Math. J. Okayama Univ. 36(1994), 105-131

STRUCTURE OF SEMIPRIME RINGS SATISFYING CERTAIN CONDITIONS

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Throughout, R will represent a ring (not necessarily with 1) with center $C = C(R)$. Let $\mathbf{Z}$ denote the ring of rational integers, and $\mathbf{Z}\langle X,Y\rangle$ the free algebra over $\mathbf{Z}$ in the indeterminates X and Y. We call a polynomial $f(X)$ in $X\mathbf{Z}[X]$ comonic if its lowest coefficient is 1 (i.e., $f(X) = X^m + X^{m+1}q(X)$ for some $m > 0$ and $q(X) \in \mathbf{Z}[X]$).

In his paper [9], Herstein introduced the concept of the hypercenter of a ring; the hypercenter T_H of the ring R is defined by $T_H(R) = \{a \in R \mid$ for each $x \in R$, there exists $n > 0$ such that $[a, x^n] = ax^n - x^n a = 0\}$. He showed that $T_H(R)$ coincides with the center of R if R has no non-zero nil ideal. By making use of this result, in [10] he also studied the rings R satisfying the condition; (h) for each $x, y \in R$, there exist $m, n > 0$ such that $[x^m, y^n] = 0$, and showed that a ring satisfying (h) has the nil commutator ideal. Further, in [11], he conjectured that a ring R has also the nil commutator ideal if for each $x, y \in R$, there exist positive integers k, m and n such that $[x^m, y^n]_k = [[\cdots[x^m, y^n], y^n], \cdots], y^n] = 0$ (k-times). Recently, Chuang and Lin [8] gave a partial answer for the conjecture. On the other hand, in connection with (h), Chacron [6] investigated structure of rings R satisfying the condition: For each $x, y \in R$ there exist polynomials $f(X)$ and $g(X)$ in $\mathbf{Z}[X]$ such that $[x^m - x^{m+1}f(x), y^m - y^{m+1}g(y)] = 0$, where m is a fixed positive integer. To generalize their results in [3,5,6,8], we consider the following subsets of a ring R:

$S^* = S^*(R) = \{a \in R \mid$ for each $x \in R$, there exist $k > 0$ and a comonic $f(X) \in X\mathbf{Z}[X]$ such that $[a, f(x)]_k = 0\}$.

$T^* = T^*(R) = \{a \in R \mid$ there exist $k > 0$ and $n > 0$ such that for each $x \in R$, $[a, x^n - x^{n+1}f(x)]_k = 0$ for some $f(X) \in \mathbf{Z}[X]\}$.

$T^*_{(n,k)} = T^*_{(n,k)}(R) = \{a \in R \mid$ for each $x \in R$ there exists $f(X) \in \mathbf{Z}[X]$ such that $[a, x^n - x^{n+1}f(x)]_k = 0\}$, where n and k are positive integers.

In the present paper, we shall show that, under appropriate hypothesis, these subsets coincide with the center of the ring R, and in connection with these subsets we shall also study structure of semiprime rings satisfying the following conditions:

(H) For each $x, y \in R$, there exist comonic polynomials $f(X)$, $g(X) \in X\mathbf{Z}[X]$ and $k > 0$ such that $[f(x), g(y)]_k = 0$.

$(\mathrm{H})'_{(m)}$ For each $x \in R$, there exist $k > 0$ and $n > 0$ such that for each $y \in R$, $[x^m - x^{m+1}f(x), y^n - y^{n+1}g(y)]_k = 0$ for some polynomials $f(X), g(X) \in \mathbf{Z}[X]$, where m is a positive integer.

$(\mathrm{H})''_{(m,n,k)}$ For each $x, y \in R$, there exist polynomials $f(X), g(X) \in \mathbf{Z}[X]$ such that $[x^m - x^{m+1}f(x), y^n - y^{n+1}g(y)]_k = 0$, where m, n and k are positive integers.

Further, we consider the following condition which is studied coupling with the condition (H) in semiprime rings:

(S)′ For each $x, y \in R$, there exist integers α, β and γ such that $xy = \alpha yx + \beta x^2 + \gamma y^2 + f(x, y)$ for some $f(X, Y) \in \mathbf{Z}\langle X, Y\rangle$ each of whose monomial terms is of length ≥ 3.

In §0, we shall characterize the class of non-commutative semiprime rings satisfying the condition (S)′. Corollary 0.2 will play an important role in the latter sections. In §1, we shall prove that a reduced ring satisfying (H) is commutative (Theorem 1.1). This generalizes the Chacron-Herstein-Montgomery result [7, Lemma 6]. In §2, we shall prove $S^*(R) = C(R)$, the center of the ring R, if R is a reduced ring (Theorem 2.1). Further, we shall consider the subsets $S^{*\prime}(R)$ and $T^{*\prime}(R)$ (see §2), and generalize some of the results in [3] and [8] (Theorem 2.2). Now, for a semiprime ring R, $S^*(R)$ and even $T^*_{(n,k)}(R)$ $(nk > 1)$ need not coincide with the center of R. In fact, since $(\mathrm{GF}(2))_2$ satisfies the identity $X^2 - X^8 = 0$, it is an obvious example such that $T^*_{(2,1)}$ does not coincide with the center of R. However, in §3, we shall show that essentially certain matrix rings are only semiprime rings in which $T^*_{(n,k)}$ and T^* need not coincide with the center, respectively (Theorems 3.3 and 3.4). In §4, by making use of Theorem 1.1 together with the results obtained in former sections, we shall study structure of semiprime rings satisfying the condition (H), $(\mathrm{H})'_{(m)}$ or $(\mathrm{H})''_{(m,n,k)}$ (Theorems 4.3, 4.4 and 4.5).

In all that follows, $D = D(R)$ denotes the commutator ideal of R, and $J = J(R)$ the Jacobson radical of R. For $x, y \in R$, define extended commutators $[x, y]_k$ as follows: Let $[x, y]_0 = x$, and proceed inductively $[x, y]_k = [[x, y]_{k-1}, y]$. For a subset U of R, we use the following notations: $\langle U\rangle$ (resp. (U)) is the subring (resp. ideal) of R generated by U. $C_R(U) = \{a \in R \mid [a, U] = 0\}$. $C^*_R(U) = \{a \in R \mid$ there exists $k > 0$ such that $[a, U]_k = 0\}$. $Ann(U) = \{a \in R \mid aU = Ua = 0\}$.

0. On non-commutative semiprime rings. In his paper [16], Streb gave a classification of non-commutative rings. In this section, we shall state the similar results for non-commutative semiprime rings.

We consider the following type of rings:

a)$_l$ $\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

a)$_r$ $\begin{pmatrix} 0 & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix}$, p a prime.

a)^1 $\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix}$, p a prime.

a)$'$ $(\mathrm{GF}(p))_2$, p a prime.

c) A non-commutative division ring.

d)$'$ A non-commutative radical domain, namely a non-commutative radical ring with no non-zero divisor of zero.

d)^1 A domain $T = \langle 1 \rangle + S$, S is a subring of T which is of type d)$'$.

e) S is a finite nilpotent ring such that $D(S)$ is the heart of R, and $SD(S) = D(S)S = 0$.

e)^1 $T = \langle 1 \rangle + S$, S is a subring of T which is of type e).

Combining [12, Proposition 1(3)] and [14, Lemma 1.4(1)(4)] with [11, Proposition 2], we can easily see the following:

Proposition A. *Let R be a non-commutative ring.*

(1) *If $xy \neq 0 = yx$ for some $x, y \in R$, then there exists a factorsubring of R (i.e., a homomorphic image of a subring of R) which is of typ e* a)$_l$, a)$_r$ *or* e).

(2) *Suppose that there exists a factorsubring of R which is of type* a)$_l$ *or* a)$_r$. *If R has the unit element* 1, *then there exists a factorsubring of R which is of type* a)^1.

(3) *Suppose that there exists a factorsubring of R which is of type* e). *If R has the unit element* 1, *then there exists a factorsubring of R which is of type* e)^1.

By the structure theorem for primitive rings, we can easily see the following.

Lemma 0.1. *If R is a primitive ring which is not a division ring, then there exists a factorsubring of R which is of type* a)$'$.

Proposition 0.1. *Let R be a non-commutative semiprime ring.*

Then there exists a factorsubring of R which is of type a)$_l$, a)$_r$, c), d)$'$ *or* e).

Proof. If R is semiprimitive, then it is a subdirect sum of primitive rings. Hence, there exists a factorsubring of R which is of type a)$'$ or c) by Lemma 0.1, and so there exists a factorsubring of type a)$_l$, a)$_r$ or c). Now, since R is a subdirect sum of prime rings, we may assume that R is a prime ring. If R is not a domain, then there exist non-zero $a, b \in R$ such that $ab = 0$. Then, $bRa \neq 0$ because R is prime, and so there exists $r \in R$ such that $bra \neq 0$. Put $x = br$ and $y = a$. Then $yx = 0 \neq xy$. By Proposition A(1), there exists a factorsubring of R which is of type a)$_l$, a)$_r$ or e). Therefore, we may assume that R is a domain with $J \neq 0$. As is well known, a domain with a non-zero commutative ideal must itself be commutative, and so J is non-commutative; thus it is of type d)$'$.

In case R has 1, from the above proof, we can see that d)$'$ can be exchanged with d)^1 in Proposition 0.1. Further, by Proposition A(2) and (3), if R is of type a)$_l$, a)$_r$, or e), then there exists a factorsubring of R which is of type a)^1 or e)^1. Therefore, we get the following:

Proposition 0.2. *Let R be a non-commutative semiprime ring with* 1. *Then there exists a factorsubring of R which is of type* a)^1, c), d)^1 *or* e)^1.

Propositio 0.3. *If R is a semiprime ring satisfying* (S)$'$, *then* $R/Ann(J)$ *is a reduced ring.*

Proof. Let a be an arbitrary element in R such that $a^2 \in Ann(J)$. Since R satisfies (S)$'$, for each $x \in J$, there exist integers α, β and γ such that

$$axa = \alpha a^2 x + \beta axax + \gamma a^2 + f(ax, a)$$

for some $f(X,Y) \in \mathbf{Z}\langle X, Y\rangle$ each of whose monomial terms is of length ≥ 3. Since $a^2 \in Ann(J)$, for each $y \in J$, we have

$$yaxa = y\beta axax + yf(ax, a). \tag{0.1}$$

On the other hand, $a^2 \in Ann(J)$ enables us to see that $f(ax, a) = g_1(ax) + g_2(ax)a + g_3(a)$ for some $g_1(X), g_3(X) \in X^3\mathbf{Z}[X]$ and $g_2(X) \in X^2\mathbf{Z}[X]$. Combining this with (0.1), We can see that

$$yaxa = y\beta axax + yg_1(ax) + yg_2(ax)a \in yaxaJ.$$

Hence we get $yaxa = 0$. We have thus seen that $(Ja)^2 = 0$, and thus $Ja = 0$ by the semiprimeness of R. That is, $a \in Ann(J)$.

Corollary 0.1. *Suppose that R satisfies* (S)$'$. *If R is a prime ring with $J \neq 0$, then R is a domain.*

Proof. Since $Ann(J) = 0$, the assertion is clear by Proposition 0.3.

The next corollary is immediate by Corollary 0.1.

Corollary 0.2. *If R is a semiprime ring satisfying* (S)$'$, *then R is a subdirect sum of domains with non-zero radical and primitive rings. In particular, R has no non-zero nil right ideal.*

Proposition 0.4. *Let R be a non-commutative semiprime ring. If R satisfies* (S)$'$, *then there exists a factorsubring of R which is of type* a)$'$, c) *or* d)$'$.

Proof. If R is primitive, then there exists a factorsubring of R which is of type a)$'$ or c) by Lemma 0.1. Therefore, in view of Corollary 0.2, we may assume that R is a non-commutative domain with $J \neq 0$. Then J is of type d)$'$.

Corollary 0.3. *Let* **P** *be a ring-property which is inherited by the factorsubrings.*

(1) *Then the following conditions are equivalent:*

(i) *For any ring R satisfying* (S)$'$ *and* **P**, *$D(R)$ is nil.*

(ii) *Every semiprime ring satisfying* (S)$'$ *and* **P** *is commutative.*

(iii) *Each ring of type* a)$'$, c) *or* d)$'$ *fails to satisfy either* (S)$'$ *or* **P**.

(2) *Let R be a semiprime ring satisfying* (S)$'$ *and* **P**. *If each ring of type* c) *or* d)$'$ *fails to satisfy* **P**, *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative ring.*

(ii) *a primitive ring which is not a division ring.*

Proof. (1) By Proposition 0.4.

(2) If R is a domain with $J \neq 0$, then J is a radical domain satisfying **P**. Hence, J is commutative by the hypothesis, and so R is commutative. Since R cannot be a non-commutative division ring by our hypothesis,

the assertion is now clear by Corollary 0.2.

In Corollary 0.3(1), we should note that rings of type a)′ satisfy (S)′. In fact, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathrm{GF}(p))_2$, then it is a root of the equation $X^2 - (a + d)X + (ad - bc) = 0$. Hence, if $a + d \neq 0$, then there exist $\alpha, \beta \in \mathbf{Z}$ such that x is a root of the equation $X + \alpha X^2 + \beta X^3 = 0$. In this case, for each $y \in (\mathrm{GF}(p))_2$, we have that $[y, x+\alpha x^2+\beta x^3] = 0$. Therefore, it is enough to consider the elements of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ in $(\mathrm{GF}(p))_2$. Let x and y are such elements in $(\mathrm{GF}(p))_2$. Then, we can easily see that there exists $\alpha \in \mathbf{Z}$ such that a pair (x, y) is a root of the equation $XY = -YX + \alpha(XY + YX)^2$.

1. The commutativity of reduced rings satisfying (H). Let $\mathcal{E}^* = \{f(X) \in X\mathbf{Z}[X] \mid f(X)$ is a comonic polynomial$\}$, and consider the following condition:

(H) For each $x, y \in R$, there exist $f(X)$, $g(X) \in \mathcal{E}^*$ and $k > 0$ such that $[f(x), g(y)]_k = 0$.

The main purpose of this section is to prove the following theorem which generalizes [7, Lemma 6]:

Theorem 1.1 *If R is a reduced ring satisfying* (H), *then R is commutative.*

In preparation for proving the above theorem, we state five lemmas. The first one, which plays an important role in our study, is an extension of [2, Theorem 1].

Lemma 1.1. *Let $\mathcal{E} = \{f(X) \in X\mathbf{Z}[X] \mid f(X)$ is a primitive polynomial (i.e., the coefficients of $f(X)$ are relatively prime)$\}$. Suppose that for each $x, y \in R$ there exist $f(X), g(X) \in \mathcal{E}$ and $k_2 > k_1 > 0$ such that $[f(x), g(y)]_{k_1} = [f(x), g(y)]_{k_2}$. If R is a torsion ring, then D is periodic.*

Proof. Let $R_p := \{x \in R \mid p^n x = 0$ for some $n > 0\}$, p a prime. As is easily seen, R_p is an ideal of R and $R = \bigoplus_{p;\mathrm{prime}} R_p$. By [2, Lemma 2], there exists a maximal periodic ideal $\mathcal{P}(R_p)$ of R_p such that $\overline{R_p} = R_p/\mathcal{P}(R_p)$ has no non-trivial periodic ideal. Since $\overline{R_p}$ has no non-zero nil ideal and $p\overline{R_p}$ is a nil ideal, we see that $\overline{R_p}$ is an algebra over $\mathrm{GF}(p)$, and also it has no non-zero algebraic ideal (see [2]). If $x, y \in \overline{R_p}$, then there exist $f(X), g(X) \in \mathcal{E}$ and positive integers k and d such that $[f(x),g(y)]_k = [f(x),g(y)]_{k+d}$. By [2, Lemma 4], we can easily see that there exist $\alpha > \beta > 0$ with $p^\alpha - p^\beta \equiv 0 \pmod d$ such that $[f(x),g(y)^{p^\alpha}] = [f(x),g(y)]_{p^\alpha} = [f(x),g(y)]_{p^\beta} = [f(x),g(y)^{p^\beta}]$, and so we get $[f(x),g(y)^{p^\alpha} - g(y)^{p^\beta}] = 0$. Since it is easily seen that $g(X)^{p^\alpha} - g(X)^{p^\beta}$ is non-zero in $X\mathrm{GF}(p)[X]$, we can see that $\overline{R_p}$ is commutative by [4, Theorem 3.6], and thus $D(R_p) \subseteq \mathcal{P}(R_p)$. Since $D(R) = \bigoplus D(R_p)$, we see that $D(R)$ is periodic.

Lemma 1.2. *Let I be an ideal of a ring R whichis a division ring or a radical domain, and the characteristic $Ch(I) = 0$. Let $a \in R$. If for each $x \in I$ there exists $k > 0$ such that $[x,a]_k = 0$, then $[I,a] = 0$.*

Proof. Suppose that I is a radical domain, and further suppose, to the contrary, that there exists $x \in I$ such that $[x,a] \neq 0$. Then, by the hypothesis, $[x,a]_s = 0$ for some $s > 1$. Let k be the minimum integer in $\{t > 1 \mid [x,a]_t = 0\}$. Put $y = [x,a]_{k-2}$. Then $y \in I$. Since I is a radical domain, there exists $y^* \in I$ which is the quasi inverse of y. Embedding R into a ring with 1, we see that $1 + y^*$ is the inverse of $1 + y$. Put $u = 1 + y$. Since $[y,a]_2 = 0 \neq [y,a]$,

$$[u,a]_2 = 0 \neq [u,a]. \tag{1.1}$$

Noting that $0 = [uu^{-1},a] = u[u^{-1},a] + [u,a]u^{-1}$, we get

$$[u^{-1},a] = -u^{-1}[u,a]u^{-1}. \tag{1.2}$$

By (1.1) and (1.2), we can easily see that $[u^{-1},a]_2 = 2(u^{-1}[u,a])^2u^{-1}$, and proceeding by induction, we get $[u^{-1},a]_n = (-1)^n n!(u^{-1}[u,a])^n u^{-1}$ for all $n > 0$. Since $Ch(I) = 0$ and also $(u^{-1}[u,a])^n u^{-1} \in I$, we see that $[u^{-1},a]_n \neq 0$ and so $[y^*,a]_n \neq 0$ for all $n > 0$. On the other hand, since $y^* \in I$, we get that $[y^*,a]_{n'} = 0$ for some $n' > 0$, a contradiction.

In case that I is a division ring, we can get the conclusion in the same way.

Lemma 1.3. *Let R be a ring satisfying* (H). *If R is torsion free, then every primitive factorsubring of R is a division ring.*

Proof. By (H), for each prime p and each $x,y \in R$, there exist $f_0(X), g_0(X) \in \mathbf{Z}[X]$ and positive integers m, n and k such that

$$[(px)^m - (px)^{m+1}f_0(px), (py)^n - (py)^{n+1}g_0(py)]_k = 0.$$

It follows that

$$\begin{aligned} 0 &= [p^m x^m - p^{m+1}x^{m+1}f(x),\ p^n y^n - p^{n+1}y^{n+1}g(y)]_k \\ &= p^{m+nk}[x^m - px^{m+1}f(x), y^n - py^{n+1}g(y)]_k, \end{aligned}$$

where $f(X) = f_0(pX)$ and $g(X) = g_0(pX)$. Hence,

$$[x^m - px^{m+1}f(x), y^n - py^{n+1}g(y)]_k = 0. \tag{1.3}$$

Let R' be a primitive factorsubring of R. Suppose that R' is not a division ring. Then we see that there exists a factorsubring of R' which is of type a)$'$ by Lemma 0.1. In (1.3), putting $x = e_{21} + e_{22}, y = e_{11} \in (\mathrm{GF}(p))_2$, we get that

$$0 = [(e_{21} + e_{22}), e_{11}]_k = [e_{21}, e_{11}]_{k-1} = e_{21} \neq 0,$$

a contradiction. Therefore, R' must be a division ring.

Lemma 1.4. *Let R be a domain. If for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^*$ such that $[f(x), g(y)] = 0$, then R is commutative.*

Proof. If $Ch(R) \neq 0$, then D is periodic by Lemma 1.1, and so D is commutative by the well known Jacobson theorem. As is well known, a domain with a non-zero commutative ideal must itself be commutative, and thus $D = 0$. We may assume therefore that $Ch(R) = 0$. If R is semiprimitive, then it is a subdirect sum of primitive rings. Since $Ch(R) = 0$, R is a subdirect sum of division rings by Lemma 1.3. We may assume therefore that R is a division ring. Then, by [5, Remark 10], for each $x, y \in R$, there exists $f(X) \in \mathcal{E}^*$ such that $[x, f(y)] = 0$. For $x, y \in R$, consider the subring $\langle x, y\rangle$ of R generated by x and y. Then we can easily see that for each $a \in \langle x, y\rangle$, there exists $f(X) \in \mathcal{E}^*$ such that $f(a) \in C(\langle x, y\rangle)$. Since $\langle x, y\rangle$ is domain, $\langle x, y\rangle$ is commutative by [7, Lemma 6], and so $[x, y] = 0$. We have thus seen that R is commutative in semiprimitive case. If R is not semiprimitive, then R has the non-zero radical J. Since J is a radical domain, we have that for each $x, y \in R$, there exists $f(X) \in \mathcal{E}^*$ such that $[x, f(y)] = 0$ again by [5, Remark 10]. Therefore, in the same way as above, we can see that J is commutative. Hence R must be commutative.

Lemma 1.5. *Let R be a domain satisfying* (H), *and $a, b \in R$. If there exists $k > 0$ such that $[a, b]_k = 0$, then $[a, b] = 0$.*

Proof. If $Ch(R) = p \neq 0$, then D is a periodic domain by Lemma 1.1, and thus D is commutative. Hence R must be commutative. We may assume therefore that $Ch(R) = 0$. Obviously, $C_R^*(b)$ is a subring of R and $b \in C_R^*(b)$. Also $C_R^*(b)$ satisfies (H) and $C_{C_R^*(b)}(b) = C_R(b) \cap C_R^*(b) = C_R(b)$. Therefore, under the hypothesis $C_R^*(b) = R$, it is enough to show $b \in C(R)$.

First, we suppose that R is semiprimitive. Since R satisfies (H) and $Ch(R) = 0$, R is a subdirect sum of division rings R_i ($i \in I$) by Lemma 1.3. Let ϕ_i be the natural epimorphism of R onto R_i, and put $b_i = \phi_i(b)$. It suffices to show that $b_i \in C(R_i)$ ($i \in I$). Since R_i satisfies (H), as we saw at the first of the proof, we may assume that $Ch(R_i) = 0$. Then, for each $x \in R_i$, there exists $k > 0$ such that $[x, b_i]_k = 0$, and so $[x, b_i] = 0$ by Lemma 1.2.

Suppose next that R has the non-zero radical J. Then, J is a radical domain, and for each $x \in J$, there exists $k > 0$ such that $[x, b]_k = 0$. Hence, by Lemma 1.2, we get that $[J, b] = 0$. Since R is a domain, as is well known, $C_R(J) \subseteq C(R)$, and so $b \in C(R)$.

Proof of Theorem 1.1. Since R is a subdirect sum of domains by [1, Theorem 2], we may assume that R is a domain. By (H), for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^*$ and $k > 0$ such that $[f(x), g(y)]_k = 0$. Hence we get that $[f(x), g(y)] = 0$ by Lemma 1.5. Therefore, R is commutative by Lemma 1.4.

2. Generalized hyper- and cohypercenter. Throughout this section, we use the following notations:

$\mathcal{E}^* = \{f(X) \in X\mathbf{Z}[X] \mid f(X)$ is a comonic polynomial$\}$.

$\mathcal{E}' = \{f(X) \in X\mathbf{Z}[X] \mid f(1) = \pm 1\}$.

$\mathcal{E}^*_{(n)} = \{X^n - X^{n+1}p(X) \mid p(X) \in \mathbf{Z}[X]\}$, where $n > 0$.

We consider the following subsets of R:

$S^* = S^*(R) = \{a \in R \mid$ for each $x \in R$, there exist $k > 0$ and $f(X) \in \mathcal{E}^*$ such that $[a, f(x)]_k = 0\}$.

$S' = S'(R) = \{a \in R \mid$ for each $x \in R$, there exist $k > 0$ and $f(X) \in \mathcal{E}'$ such that $[a, f(x)]_k = 0\}$.

$S^{*\prime} = S^{*\prime}(R) = \{a \in R \mid$ for each $x \in R$, there exist $k > 0$ and $f(X) \in \mathcal{E}^* \cap \mathcal{E}'$ such that $[a, f(x)]_k = 0\}$.

$T^* = T^*(R) = \{a \in R \mid$ there exist $k > 0$ and $n > 0$ such that for each

$x \in R$, $[a, f(x)]_k = 0$ for some $f(X) \in \mathcal{E}^*_{(n)}\}$.

$T^{*\prime} = T^{*\prime}(R)=\{a \in R \mid$ there exist $k > 0$ and $n > 0$ such that for each $x \in R$, $[a, f(x)]_k = 0$ for some $f(X) \in \mathcal{E}^*_{(n)} \cap \mathcal{E}'\}$.

Obviously, $T^* \subseteq S^*$. Further, S^* and T^* include the hypercenter ([9]) and the cohypercenter ([5]), respectively.

The main purpose of this section is to prove the following theorems:

Theorem 2.1. *If R is a reduced ring, then $S^* = C$.*

Theorem 2.2. (1) *If R is a ring with no non-zero nil right ideal, then $S^{*\prime} = C$.*

(2) *If R is a semiprime ring, then $T^{*\prime} = C$.*

Note that Theorem 2.2(1) and Theorem 2.2(2) generalize [8, Theorem 4] and [3. Theorem 2], respectively.

In preparation for proving our theorems, we state the following lemmas.

Lemma 2.1. *All of S^*, S', $S^{*\prime}$, T^* and $T^{*\prime}$ are subrings of R.*

Proof. If $a \in S^{*\prime}$, then for each $x \in R$, there exist $k_1 > 0$ and $p(X) \in \mathcal{E}^* \cap \mathcal{E}'$ such that $[a, p(x)]_{k_1} = 0$, and if $b \in S^{*\prime}$, then for $p(x)$, there exist $k_2 > 0$ and $q(X) \in \mathcal{E}^* \cap \mathcal{E}'$ such that $[b, q(p(x))]_{k_2} = 0$. Since $[a, -p(x)]_{k_1} = 0$, we may assume that $p(1) = 1$, and also that $q(p(X)) \in \mathcal{E}^* \cap \mathcal{E}'$. Putting $k = max\{k_1, k_2\}$ and $h(X) = q(p(X))$,

$$[a, h(x)]_k = [a, q(p(x))]_k = \sum_i M_i(x)[a, p(x)]_k N_i(x) = 0,$$

where $M_i(X), N_i(X) \in X\mathbf{Z}[X]$. Hence,

$$[a, h(x)]_k = 0 = [b, h(x)]_k.$$

Furthermore, since $[a+b, h(x)]_k = [a, h(x)]_k + [b, h(x)]_k$ and $[ab, h(x)]_{2k} = \sum_{i=0}^{2} k\binom{2k}{i}[a, h(x)]_i[b, h(x)]_{2k-i} = 0$, we can easily see that $\langle a, b\rangle \subseteq S^{*\prime}$, and so $S^{*\prime}$ is a subring of R. Similarly, S^*, S', T^* and $T^{*\prime}$ are subrings of R.

Lemma 2.2. *Let p be a prime integer, R an algebra over* GF(p), *and $A(R)$ the algebraic hypercenter of R (see [4]). Then $S^*(R) = A(R)$.*

Proof. Obviously, $A(R) \subseteq S^*(R)$. On the other hand, since $[x,y]_{p^\alpha} = [x, y^{p^\alpha}]$ for all $x, y \in R$ and all $\alpha > 0$, we can easily see that $S^*(R) \subseteq A(R)$.

Lemma 2.3. *Let R be a prime ring with no non-zero nil ideal. If R has the non-zero radical J and $Ch(R) = p \neq 0$, then $S^* = C$.*

Proof. First, we claim that R has no non-zero periodic ideal. Suppose, to the contrary, that R has a periodic ideal $I \neq 0$. Since R is a prime ring, $I \cap J \neq 0$. For each $x \in I \cap J$, there exist positive integers n and d such that $x^n = x^{n+d}$, and so $x^n \in x^n J$. We see that $x^n = 0$. This implies a contradiction that $I \cap J$ is a non-zero nil ideal. Hence R has no non-zero periodic ideal as claimed. R is an algebra over GF(p) and it has no non-zero periodic ideal; thus it has no non-zero algebraic ideal. Then C = the algebraic hypercenter of R by [4, Theorem 1.6]. On the other hand, we see that S^* = the algebraic hypercenter of R by Lemma 2.2. Therefore, we get that $S^* = C$.

Lemma 2.4. *If R is a division ring, then $S^* = C$.*

Proof. By Lemma 2.1, S^* is a subring of R. Since S^* satisfies (H), it is commutative by Theorem 1.1. Let K be the subfield of R generated by S^*. Then, it is clear that K is preserved by all automorphisms of R, and so $S^* = C$ by Cartan-Brauer-Hua theorem.

Lemma 2.5. *If R is a primitive ring which is not a division ring, then $S' = C$.*

Proof. By the density theorem, R acts densely on a vector space V over the division ring Δ with $dim V > 1$. Suppose that there exist $v \in V$ and $x \in S'$ such that v and vx are linearly independent. By the density action of R, there exists $y \in R$ such that $vy = 0$ and $vxy = vx$. Since $x \in S'$, there exist $k > 0$ and $f(X) \in X\mathbb{Z}[X]$ with $f(1) = \pm 1$ such that $[x, f(y)]_k = 0$. On the other hand, we see that

$$v[x, f(y)] = v(xf(y) - f(y)x) = f(1)vx,$$

and by an easy induction, we have that $v[x, f(y)]_k = f(1)^k vx$. Hence we get that $0 = v[x, f(y)]_k = f(1)^k vx = \pm vx$, a contradiction. Therefore, for each $v \in V$ and $x \in S'$, there exists $\lambda \in \Delta$ such that $vx = \lambda v$, and so $x \in C$ by [5, Remark 13]. We have thus seen that $S' \subseteq C$, and thus $S' = C$.

Lemma 2.6. *Let R be a semiprimitive ring. If R is torsion free, then $S^* = C$.*

Proof. R is a subdirect sum of primitive rings R_i $(i \in I)$. Let ϕ_i be the natural epimorphism of R onto R_i. It suffices to show that $\phi_i(S^*) \subseteq C(R_i)$ for all $i \in I$. If $a \in S^*$, as we saw in the proof of Lemma 1.3, then for each prime p and $y \in R$, there exist $m > 0$, $k > 0$ and $f(X) \in \mathbf{Z}[X]$ such that

$$[a, y^m - py^{m+1}f(y)]_k = 0. \tag{2.1}$$

If R_i is a division ring, then $S^*(R_i) = C(R_i)$ by Lemma 2.4, and so $\phi_i(S^*) \subseteq S^*(R_i) = C(R_i)$. We may assume therefore that R_i is not a division ring. Consider the case that $Ch(R_i) = p \neq 0$. If $a \in S^*$, then for each $y \in R_i$, there exists $b \in R$ such that $y = \phi_i(b)$ and $[\phi_i(a), y^m]_k = 0$ for some $m > 0$ by (2.1). Therefore, we see that $\phi_i(S^*) \subseteq S'(R_i)$. Hence, we get that $\phi_i(S^*) \subseteq C(R_i)$ by Lemma 2.5. Next, consider the case that $Ch(R_i) = 0$. Since R_i is a primitive ring which is not a division ring, by the density theorem, R_i acts densely on a vector space V_i over the division ring Δ_i with $dimV_i > 1$. Suppose that there exist $v \in V_i$ and $x \in \phi_i(S^*)$ such that v and vx are linearly independent. By the density action of R_i, there exists $y \in R_i$ such that $vy = 0$ and $vxy = vx$. Since $x \in \phi_i(S^*)$ and $\phi_i(R) = R_i$, there exist $m, k > 0$ and $f(X) \in \mathbf{Z}[X]$ such that $[x, y^m - 2y^{m+1}f(y)]_k = 0$ by (2.1). Put $g(X) = X^m - 2X^{m+1}f(X)$. Then $g(1) = 1 - 2f(1) \neq 0$. On the other hand, as we saw in the proof of Lemma 2.5, we have that $v[x, g(y)]_k = g(1)^k vx$, and thus $0 = v[x, g(y)]_k = g(1)^k vx \neq 0$, a contradiction. Hence, for each $v \in V_i$ and $x \in \phi_i(S^*)$, there exists $\lambda \in \Delta_i$ such that $vx = \lambda v$, and so $x \in C(R_i)$ by [5, Remark 13]. We have thus seen that $\phi_i(S^*) \subseteq C(R_i)$.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. R is a subdirect sum of domains R_λ $(\lambda \in \Lambda)$ by [1, Theorem 2]. Let ϕ_λ be the natural epimorphism of R onto R_λ. Since $\phi_\lambda(S^*(R)) \subseteq S^*(R_\lambda)$, it is enough to show that $S^*(R_\lambda) = C(R_\lambda)$ for all $\lambda \in \Lambda$. That is to say, we may assume that R is a domain.

First, we suppose that $Ch(R) = p \neq 0$. If there exists a non-zero periodic ideal I, then the domain I is commutative by the well-known Jacobson theorem. As is well known, a domain with a non-zero commutative ideal must itself be commutative, and thus $S^* = C = R$. Therefore, we may assume that R has no non-zero periodic ideal and thus it has no non-zero algebraic ideal over GF(p). Then it follows that $S^* = C$ by [4, Theorem 1.6] and Lemma 2.2.

Suppose next that $Ch(R) = 0$. Since S^* is a domain satisfying (H), it is a commutative subring of R by Theorem 1.1. Therefore, if R has the non-zero radical J, then $S^* = C$ by [5, Remark 9]. On the other hand, if R is semiprimitive, then $S^* = C$ by Lemma 2.6.

To prove Theorem 2.2, we shall state some more lemmas.

Lemma 2.7. (1) *If R is a prime ring with $J \neq 0$, then T^* is a reduced ring.*

(2) *Let R be a ring with no non-zero nil right ideal. If there exists a non-zero $a \in S^*$ with $a^2 = 0$ then $aJ = 0$.*

Proof. (1) Suppose that there exists a non-zero element $a \in T^*$ such that $a^2 = 0$. Then, since R is prime, aJ is non-zero. By the property of T^*, there exist $k > 0$ and $n > 0$ such that for each $x \in J$, $[a, (ax)^n - (ax)^{n+1}f(ax)]_k = 0$ for some $f(X) \in \mathbf{Z}[X]$. We get that

$$0 = ((ax)^n - (ax)^{n+1}f(ax))^k a = ((ax)^n - (ax)^{n+1}f(ax))^k ax$$

because of $a^2 = 0$. Hence, there exists $m > 0$ such that for each $y \in aJ$, $y^m - y^{m+1}h(y) = 0$ for some $h(X) \in \mathbf{Z}[X]$. Then, $y^m = y^{m+1}h(y) = y^m(yh(y))$, and so $y^m \in y^m J$. Hence $y^m = 0$ for all $y \in aJ$. Therefore, R has a non-zero nilpotent ideal by the well-known Levitzki's result, but this contradicts the hypothesis that R is prime. We have thus seen that T^* is a reduced ring.

(2) Suppose, to the contrary, that $aJ \neq 0$. Since $a \in S^*$, for each $x \in J$, $[a, f(ax)]_k = 0$ for some $k > 0$ and $f(X) \in \mathcal{E}^*$. As we saw in the proof of (1), we can get that $(ax)^m = 0$ for some $m > 0$ depending on x. This implies a contradiction that aJ is a non-zero nil right ideal, and thus $aJ = 0$.

The next one has proved essentially at [8, Lemma 22].

Lemma 2.8. *Let R be a ring with the non-zero radical J. If A is a commutative reduced subring of R which is preserved by all quasi-inner automorphisms, then $2[A, J] = 0$.*

Proof. Let $a \in A$ and $x \in J$. Put $a_1 = (1+x)a(1+x)^{-1}$ and $a_2 = (1+2x)a(1+2x)^{-1}$. Then a_1 and a_2 are in A because of the hypothesis of A. Write $(1+x)a = a_1(1+x)$ and $(1+2x)a = a_2(1+2x)$. Compute

$$\begin{aligned} a &= 2(1+x)a - (1+2x)a = 2a_1(1+x) - a_2(1+2x) \\ &= (2a_1 - a_2) + 2(a_1 - a_2)x. \end{aligned}$$

Since A is commutative, we have that

$$0 = [a, a] = [a, (2a_1 - a_2) + 2(a_1 - a_2)x] = 2(a_1 - a_2)[a, x]. \tag{2.2}$$

Noting that

$$\begin{aligned} a_1 - a_2 &= (1+x)a(1+x)^{-1} - (1+2x)a(1+2x)^{-1} \\ &= \{(1+x)a(1+2x) - (1+2x)a(1+x)\}(1+x)^{-1}(1+2x)^{-1} \end{aligned}$$

and $(1+x)a(1+2x) - (1+2x)a(1+x) = [a, x]$, we get that

$$a_1 - a_2 = [a, x](1+x)^{-1}(1+2x)^{-1}. \tag{2.3}$$

Hence, $2(a_1 - a_2)^2 = 2(a_1 - a_2)[a, x](1+x)^{-1}(1+2x)^{-1} = 0$ by (2.2). Since A is a reduced ring, we see that $2(a_1 - a_2) = 0$. Going back to (2.3), we get that $2[a, x] = 0$. We have thus seen $2[A, J] = 0$.

Lemma 2.9. *If R is semiprimitive, then $S^{*\prime} = C$.*

Proof. By Theorem 2.1, if R is a division ring, then $S^{*\prime} = C$. In case that R is a primitive ring which is not a division ring, since $S^{*\prime} \subseteq S'$, we get that $S^{*\prime} = C$ by Lemma 2.5. Since a semiprimitive ring is a subdirect sum of primitive rings, we have thus shown that $S^{*\prime} = C$ in semiprimitive case.

Lemma 2.10. *Let R be a prime ring, and J the Jacobson radical of R.*

(1) *If R satisfies (S)′ and $J \neq 0$, then $S^* = C$.*
(2) *If $J \neq 0$, then $T^* = C$.*

Proof. (1) R is a domain by Corollary 0.1. Hence $S^* = C$ by Theorem 2.1.

(2) By Lemma 2.7(1), T^* is a reduced ring. Hence, T^* is commutative by Theorem 2.1. Therefore, in view of [5, Remark 9], we get $T^* = C$.

Proof of Theorem 2.2. (1) Since R has no non-zero nil ideal, R is a subdirect sum of prime rings R_i $(i \in I)$ each of which has no non-zero nil ideal. Let ϕ_i be the natural homomorphism of R onto R_i. It suffices to show that $\phi_i(S^{*\prime}) \subseteq C(R_i)$ for all $i \in I$. By Lemma 2.9, we may assume that both R and R_i have the non-zero radical. If $Ch(R_i) = p \neq 0$, then $S^{*\prime}(R_i) = C(R_i)$ by Lemma 2.3, and so $\phi_i(S^{*\prime}) \subseteq S^{*\prime}(R_i) = C(R_i)$ as desired. We may assume henceforth that $Ch(R_i) = 0$. By Lemma 2.9, $S^{*\prime}(R/J) = C(R/J)$. Then, we claim that $S^{*\prime}$ is a reduced ring. Assume, to the contrary, that there exists a non-zero $a \in S^{*\prime}$ with $a^2 = 0$. Then, by Lemma 2.7(2), $aJ = 0$ and $a \notin J$. Put $\bar{R} = R/J$, and $\bar{a} = a + J$. Since $S^{*\prime}(\bar{R}) = C\bar{R})$, $\bar{a}\bar{R}$ is a nil ideal of $\bar{R}$. Hence $\bar{a}\bar{R} = 0$ and thus $ax \in J$ for all $x \in R$. Therefore, $aRa \subseteq Ja = 0$. This contradicts the semiprimeness of R. We have thus seen that $S^{*\prime}$ is reduced. Hence $S^{*\prime}$ is a commutative reduced ring by Theorem 2.1. By Lemma 2.8, we can see that $2[S^{*\prime}, J] = 0$, and that $2[\phi_i(S^{*\prime}), \phi_i(J)] = 0$. Hence, $[\phi_i(S^{*\prime}), \phi_i(J)] = 0$, and thus $\phi_i(S^{*\prime}) \subseteq C_{R_i}(\phi_i(J))$. If $\phi_i(J) \neq 0$, then it is a non-zero ideal of prime ring R_i. As is well known, $C_{R_i}(\phi_i(J)) \subseteq C(R_i)$, and so $\phi_i(S^{*\prime}) \subseteq C(R_i)$. If $\phi_i(J) = 0$, then, $[R, S^{*\prime}] \subseteq J$ implies that

$$[R_i, \phi_i(S^{*\prime})] = [\phi_i(R), \phi_i(S^{*\prime})] \subseteq \phi_i(J) = 0.$$

Therefore, we get that $\phi_i(S^{*\prime}) \subseteq C(R_i)$.

(2) Since R is a subdirect sum of prime rings, we may assume that R is a prime ring, and further, that R has the non-zero radical J by (1). Since $T^{*\prime} \subseteq T^*$, we get the conclusion by Lemma 2.10(2).

Corollary 2.1. *If R is a semiprime ring satisfying* (S)$'$, *then* $S^{*\prime} = C$.

Proof. By Corollary 0.2, R has no non-zero nil right ideal. Then we get the conclusion by Theorem 2.2(1).

The next is a special case of the above corollary.

Corollary 2.2. *Let* $H = \{a \in R \mid$ *for each* $x \in R$, *there exist* $k > 0$

and $m > 0$ *such that* $[a, x^m]_k = 0\}$. *If* R *is a semiprime ring satisfying* (S)$'$, *then* $H = C$.

3. The subsets S^*, T^* and $T^*_{(n,k)}$ in semiprime rings. In section 2, we saw that $S^*(R) = C(R)$ if R is a reduced ring. However, if R is a semiprime ring, (or a ring with no non-zero nil ideal) then $S^*(R)$ and even $T^*_{(n,k)}(R)$ ($nk > 1$) need not coincide with $C(R)$ in general. In this section, we shall study structure of semiprime rings R in which $S^*(R)$, $T^*(R)$ and $T^*_{(n,k)}$ ($nk > 1$) do not coincide with $C(R)$, respectively.

Throughout this section, for $n > 0$, R_n will denote the ring of $n \times n$-matrices over a ring R. We call a field F periodic if it is an algebraic extension field over a finite field.

Let n and k be positive integers, and consider the following subsets:

$S^* = S^*(R) = \{a \in R \mid$ for each $x \in R$, there exist $k' > 0$ and $f(X) \in \mathcal{E}^*$ such that $[a, f(x)]_{k'} = 0\}$.

$T^* = T^*(R) = \{a \in R \mid$ there exist $k' > 0$ and $n' > 0$ such that for each $x \in R$, $[a, f(x)]_{k'} = 0$ for some $f(X) \in \mathcal{E}^*_{(n')}\}$.

$T^*_{(n,k)} = T^*_{(n,k)}(R) = \{a \in R \mid$ for each $x \in R$ there exists $f(X) \in \mathcal{E}^*_{(n)}$ such that $[a, f(x)]_k = 0\}$.

Obviously, $T^*_{(n,k)} \subseteq T^* \subseteq S^*$. The main purpose of this section is to prove the following theorems:

Theorem 3.1. *If* R *is a semiprime ring satisfying* (S)$'$, *then* R *is a subdirect sum of rings each of which has one of the following types.*

(i) *a prime ring* R' *with* $S^*(R') = C(R')$.

(ii) *a dense ring of linear transformations on a vector space* V *over* F, *where* F *is a periodic field and* $\dim_F V > 1$.

Theorem 3.2. *If* R *is a prime ring, then one of the following properties hold*:

(i) $T^*(R) = C(R)$.

(ii) R *is isomorphic to* F_t, *where* F *is a periodic field and* $t > 1$ *an integer.*

Theorem 3.3. *Let* n *and* k *be positive integers. If* R *is a prime ring, then one of the following properties hold*:

(i) $T^*_{(n,k)}(R) = C(R)$.

(ii) *R is isomorphic to F_t, where F is a periodic field and $1 < t \leq kn$.*

If R is n-algebraic over a subring A (see [6]), then $C_R(A) \subseteq T_{(n,1)}(R)$. On account of this, we can say that Theorem 3.3 improves [6, Theorem 3]. Needless to say that Theorem 3.2 and Theorem 3.3 can be extended to the results for semiprime rings, respectively: If R is a semiprime ring, then R is a subdirect sum of rings each of which is of type (i) or (ii).

In preparation for proving the above theorems, we state the following lemmas.

Lemma 3.1 *Let R be a ring satisfying* (H). *If $a \in R$ with $a^2 = 0$, then for each $x \in R$, there exists $f(X) \in \mathcal{E}^*$ such that $f(ax) = 0$.*

Proof. For $x \in R$, we put $y = a + ax$. By (H), we have that $[p(y), q(ax)]_k = 0$ for some $p(X), q(X) \in \mathcal{E}^*$ and $k > 0$. Since $y^t = (ax)^t + (ax)^{t-1}a$ for all $t > 0$ because of $a^2 = 0$, we can see that $p(y) = p(ax) + p_0(ax)a$, where $p(X) = Xp_0(X)$. Hence, we have that

$$0 = [p(ax) + p_0(ax)a, q(ax)]_k = [p_0(ax)a, q(ax)]_k = \pm q(ax)^k p_0(ax)a,$$

and so $q(ax)^k p_0(ax)ax = 0$. Put $f(X) = q(X)^k p(X)$. Then we get $f(X) \in \mathcal{E}^*$ and $f(ax) = 0$.

From the above proof, we can easily see the following remark which is used in section 4.

Remark 3.1. Let m, n and k be positive integers, and R a ring satisfying $(\mathrm{H})''_{(m,n,k)}$. If $a \in R$ with $a^2 = 0$, then for each $x \in R$, there exists $f(X) \in \mathcal{E}^*_{(nk+m)}$ such that $f(ax) = 0$.

Lemma 3.2. *Let R be a primitive ring. If $S^* \neq C$, then R is a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F V > 1$.*

Proof. By the density theorem, R is a dense ring of linear transformations on a vector space V over a division ring F. Since R is not a division ring by Theorem 1.1, we see that $dim_F V > 1$, and also that $Ch(F) = p$ is non-zero by Lemma 2.6. Let $a \in S^* \backslash C$. By [5, Remark 13], there exists $v \in V$ such that va and v are linearly independent over F. Let α be an arbitrary element in F. By the density of R, there exists $x \in R$ such that $vax = \alpha va$ and $vx = 0$. Then there exist $f(X) \in \mathcal{E}^*$ and k such that $[a, f(x)]_k = 0$ because of $a \in S^*$. On the other hand, as we saw in the proof of Lemma 2.5, we have that

$$0 = v[a, f(x)]_k = f(\alpha)^k va,$$

and so $f(\alpha)^k = 0$. Hence, by Theorem 1.1 (or the well-known Herstein theorem for division rings), F is commutative, and also F is a periodic field over GF(p).

Lemma 3.3. *Let p be a prime number, and R an algebra over* GF(p). *If $a \in T^*(R)$, then there exists a positive integer k such that R is k-algebraic over the subring $C_R(a)$ (see* [6]).

Proof. If $a \in T^*(R)$, then there exist positive integers k_0 and n_0 such that for each $x \in R$, $[a, f(x)]_{k_0} = 0$ for some $f(X) \in \mathcal{E}^*_{(n_0)}$. Choose $k_1 > k_0$ such that k_1 is a power of p. Since $Ch(R) = p$, we see that

$$[a, f(x)]_{k_0} = 0 = [a, f(x)]_{k_1} = [a, f(x)^{k_1}].$$

Let $k = k_1 n_0$. We have seen that for each $x \in R$, $g(x) \in C_R(a)$ for some $g(X) \in \mathcal{E}^*_{(k)}$. Hence, R is k-algebraic over the subring $C_R(a)$.

Lemma 3.4. *Let R be a dense ring of linear transformations on a vector space V over a division ring F. Suppose that $dim_F V > m$ for some $m > 0$. If v_1, $v_2 \in V$ and they are linearly independent over F, then there exists $a \in R$ such that $v_1 a^m = v_2$ and $v_1 a^{m+1} = v_2 a = 0$.*

Proof. Since $dim_F V > m$, we can choose linearly independent elements $w_1, \cdots, w_{m+1}$ in V such that $w_1 = v_1$ and $w_{m+1} = v_2$. By the density of R, there exists $a \in R$ such that $w_1 a = w_2$, $w_2 a = w_3$, $\cdots$, $w_m a = w_{m+1}$ and $w_{m+1} a = 0$. Then, we can easily see that

$$v_1 a^m = w_1 a^m = w_{m+1} = v_2 \qquad \text{and}$$
$$v_1 a^{m+1} = w_1 a^{m+1} = w_{m+1} a = v_2 a = 0.$$

We are now in a position to prove our theorems.

Proof of Theorem 3.1. Since R is a subdirect sum of prime rings, we may assume that R is a prime ring. If R has the non-zero radical J, then

$S^* = C$ by Lemma 2.10(1). Suppose that R is primitive. If R is a division ring, then $S^* = C$ by Theorem 2.1. If R is not a division ring, then either $S^* = C$ or R is of type (ii) by Lemma 3.2. Since a semiprimitive ring is a subdirect sum of primitive rings, we have done.

For a non-empty subset U of R, let $C_R^{**}(U) = \{a \in R \mid$ for each $y \in U$, there exists $k > 0$ such that $[a, y]_k = 0\}$.

Corollary 3.1. *Let R be a semiprime ring satisfying* (S)′, *and U a subset of R. If for each $x \in R$, there exists $f(X) \in \mathcal{E}^*$, such that $f(x) \in U$, then R is a subdirect sum of rings R_i ($i \in I$) each of which satisfies one of the followings*:

(i) $C_{R_i}^{**}(U_i) = C(R_i)$, *where $U_i = \phi_i(U)$ and ϕ_i is the natural epimorphism of R onto R_i.*

(ii) *R_i is a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F V > 1$.*

Proof. Since $C_{R_i}^{**}(U_i) \subseteq S^*(R_i)$, the assertion is immediate by Theorem 3.1.

Proof of Theorem 3.2. By Lemma 2.10(2), we may assume that R is semiprimitive. First, suppose $Ch(R) = 0$. Then, we see that $T^* = C$ by Lemma 2.6. Next, suppose $Ch(R) = p \neq 0$. We further suppose that R is not of type (ii). Let a be an arbitrary element of T^*. Since R is an algebra over GF(p), we see that R is k-algebraic over $C_R(a)$ for some $k > 0$ by Lemma 3.3. By [6, Theorem 3], we see that $C_R(C_R(a)) = C$, and thus $a \in C_R(C_R(a)) = C$. Therefore, we get $T^* = C$.

Corollary 3.2. *Let R be a prime ring, U a subset of R, and n a positive integer. If for each $x \in R$, there exists $f(X) \in \mathcal{E}_{(n)}^*$ such that $f(x) \in U$, then one of the following properties hold*:

(i) $C_R^*(U) = C(R)$.

(ii) *R is isomorphic to F_t, where F is a periodic field and $t > 1$ an integer.*

Proof. Since $C_R^*(U) \subseteq T^*(R)$, the assertion is clear by Theorem 3.2.

Proof of Theorem 3.3. If $T_{(n,k)}^* \neq C$, then R is isomorphic to F_t for some periodic field F and $t > 1$ by Theorem 3.2. Hence, it suffices to show that $t \leq nk$. Suppose, to the contrary, that $t > nk$. Let $a \in T^*_{(n,k)} \backslash C$. Let V and F be as in the proof of Theorem 3.2. By [5, Remark 13], there exists $v \in V$ such that v and va are linearly independent over F. Since $t > nk$, there exists $b \in F_t$ such that $vab^{nk} = v$ and $vab^{nk+1} = vb = 0$ by Lemma 3.4. Since $a \in T^*_{(n,k)}$, there exists $f(X) \in \mathcal{E}^*_{(n)}$ such that $[a, f(b)]_k = 0$, and so $v[a, f(b)]_k = 0$. Let $f(X)^k = X^{nk} + g(X)$, where $g(X) \in X^{nk+1}\mathbf{Z}[X]$. Then, since $vab^{nk} = v \neq 0$ and $vab^{nk+1} = 0 = vb$, we see that

$$v[a, f(b)]_k = vaf(b)^k = vab^{nk} + vag(b) = v.$$

This implies a contradiction that $v = 0$.

For a non-empty subset U of R, let $C^k_R(U) = \{a \in R \mid [a, U]_k = 0\}$.

Corollary 3.3. *Let n and k be positive integers, R a prime ring, and U a subset of R. If for each $x \in R$, there exists $f(X) \in \mathcal{E}^*_{(n)}$ such that $f(x) \in U$, then one of the following properties hold:*

(i) $C^k_R(U) = C$.

(ii) *R is isomorphic to F_t, where F is a periodic field and $1 < t \leq kn$.*

Proof. Since $C^k_R(U) \subseteq T^*_{(n,k)}$, the assertion is clear by Theorem 3.3.

4. Structure of semiprime rings satisfying (H). Let m, n and k be positive integers, and we consider the following conditions:

(H) For each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^*$ and $k > 0$ such that $[f(x), g(y)]_k = 0$.

$(\mathrm{H})'_{(m)}$ For each $x \in R$, there exist $k > 0$ and $n > 0$ such that for each $y \in R$, $[f(x), g(y)]_k = 0$ for some $f(X) \in \mathcal{E}^*_{(m)}$ and $g(X) \in \mathcal{E}^*_{(n)}$.

$(\mathrm{H})''_{(m,n,k)}$ For each $x, y \in R$, there exist $f(X) \in \mathcal{E}^*_{(m)}$ and $g(X) \in \mathcal{E}^*_{(n)}$ such that $[f(x), g(y)]_k = 0$.

Obviously, $(\mathrm{H})''_{(m,n,k)}$ implies $(\mathrm{H})'_{(m)}$, and $(\mathrm{H})'_{(m)}$ implies (H).

In this section, we shall study structure of semiprime rings satisfying the above conditions, and consequently, give generalizations of results in [6] and [8].

In section 1, we saw that a reduced ring satisfying (H) is commutative. However, a ring satisfying (H) with no non-zero nil ideal need not be commutative. We begin this section by stating the following conjecture:

Conjecture 4.1. *Let R be a ring with no non-zero nil ideal. If R satisfies* (H), *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $\dim_F V > 1$.*

We claim that if the answer of Köthe conjecture (i.e., a ring which has a non-zero one-sided nil ideal contains a non-zero two-sided nil ideal) is positive, then the answer of our conjecture is also positive (see Theorem 4.2). The proof of our conjecture seems to be out of reach, however, by making use of Theorem 1.1, we can prove the following theorems with respect to rings satisfying (H), which includes a generalization of [8, Theorem 3]:

Theorem 4.1. *Let R be a ring with no non-zero nil ideal. If R is a torsion ring satisfying* (H), *then the conclusion of Conjecture* 4.1 *holds.*

Theorem 4.2. *Let R be a ring with no non-zero nil right ideal. If R satisfies* (H), *then the conclusion of Conjecture* 4.1 *holds.*

Theorem 4.3. *Suppose that R satisfies* (S)$'$ *and* (H). *If R is a semiprime ring, then the conclusion of Conjecture* 4.1 *holds.*

Note that Theorem 4.2 generalizes [8, Theorem 3].

In preparation for proving the above theorems, we state the following lemmas.

Lemma 4.1. *Let R be a ring satisfying* (H). *If $a \in R$ with $a^2 = 0$, then aJ is a nil right ideal, where J is the Jacobson radical of R.*

Proof. By Lemma 3.1, for each $x \in J$, there exist $f(X) \in \mathbf{Z}[X]$ and $m > 0$ such that $(ax)^m - (ax)^{m+1}f(ax) = 0$, and so $(ax)^m \in (ax)^m J$, which implies $(ax)^m = 0$. Therefore, aJ is a nil right ideal.

Lemma 4.2. *Let R be a primitive ring which is not a division ring. If R satisfies* (H), *then R is a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $\dim_F V > 1$.*

Proof. By hypothesis, R is a dense ring of linear transformations on a vector space V over F, where F is a division ring and $dim_F V > 1$. Then $Ch(R) = p \neq 0$ by Lemma 1.3. By the structure theorem for primitive rings, there exists a subring R' of R which maps onto F_2. Then, F_2 and also F satisfy (H). By Theorem 1.1, F is a commutative field. Let α be an arbitrary element in F. For $\alpha(e_{21} + e_{22}), \alpha e_{11} \in F_2$, there exist $f(X), g(X) \in \mathcal{E}^*$ and $k > 0$ such that

$$\begin{aligned} 0 &= [f(\alpha(e_{21} + e_{22})), g(\alpha e_{11})]_k \\ &= f(\alpha)g(\alpha)^k[e_{21} + e_{22}, e_{11}]_k = f(\alpha)g(\alpha)^k e_{21}. \end{aligned}$$

Hence we get that $f(\alpha)g(\alpha)^k = 0$. Since $h(X) = f(X)g(X)^k \in \mathcal{E}^*$, we see that α is algebraic over GF(p).

Lemma 4.3 *If R is a semiprimitive ring satisfying* (H), *then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative field.*

(ii) *a dense ring of linear transformations on a vector space V over F, where F is a periodic field and $dim_F V > 1$.*

Proof. R is a subdirect sum of primitive rings. If R is a division ring, then R is a commutative field by Theorem 1.1. Hence the conclusion is now clear by Lemma 4.2.

Proof of Theorem 4.1. As is well known, R is a subdirect sum of prime rings with no non-zero nil ideal. We may assume henceforth that R is a prime ring with no non-zero nil ideal, and also the $Ch(R) \neq 0$. Since D is periodic by Lemma 1.1, as we saw in the proof of Lemma 2.3, if both D and J are non-zero, then $D \cap J$ is a non-zero nil ideal. But it is impossible by our hypothesis. Hence R is either commutative or semiprimitive. Then we get the conclusion by Lemma 4.3.

Proof of Theorem 4.2. First, we shall show that $\bar{R} = R/Ann(J)$ is a reduced ring. Since R is semiprime, we have

$$J \cap Ann(J) = 0. \tag{4.1}$$

Let $\bar{a} \in \bar{R}$ such that $\bar{a}^2 = 0$. By (4.1) we can see that aJ is isomorphic to $\bar{a}\bar{J}$. Then, by Lemma 4.1, $\bar{a}\bar{J}$ is a nil right ideal, and so is aJ. By the hypothesis we see that $aJ = 0$ and thus $\bar{a} = 0$. We have thus seen that $\bar{R}$ is reduced. Hence $\bar{R}$ is a subdirect sum of commutative domains by Theorem 1.1 and [1, Theorem 2], and so R is a subdirect sum of semiprimitive rings and commutative domains by (4.1). Then we get the conclusion by Lemma 4.3.

Proof of Theorem 4.3. By Corollary 0.2, R has no non-zero nil right ideal. Then we get the conclusion by Theorem 4.2.

Corollary 4.1. (1) ([5, Theorem 2]) *Let R be a semiprime ring. If for each $x, y \in R$, there exist $f(X), g(X) \in X^2\mathbf{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$, then R is commutative.*

(2) *Let R be a ring with no non-zero nil right ideal. If for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^* \cap \mathcal{E}'$ and $k > 0$ such that $[f(x), g(y)]_k = 0$, then R is commutative.*

(3) *Let R be a semiprime ring satisfying* (S)$'$. *If for each $x, y \in R$, there exist $f(X), g(X) \in \mathcal{E}^* \cap \mathcal{E}'$ and $k > 0$ such that $[f(x), g(y)]_k = 0$, then R is commutative.*

Proof. (1) It is clear that R satisfies (S)$'$. Hence, by Theorem 4.3 and Lemma 0.1, if R is non-commutative, then there exists a factorsubring of R which is of type $(\mathrm{GF}(p))_2$ for some prime p. However, noting that for $e_{12}, e_{21} \in (\mathrm{GF}(p))_2$,

$$[e_{12} - f(e_{12}), e_{21} - g(e_{21})] = e_{11} - e_{22} \neq 0$$

for all $f(X), g(X) \in X^2\mathbf{Z}[X]$, we see that R is commutative.

(2) Noting that for $e_{21} + e_{22}, e_{11} \in (\mathrm{GF}(p))_2$,

$$[f(e_{21} + e_{22}), g(e_{11})] = \pm e_{21} \neq 0$$

for all $f(X), g(X) \in \mathcal{E}'$, we can see that R is commutative by Theorem 4.2 and Lemma 0.1.

(3) Since R has no non-zero nil right ideal by Corollary 0.2, we get the conclusion by (2).

Corollary 4.2. *Let R be a semiprime ring satisfying* (S)$'$. *If for each $x, y \in R$, there exist positive integers m, n and k such that $[x^m, y^n]_k = 0$, then R is commutative.*

Next, we shall prove the following two theorems concerning rings satisfying $(\mathrm{H})'_{(m)}$ and $(\mathrm{H})''_{(m,n,k)}$, respectively.

Theorem 4.4. *Let R be a semiprime ring, and m a positive integer. If R satisfies $(\mathrm{H})'_{(m)}$, then R is a subdirect sum of rings each of which has one of the following types.*

(i) *a commutative domain.*

(ii) *F_t, where F is a periodic field and $t > 1$ an integer.*

Theorem 4.5. *Let R be a prime ring, and m, n and k positive integers. If R satisfies $(\mathrm{H})''_{(m,n,k)}$, then R is of one of the following forms:*

(i) *a commutative domain.*

(ii) *F_t, where F is a periodic field and $1 < t \leq max\{kn, m\}$.*

Taking $k = 1$ and $m = n$ in Theorem 4.5, we obtain [6, Corollary 5.3].

Lemma 4.4 *Let R be a primitive ring which is not a division ring. If R satisfies $(\mathrm{H})'_{(m)}$ for some $m > 0$, then $R = F_t$ for some $t > 1$, where F is a periodic field.*

Proof. By Lemma 4.2, R is a dense ring of linear transformations on a vector space V over a periodic field F and $dim_F V > 1$. Therefore, it suffices to show that $dim_F V$ is finite. We may assume that $dim_F V > m$. Let $v_1, v_2 \in V$ such thatthey are linearly independent over F. Then, by Lemma 3.4, there exists $a \in R$ such that $v_1 a^m = v_2 \neq 0 = v_1 a^{m+1}$. Now, by $(\mathrm{H})'_{(m)}$, there exist positive integers k and n such that for each $x \in R$,

$$(4.2) \quad [f(a), g(x)]_k = 0 \quad \text{forsome } f(X) \in \mathcal{E}^*_{(m)} \text{ and } g(X) \in \mathcal{E}^*_{(n)}.$$

Since $v_1 a^m = v_2$, we see that $v_1 a^m$ and v_1 are linearly independent over F. If $dim_F V > nk$, there exists $b \in R$ such that

$$v_1 a^m b^{nk} \neq 0, \quad v_1 a^m b^{nk+1} = 0 \quad \text{and} \quad v_1 b = 0,$$

again by Lemma 3.4. By (4.2), there exist $f(X) \in \mathcal{E}^*_{(m)}$ and $g(X) \in \mathcal{E}^*_{(n)}$ such that $[f(a), g(b)]_k = 0$. This forces a contradiction

$$0 = v_1[f(a), g(b)]_k = v_1 f(a) g(b)^k = v_1 a^m b^{nk} \neq 0.$$

Hence $dim_F V \leq nk$.

The above proof enables us to see the following:

Lemma 4.5. *Let R be a primitive ring which is not a division ring, and let m, n and k be positive integers. If R satisfies* $(\mathrm{H})''_{(m,n,k)}$, *then $R = F_t$, where F is a periodic field and $1 < t \leq max\{nk, m\}$.*

Proof of Theorem 4.4. Since R is a subdirect sum of prime rings, we may assume that R is a prime ring. If R is a division ring, then R is commutative by Theorem 1.1. Hence, if R is semiprimitive, then it is a subdirect sum of commutative fields and primitive rings which are not division rings. Then each of those primitive rings is of type (ii) by Lemma 4.4. Therefore, it is enough to consider the case that R has the non-zero radical J. In this case, we shall show that R is a reduced ring. Suppose, to the contrary, that there exists a non-zero $a \in R$ with $a^2 = 0$. By Lemma 4.1, aJ is a nil right ideal, and $aJ \neq 0$ because of primeness of R. Since R has no non-zero nilpotent ideal, aJ cannot be of bounded index of nilpotency by the well-known Levitzki's result. Hence there exists $b \in aJ$ such that $b^{2m} = 0 \neq b^m$. By $(\mathrm{H})'_{(m)}$, there exist $k, n > 0$ such that for each $y \in J$,

$$[b^m - b^{m+1}f(b), g(b^m y)]_k = 0$$

for some $f(X) \in \mathbf{Z}[X]$ and $g(X) \in \mathcal{E}^*_{(n)}$. Since $b^{2m} = 0$, we get that

$$0 = g(b^m y)^k (b^m - b^{m+1}f(b)) = g(b^m y)^k b^m (1 - b^{m+1}f(b)),$$

and so $g(b^m y)^k b^m = 0$ because of $b \in J$. Hence we have

$$g(b^m y)^k b^m y = 0. \tag{4.3}$$

We can put $g(X)^k = X^{kn} - X^{kn+1}g_0(X)$ for some $g_0(X) \in \mathbf{Z}[X]$. By (4.3), we have that

$$0 = (b^m y)^{kn+1} - (b^m y)^{kn+2} g_0(b^m y) = (b^m y)^{kn+1}(1 - (b^m y) g_0(b^m y)).$$

Since $b^m y \in J$, we see that $(b^m y)^{kn+1} = 0$ for all $y \in J$. We get a contradiction that $b^m J$ is a non-zero nil right ideal of bounded index. We have thus seen that R is a reduced ring. Therefore, R is a commutative domain by Theorem 1.1.

Proof of Theorem 4.5. In view of Theorem 1.1, it is enough to consider the case that R is not reduced. Let $0 \neq a \in R$ with $a^2 = 0$, and $s = nk + m$. By Remark 3.1, for each $y \in aR$, there exists $f(X) \in \mathbf{Z}[X]$ such that

$$y^s - y^{s+1}f(y) = 0, \tag{4.4}$$

and thus

$$y^s = y^{s+1}f(y) = y^{s+2}f(y)^2 = \cdots = y^{s+i}f(y)^i$$

for all $i > 0$. Hence, if aR is nil, then aR is a non-zero nil right ideal of index s, whence aR contains a non-zero nilpotent ideal of R, a contradiction. Therefore, we can see that there exists non-zero idempotent $e = y^s f(y)^s$. We consider the ring eRe. Note that the primeness of R is inherited by eRe, and that eRe satisfies $(\mathrm{H})''_{(m,n,k)}$. By Theorem 4.4, eRe is a subdirect sum of commutative rings and F_t's, where F is a commutative field and $t \leq max\{nk, m\} = t'$ by Lemma 4.5. By the well-known Amitsur, Levitzki result, F_t satisfies the standard identity of degree $2t'$, and so does eRe. Therefore, by the well-known Posner's theorem, eRe is an order in Δ_l for some division ring Δ and $l > 0$. Since $eRe \subseteq aR$, eRe satisfies (4.4). Hence all the regular elements of eRe are invertible in eRe, which implies $eRe = \Delta_l$. Then we can see that R has the non-zero socle. As is well known, a prime ring with the non-zero socle is a primitive ring. Therefore, again by Lemma 4.5 and Theorem 1.1, we get that either R is a commutative field F or $R = F_t$ for some periodic field and $1 < t \leq max\{nk, m\}$.

Corollary 4.3. *Let m, n and k be positive integers, and R a semiprime ring. If R satisfies $(\mathrm{H})''_{(m,n,k)}$, then R satisfies the standard identity of degree $2t$, where $t = max\{kn, m\}$.*

Proof. This follows from Theorem 4.5 and the well-known Amitsur-Levitzki result for the polynomial identity satisfied by matrices.

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(*Received October 4, 1993*)

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