sigma\} \oplus Z_2\{i\nu* + \tilde{\sigma}'\sigma\} \oplus Z_2\{\tilde{\kappa}'\}$.

By $(6)_3$, we have $\pi_{20}^S(CP^4) \cong \pi_{20}^S(CP^3)$. By Lemma 3.6 of [13] and Lemma 4.1, we have $\omega_2\zeta \in -\langle\omega_2, \sigma, \nu\rangle \circ 8\iota \subset 8\pi_{21}^S(Q_2) = 0$. So, by $(6)_4$, we have $\pi_{20}^S(CP^5) = Z_4\{i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\nu* + i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\tilde{\kappa}'\} \oplus Z_2\{\gamma_5\mu\}$. In $(6)_5$, by Lemma 2.9 and Proposition 4.18 of [13], we have $\gamma_5\nu^3 = 0$ and $\gamma_5\eta^2\sigma = \tilde{\sigma}''\eta^3 = 4\tilde{\sigma}''\nu \in 4\pi_{20}^S(CP^5) = 0$. So, we have $\pi_{20}^S(CP^6) = Z_4\{i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\nu* + i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\tilde{\kappa}'\}$. We show

Lemma 4.3. $\omega_3\sigma = \pm 2i\tilde{\sigma}\sigma = \pm 2i\nu*$.

Proof. By Lemma 2.6 of [13], $\omega_3 = 2ai\tilde{\sigma} + 2b\widetilde{2\nu}''$ for odd integers a and b. We have $\widetilde{2\nu}''\circ\sigma = \widetilde{2\nu}'\circ\sigma - 2i\tilde{\sigma}\sigma$ and $\widetilde{2\nu}'\circ\sigma \in \langle i, \overline{2\nu}, \widetilde{2\nu}\rangle\circ\sigma$. We recall $\widetilde{2\nu} = \tilde{\bar{\eta}}\circ\tilde{\eta}$ and so $\widetilde{2\nu}\circ\sigma \in -i\langle\nu, \bar{\eta}, \tilde{\eta}\sigma\rangle$. Since $\langle\nu, \bar{\eta}, \tilde{\eta}\sigma\rangle \subset \langle\nu, 2\nu, \sigma\rangle \ni 0$

$\bmod \sigma^2$, $\sigma\langle\nu,\bar{\eta},\tilde{\eta}\sigma\rangle = -\langle\sigma,\nu,\bar{\eta}\rangle\circ\tilde{\eta}\sigma \subset \{M^{13},S^0\}\circ\tilde{\eta}\sigma = 0$ and $\sigma^3 \neq 0$ ([5]), we have $\widetilde{2\nu}\circ\sigma = 0$ and $\widetilde{2\nu}'\circ\sigma \in -i\langle\overline{2\nu},\widetilde{2\nu},\sigma\rangle \bmod 0$. Since $\nu\nu* = \sigma^3$ ([5]) and $\langle\overline{2\nu},\widetilde{2\nu},\sigma\rangle\circ\nu = -\overline{2\nu}\langle\widetilde{2\nu},\sigma,\nu\rangle \subset -\langle 2\nu,\nu,\langle 2\nu,\sigma,\nu\rangle\rangle = \langle 2\nu,\nu,0\rangle \ni 0 \bmod 0$, we have $\langle\overline{2\nu},\widetilde{2\nu},\sigma\rangle \ni 2c\nu* \bmod \eta\bar{\mu}$ with an integer c. Therefore we have $\omega_3\sigma = 2(a-2b)i\tilde{\sigma}\sigma + 4bci\nu*$. This and Lemma 4.1 complete the proof.

Since $\omega_4\nu \neq 0$ and $2\omega_4\nu = 0$ by Lemma 1.4 of [13], we have that $\omega_4\nu$ is of order 2. In $(6)_6$, we have $p_7(t_4\omega_4\nu) = \nu^2$ and $\gamma_6\sigma = it_3\omega_3\sigma = 2i\tilde{\sigma}'\sigma$ by Lemma 4.3. So we have $\pi_{20}^S(CP^7) = Z_2\{t_4\omega_4\nu\} \oplus Z_2\{i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\nu*\} \oplus Z_2\{i\bar{\kappa}'\}$. By $(6)_7$, we have $\pi_{20}^S(CP^8) \cong \pi_{20}^S(CP^7)$. In $(6)_8$, we have $\gamma_8\nu = it_4\omega_4\nu$ and $p_9(\gamma_9\eta) = \eta^2$. So, by $(6)_9$, we have the following

Proposition 4.4. *$\pi_{20}^S(CP^9) = Z_2\{\gamma_9\eta\} \oplus Z_2\{i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\bar{\kappa}'\} \oplus Z_2\{i\nu*\}$ and $\pi_{20}^S(CP^{10}) = Z\{\zeta_{10}\} \oplus Z_2\{i\tilde{\sigma}'\sigma\} \oplus Z_2\{i\bar{\kappa}'\} \oplus Z_2\{i\nu*\}$.*

5. The image of the S^1-transfer in $\pi_k^S(S^0)$ for $k = 20, 21$.

First of all we show

Lemma 5.1. *$g_3\widetilde{\eta\kappa} = \pm 2\bar{\kappa}$, $g_5\tilde{\zeta} \equiv 0 \bmod 2\bar{\kappa}$, $g_6\tilde{\mu} \equiv 0 \bmod 4\bar{\kappa}$ and $g_7\widetilde{2\sigma}' \equiv 0 \bmod 2\bar{\kappa}$.*

Proof. We know the first relation. Assume that $g_5\tilde{\zeta} = \bar{\kappa}$. Then we have $\eta\bar{\kappa} = g_5\tilde{\zeta}\eta \in -g_4\langle\gamma_3,\zeta,\eta\rangle$. Since $p_3\langle\gamma_3,\zeta,\eta\rangle \subset \langle\eta,\zeta,\eta\rangle = 0$, we have $\langle\gamma_3,\zeta,\eta\rangle \subset i_*\pi_{20}^S(CP^2) = \{i\nu*\}$. So we have $g_5\tilde{\zeta}\eta = a\nu\nu* = a\sigma^3$ for $a = 0$ or 1. This contradicts the fact that $\pi_{21}^S(S^0) = Z_2\{\eta\bar{\kappa}\} \oplus Z_2\{\sigma^3\}$ [5]. Therefore we have $g_5\tilde{\zeta} \equiv 0 \bmod 2\bar{\kappa}$. By the proof of Proposition 1.4, we have $2g_6\tilde{\mu} = g_6(i\zeta_2\rho) = 0$. So we obtain the third relation. Assume that $g_7\widetilde{2\sigma}' = \bar{\kappa}$. Then $\eta\bar{\kappa} = g_7\widetilde{2\sigma}\circ\eta \in -g_6\langle\gamma_5,2\sigma,\eta\rangle$. By Proposition 4.18 of [13], we have $\langle\gamma_5,2\sigma,\eta\rangle \supset \langle\gamma_5\sigma,2\iota,\eta\rangle = \langle\tilde{\sigma}''\eta + ai\eta*,2\iota,\eta\rangle \subset \langle\tilde{\sigma}''\eta,2\iota,\eta\rangle + \langle ai\eta*,2\iota,\eta\rangle \ni 2\tilde{\sigma}''\nu \pm 2ai\nu* \bmod \{\gamma_5\mu\} + \pi_{19}^S(CP^5)\circ\eta$ with $a = 0$ or 1. Therefore, by Proposition 1.4 and by the above conclusion, we have $g_7\widetilde{2\sigma}\circ\eta \in 2\pi_{20}^S(S^0)\circ\eta = 0$. This is a contradiction and completes the proof.

Lemma 5.2. *$g_9\widetilde{4\nu} \neq c\bar{\kappa}$ and $g_9\widetilde{4\nu}' \neq c'\bar{\kappa}$ for odd integers c and c'.*

Proof. It suffices to prove the first relation. By Lemma 2.6 of [13], we have $\gamma_6 = it_3\omega_3 = 2ai\tilde{\sigma}' + 2bi\widetilde{2\nu}'''$ for odd integers a and b. So we have $\langle\gamma_6, \eta, 4\nu\rangle \supset (ai\tilde{\sigma}' + bi\widetilde{2\nu}''')\circ\langle 2\iota, \eta, 4\nu\rangle = 0 \bmod \pi_{15}^S(CP^6)\circ 4\nu = 0$ by Proposition 4.18 of [13]. Since $0 = g_8\gamma_7 \in \langle g_7, \gamma_6, \eta\rangle$ and $\langle\gamma_6, \eta, \nu\rangle = 0$, we can define a tertiary composition $\langle g_7, \gamma_6, \eta, 4\nu\rangle$ ([4]). Its indeterminancy consists of $G = g_{8*}\pi_{19}^S(CP^7) + \{\Sigma^{13}CP^2, S^0\}\circ\alpha$, where $\alpha \in \langle i, \eta, 4\nu\rangle$ is a coextension of 4ν. Obviously we have $\{\Sigma^{13}CP^2, S^0\} = Z_{32}\{\overline{2\rho}\} \oplus Z_4\{\overline{\eta\kappa}\}$, where $\overline{2\rho} \in \langle 2\rho, \eta, p\rangle$, $\overline{\eta\kappa} \in \langle\eta\kappa, \eta, p\rangle$, $16\overline{2\rho} = \bar{\mu}p$ and $2\overline{\eta\kappa} = \nu\kappa p$. We have that $\overline{2\rho}\circ\alpha \in \langle 2\rho, \eta, 4\nu\rangle \supset \rho\langle 2\iota, \eta, 4\nu\rangle = 0 \bmod 0$ and that $\overline{\eta\kappa}\circ\alpha \in \langle\eta\kappa, \eta, 4\nu\rangle \supset \langle\eta\kappa, \eta, 2\iota\rangle\circ 2\nu \ni 0 \bmod 0$. Therefore, by Proposition 2.4 and Lemma 5.1, we have $G = Z_4\{2\bar{\kappa}\}$. From the definition, we have $g_9\widetilde{4\nu} \in \langle g_8, \gamma_7, 4\nu\rangle = \langle g_7, \gamma_6, \eta, 4\nu\rangle$. By [4] and [6], we have $2\langle g_7, \gamma_6, \eta, 4\nu\rangle = -g_7\langle\gamma_6, \eta, 4\nu, 2\iota\rangle \supset g_7(ai\tilde{\sigma}' + bi\widetilde{2\nu}''')\circ\langle 2\iota, \eta, 4\nu, 2\iota\rangle \subset \pi_{14}^S(S^0)\circ\pi_6^S(S^0) = \{4\bar{\kappa}\}$. This shows that $2g_9\widetilde{4\nu} \neq \pm 2\bar{\kappa}$, which completes the proof.

On the 3-primary components, we have $g_4\tilde{\alpha}_1\beta_1 \in \langle g_3, \gamma_2, \alpha_1\rangle\circ\beta_1 \supset \pm\langle\alpha_1, \alpha_1, \alpha_1\rangle\circ\beta_1 = \beta_1^2 \bmod g_{3*}\pi_9^S(CP^2; 3)\circ\beta_1 = \{\alpha_1\alpha_2\beta_1\} = 0$. Thus we have the first assertion of iii) of Theorem. We have $g_4\tilde{\kappa}' = g_2(H)\tilde{\kappa} \in \langle\nu, 2\nu, \kappa\rangle \ni \eta\bar{\kappa} \bmod 0$ and $g_4\tilde{\sigma}'\sigma = \sigma^3$ by the proof of Lemma 5.3 of [13]. This leads us to the second assertion of iii), which completes the proof of Theorem.

6. Appendix: The image of the S^3-transfer. Let $g_n' = g_n(H)\colon \Sigma^{4n}Q_n \to S^{4n}$ be the S^3-transfer map [13]. We set $C_k = \mathrm{Im}\{g_{n*}'\colon \pi_k^S(Q_n) \to \pi_k^S(S^0)$ for $k \leq 4n-1\}$. First we recall the following

Theorem 6.1 (Morisugi [8]). *Let $n = 2^t$ for some integer t. Then there exists an element $\tilde{\eta}_n \in \pi_{4n}^S(Q_n)$ such that $g_n'\tilde{\eta}_n \in \pi_{4n}^S(S^0)$ is Mahowald's element.*

By [9], [12] and the James splitting theorem ([2]), we have the following

Lemma 6.2. *Let $b(n) = 1$ for odd n and $b(n) = 2$ for even n. Then there exists an element $\zeta_n' \in \pi_{4n-1}(Q_n)$ such that $t_n\zeta_n' = b(n)\zeta_{2n-1}$ in the stable range.*

By [11], [13] and by Lemma 6.2, we have the following: $C_3 = B_3 =$

$Z_{24}\{\nu\}$; $C_7 = Z_{60}\{4\sigma\}$; $C_{11} = B_{11} = Z_{504}\{\zeta\}$; $4\rho \in C_{15}$; $\bar{\zeta} \in C_{19}$. By [11], [13] and by Theorem 6.1, we have the following: $C_8 = B_8 = Z_2\{\bar{\nu}\}$; $C_{16} = B_{16} = Z_2\{\omega *\}$. We show

Proposition 6.3. *$C_6 = B_6 = \pi_6^S(S^0)$; $C_9 = Z_2\{\nu^3\}$; $C_{10} = B_{10} = Z_3\{\beta_1\}$; $C_{13} = B_{13} = \pi_{13}^S(S^0)$; $C_{14} = B_{14} = \pi_{14}^S(S^0)$; $C_{15} = Z_{120}\{4\rho\} \oplus Z_2\{\eta\kappa\}$; $C_{17} = Z_2\{\eta\omega *\} \oplus Z_2\{\nu\kappa\}$; $C_{18} = B_{18} = Z_8\{\nu *\}$; $C_{19} = Z_{264}\{\bar{\zeta}\}$; $C_{20} = B_{20} = Z_4\{2\bar{\kappa}\} \oplus Z_3\{\beta_1^2\}$ and $C_{21} = B_{21} = \pi_{21}^S(S^0)$.*

Proof. Since $\pi_6^S(Q_2) = Z_2\{i\nu\}$, we have $g_2'(i\nu) = \nu^2$. Since $\pi_9^S(Q_3) \cong \pi_9^S(Q_2) = Z_2\{\tilde{\eta}'\eta\} \oplus Z_2\{i\nu^2\}$, we have $g_3'(i\tilde{\eta}'\eta) = g_2'(\tilde{\eta}'\eta) = \bar{\nu}\eta = \nu^3$ and $g_3'(i\nu^2) = \nu^3$. By a suitable choice of $\tilde{\nu}' \in \langle i, 2\nu, \nu\rangle$, we have $\pi_{10}^S(Q_2) = Z_{72}\{\tilde{\nu}'\} \oplus Z_{80}\{3i\sigma\}$. So we have $g_2'\tilde{\nu}' \in \langle \nu, 2\nu, \nu\rangle \ni \pm\beta_1 \bmod 0$. We have $g_2'(i\beta_1) = \alpha_1\beta_1$. In §5 of [13], we obtain the relations $g_2'\tilde{\sigma} = \sigma^2$, $g_3'\widetilde{2\nu}' \equiv \kappa \bmod \sigma^2$ and $g_3'\tilde{\sigma}' = \nu *$, where $\tilde{\sigma} \in \langle i, 2\nu, \sigma\rangle$, $\widetilde{2\nu}' \in \langle i, \overline{2\nu}, \widetilde{2\nu}\rangle$ and $\tilde{\sigma}' \in \langle i, \omega_2, \sigma\rangle$. By the last argument of §5 , we have $C_{21} = B_{21}$. By Lemmas 1.2 and 6.2, we have $C_{19} = \{\bar{\zeta}\}$ since $g_5'(i\eta *) = \nu\eta * = 0$. By Lemma 1.2, we have $\pi_{20}^S(Q_6) = Z_4\{i\widetilde{\eta\varepsilon}\} \oplus Z_2\{i\eta\eta *\} \oplus Z_2\{i\bar{\mu}\} \oplus Z_3\{i\tilde{\nu}'\beta_1\}$. From the definition, we have $g_3'\widetilde{\eta\varepsilon} \in \langle g_2', \omega_2, \eta\varepsilon\rangle$. By Lemma 1.2 and [6], we have $\langle g_2', \omega_2, \eta\varepsilon\rangle \subset \langle g_2', i\eta\kappa, \eta\rangle \supset \langle \nu, \eta\kappa, \eta\rangle \supset \langle \nu, \eta, \eta\kappa\rangle \ni 2\bar{\kappa} \bmod g_{2*}'\pi_{20}^S(Q_2) = \{4\bar{\kappa}\}$. So we have $g_3'\widetilde{\eta\varepsilon} = \pm 2\bar{\kappa}$. We have $g_2'\tilde{\nu}'\beta_1 = \beta_1^2$. Therefore we have $C_{20} = B_{20} = Z_4\{2\bar{\kappa}\} \oplus Z_3\{\beta_1^2\}$. This completes the proof.

Summarizing the above argument, we have the following

Theorem 6.4. *Let $k \leq 21$.*

i) *$C_k = B_k$ except for $k = 7, 9, 15, 17$ and 19.*

ii) *$C_7 = 4\pi_7^S(S^0)$, $C_9 = Z_2\{\nu^3\}$, $C_{15} = Z_{120}\{4\rho\} \oplus Z_2\{\eta\kappa\}$, $C_{17} = Z_2\{\eta\omega *\} \oplus Z_2\{\nu\kappa\}$ and $C_{19} = Z_{264}\{\bar{\zeta}\}$.*

iii) *The elements 2σ, $\eta^2\sigma$, $\eta\varepsilon$, 2ρ, $\eta^2\rho$ and $\bar{\sigma}$ are in the image of the S^1-transfer but not in that of the S^3-transfer.*

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