Math. J. Okayama Univ. 36(1994), 85–103

STRUCTURE OF RINGS SATISFYING CERTAIN POLYNOMIAL IDENTITIES AND COMMUTATIVITY THEOREMS

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0. Introduction. Throughout, all rings will mean associative rings which are not necessarily commutative. Moreover, Z will represent the ring of rational integers, and by $Z\langle X,Y\rangle$ will be meant the free algebra over Z in two indeterminates. For positive integers $n_1,\cdots,n_r$ their greatest common divisor is denoted by $(n_1,\cdots,n_r)$.

In [5], Y. Kobayashi defined an additive map Φ of $Z\langle X,Y\rangle$ to Z, and indicated that for $f(X,Y)\in Z\langle X,Y\rangle$, $\Phi(f(X,Y))$ is closely related with the commutativity of rings with 1 and satisfying the polynomial identity $f(X,Y)=0$, where Φ will be defined later. In [6], he turned his attention to the fact that $\Phi((XY)^n-X^nY^n)=-n(n-1)/2$ for $n>1$, and investigated the structure of $n(n-1)/2$-torsion free rings with 1 and satisfying the polynomial identity $(XY)^n-X^nY^n=0$. Coincidentally, he proved the following ([6, Theorem]): Let R be a ring with 1. If $E(R)=\{n\in Z\mid n>0 \text{ and } (xy)^n=x^ny^n \text{ for all } x,y\in R\}$ contains integers $n_1,\cdots,n_r\geq 2$ such that $(n_1(n_1-1)/2,\cdots,n_r(n_r-1)/2)=1$ and some of n_i's is even, then R is commutative. In connection with the above theorem, Y. Kobayashi and the present author raised respectively the following conjectures:

Conjecture 0.1 ([7, Conjecture 1]). Let R be a ring with 1. If $E(R)$ contains integers $n_1,\cdots,n_r\geq 2$ such that R is $(n_1(n_1-1)/2,\cdots, n_r(n_r-1)/2)$-torsion free and some of n_i's is even, then R is commutative.

Conjecture 0.2 ([16, Conjecture (I)]). Let R be a ring with 1. If for each $x,y\in R$, there exist integers $n_i\geq 2$ $(i=1,\cdots,r)$ such that $(n_1(n_1-1)/2,\cdots,n_r(n_r-1)/2)=1$ and some of n_i's is even and such that $(xy)^{n_i}=x^{n_i}y^{n_i}$ $(i=1,\cdots,r)$, then R is commutative.

In [8] and [9], Y. Kobayashi gave partial affirmative answers to the above conjectures. In §2 and §4 of the present paper, those results will be improved more precisely and satisfactorily.

Meanwhile, J. Grosen [2] generalized some known commutativity theorems for a ring with 1 and satisfying certain polynomial identities by assuming that the identities hold merely for the elements of a certain subset of the ring rather than for all elements of the ring. Almost all the results obtained in [2] have been improved and sharpened in [13]. In §3 of the present paper, we shall prove some commutativity theorems for a ring with 1 and satisfying polynomial identities of the form $(XY)^n - X^nY^n = 0$ merely for the elements of a certain subset of the ring.

Recently, W. Streb [17] gave a classification of non-commutative rings. H. Komatsu and H. Tominaga applied the classification to the proof of some commutativity theorems, in [11], [12], [13] and [14]. In our subsequent study, we shall use frequently several results obtained in [12] and [14], which will be summarized in §1 together with notations employed in the present paper.

1. Preliminaries. Throughout the present paper, R will represent a ring with 1. We use the following notations. Let M be a non-empty subset of R, and k a positive integer.

$C = C(R)$ = the center of R.

$D = D(R)$ = the commutator ideal of R.

$N = N(R)$ = the set of all nilpotent elements in R.

$N^* = N^*(R) = \{x \in R \mid x^2 = 0\}$.

$J = J(R)$ = the Jacobson radical of R.

$U = U(R)$ = the set of units in R.

Q = the intersection of the set of non-units in R with the set of quasi-regular elements in $R = (1+U)\backslash U$ $(\supseteq N \cup J)$.

$C_R(M)$ = the centralizer of M in R.

$\mathrm{Ann}_R(M) = \{x \in R \mid xM = Mx = 0\}$.

As usual, for $x, y \in R$, let $[x,y]_1 = [x,y] = xy - yx$, and define, recursively $[x,y]_k = [[x,y]_{k-1}, y]$ for all $k > 1$.

$\boldsymbol{Z}\langle X,Y\rangle$ = the free algebra over $\boldsymbol{Z}$ in the indeterminates X and Y.

$\boldsymbol{K} = \boldsymbol{Z}\langle X,Y\rangle[X,Y]\boldsymbol{Z}\langle X,Y\rangle$.

$\boldsymbol{K}_k$ = the set of all $f(X,Y) \in \boldsymbol{K}$ each of whose monomial terms is of length $\geq k$ (together with 0).

$\boldsymbol{W}$ = the set of all words in X and Y, namely products of factors each of which is X or Y (together with 1).

As is well-known, $\boldsymbol{K} = \boldsymbol{K}_2$ coincides with the kernel of the natural homomorphism of $\boldsymbol{Z}\langle X,Y\rangle$ onto $\boldsymbol{Z}[X,Y]$. Let $f(X,Y) = \sum f_{ij}(X,Y)$ be a polynomial in $\boldsymbol{Z}\langle X,Y\rangle$, where $f_{ij}(X,Y)$ is a homogeneous polynomial with degree i in X and degree j in Y. Then we can easily see that $f(X,Y)$

is in $\boldsymbol{K}$ if and only if for each i, j, the sum of the coefficients of $f_{ij}(X, Y)$ equals zero.

Following [5], we denote by Φ the additive map of $\boldsymbol{Z}\langle X, Y\rangle$ to $\boldsymbol{Z}$ defined as follows: For each monic monomial $X_1 \cdots X_r$ (X_i is either X or Y), $\Phi(X_1 \cdots X_r)$ is the number of pairs (i, j) such that $1 \leq i < j \leq r$ and $X_i = X$, $X_j = Y$. We can easily see that, for any $f(X,Y) \in Z\langle X,Y\rangle$, $\Phi(f(X,Y))$ equals the coefficient of XY occurring in $f(1 + X, 1 + Y)$. Now, let $f(X,Y) \in \boldsymbol{K}$. Then $f(1 + X, 1 + Y) \in \boldsymbol{K}$, and so there exists $g(X,Y) \in \boldsymbol{K}_3$ such that $f(1 + X, 1 + Y) = \Phi(f(X,Y))[X,Y] + g(X,Y)$.

Further, we put

$$e(k) = \begin{cases} k & \text{if } k \text{ is even}, \\ k-1 & \text{if } k \text{ is odd}. \end{cases}$$

We consider the following conditions:

(S) For each $x, y \in R$, there exists $f(X,Y) \in \boldsymbol{K}_3$ such that $[x, y] = f(x, y)$.

$Q(k)$ If $x, y \in R$ and $k[x, y] = 0$ then $[x, y] = 0$.

By [12, Theorem 1.2, Proposition 1.6 and Proposition 1.7], we obtain the next

Theorem 1.1. *Let R be a non-commutative ring with 1. Then there exists a factorsubring of R which is of type* a)[1], b), c), d)[1] *or* e)[1]:

a)[1] $\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix}$, *where p a prime number.*

b) $\mathrm{M}_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \ \alpha, \beta \in K \right\}$, *where K is a finite field with a non-trivial automorphism σ.*

c) *A non-commutative division ring.*

d)[1] *A domain which is generated by* 1 *and a simple radical subring.*

e)[1] *A ring $B = \langle 1, x, y\rangle$ with* 1 *such that $D(B)$ is the heart of B and $x, y \in \mathrm{Ann}_B(D(B))$.*

Now, let $\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix}$ be an element of $\mathrm{M}_\sigma(K)$. Let $K^\sigma = \{\gamma \in K \mid \sigma(\gamma) = \gamma\}$. Then

$$(1.1) \quad \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix}^k = \begin{cases} \begin{pmatrix} \alpha^k & (\sigma(\alpha^k) - \alpha^k)(\sigma(\alpha) - \alpha)^{-1}\beta \\ 0 & \sigma(\alpha^k) \end{pmatrix} & \text{if } \alpha \notin K^\sigma, \\ \begin{pmatrix} \alpha^k & k\alpha^{k-1}\beta \\ 0 & \alpha^k \end{pmatrix} & \text{if } \alpha \in K^\sigma. \end{cases}$$

This formula will be used repeatedly in §2 and §4.

By [12, Proposition 1.3(2), Lemma 1.4(1) and (4), and Proposition 1.7], we obtain

Lemma 1.2. *Let R be a ring with 1. If $xy \neq 0 = yx$ for some $x, y \in R$, then there exists a factorsubring of R which is of type* a)^1 *or* e)^1.

Lemma 1.3 ([12, Lemma 2.1]). *Let R be a ring satisfying* (S) *such that $D \subseteq N$. Then there hold the following:*

(1) *N is a commutative ideal of R.*

(2) *$C_R(N^*)$ is a maximal commutative subring of R.*

(3) $\mathrm{Ann}_R([N^*, R])$ *is the largest commutative ideal of R and is contained in $C_R(N^*)$.*

(4) *For any non-empty subset M of N, $R/\mathrm{Ann}_R([M, R])$ has no non-zero nil ideals.*

(5) *Let $c \in N$, $x \in R$, k a positive integer, and p a prime number.*

(i) *If $x^k[c, x] = 0 = [c, x]x^k$ then $[c, x] = 0$.*

(ii) *If $[c, x]_k = 0$ then $[c, x] = 0$.*

(iii) *If $[c, px] = 0 = [c, x^p]$, then $[c, x] = 0$.*

(iv) *If the additive order of $[c, x]$ is finite, then it is square-free.*

The next is included in [14, Proposition 2.9(2)].

Lemma 1.4. *Let R be a non-commutative subdirectly irreducible ring satisfying* (S). *Suppose that R satisfies the identity $[(XY)^n - X^nY^n, X] = 0$ with some $n > 1$. Then R is isomorphic to some $\mathrm{M}_\sigma(K)$.*

The next is included in [14, Lemma 2.10(2)].

Lemma 1.5. *Let n be a positive integer. Let $R = \mathrm{M}_\sigma(K)$, and put $t = (|K| - 1)/(|K^\sigma| - 1)$. If R satisfies the identity $[(XY)^{n+1} - X^{n+1}Y^{n+1}, X] = 0$, then t divides n or $n+1$.*

Theorem 1.6 ([12, Theorem 3.6]). *Let R be a ring with 1, and n a positive integer. Then the following conditions are equivalent:*

1) *R satisfies the identities $[X^n, Y^n] = 0$ and $[X - X^m, Y - Y^m] = 0$ for some $m > 1$.*

2) *R satisfies* (S) *and the identity $[X^n, Y^n] = 0$.*

3) *R is a subdirect sum of rings each of which has one of the following*

types:

i) *A commutative ring.*

ii) $M_\sigma(K)$, *where* $(|K|-1)/(|K^\sigma|-1)$ *divides* n.

The next is included in [14, Theorem 2.12(II)].

Theorem 1.7. *Let R be a ring with* 1, *and n a positive integer. If $k = n(n+1)/2$ is odd, then the following conditions are equivalent*:

1) *R satisfies $Q(k)$ and the identity* $(XY)^n - Y^nX^n = 0$.

2) *R satisfies $Q(k)$ and the identity* $(XY)^{n+1} - X^{n+1}Y^{n+1} = 0$.

3) *R is a subdirect sum of rings each of which has one of the following types*:

i) *A commutative ring.*

ii) $M_\sigma(K)$, *where* $(|K|-1)/(|K^\sigma|-1)$ *divides* $e(n+1)$ *and* $2K = 0$.

2. On Conjecture 0.1. Given $x, y \in R$, we denote by $E(x,y)$ the set of integers $n > 1$ such that $(xy)^n = x^ny^n$; and $\tilde{E}(x,y) = E(x,y) \cap E(y,x)$. For a positive integer n, an element x of a module G is said to be *n-torsion free* if the order of x is infinite or relatively prime to n. Obviously, every element of G is n-torsion free if and only if $nx = 0$ implies $x = 0$ for any $x \in G$.

The purpose of this section is to give a complete answer to Conjecture 0.1. In [8], Kobayashi proved the following theorem which is a partial answer to Conjecture 0.1.

Theorem A. *Let R be a ring with* 1. *If for any $x, y \in R$, $\tilde{E}(x,y)$ contains (at least one) even integers $n_1, \cdots, n_s$ and odd integers $n_{s+1}, \cdots, n_r$ $(r \geq s \geq 1)$ such that $(n_1, \cdots, n_s, n_{s+1} - 1, \cdots, n_r - 1)$ is* 2 *(or a multiple of* 4) *and $[x,y]$ is $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2)$-torsion free, then R is commutative.*

In connection with the above theorem, in [9], he determined the structure of $n(n-1)/2$-torsion free rings with 1 satisfying the identity $(xy)^n = x^ny^n$, when n is a positive even integer. Recently, this result has been generalized by Komatsu and Tominaga (see [14, Theorem 2.12]). The main theorems of this section can be stated as follows:

Theorem 2.1. *Let R be a ring with* 1. *Suppose that, for each $x, y \in$*

R, $\tilde{E}(x,y)$ contains $n_1,\cdots,n_s$ such that $(e(n_1),\cdots,e(n_s)) \equiv 0 \pmod 4$ and $[x,y]$ is $(n_1(n_1-1)/2,\cdots,n_s(n_s-1)/2)$-torsion free. Then R is commutative.

Theorem 2.2. *Let R be a ring with 1, and n a positive integer such that $n \equiv 2 \pmod 4$. Then the following conditions are equivalent:*

1) *R satisfies $Q(n(n-1)/2)$ and the identity $(XY)^n - X^nY^n = 0$.*

2) *R satisfies $Q(n(n+1)/2)$ and the identity $(XY)^n - Y^nX^n = 0$.*

3) *R satisfies $Q(n(n+1)/2)$ and the identity $(XY)^{n+1} - X^{n+1}Y^{n+1} = 0$.*

4) *For each $x,y \in R$, $\tilde{E}(x,y)$ contains $n_1,\cdots,n_s$ and $m_1,\cdots,m_r$ such that $(e(n_1),\cdots,e(n_s)) = n$ and $[x,y]$ is $(m_1(m_1-1)/2,\cdots,m_r(m_r-1)/2)$-torsion free.*

5) *R is a subdirect sum of rings each of which has one of the following types:*

i) *A commutative ring.*

ii) *$\mathrm{M}_\sigma(K)$, where K is a finite field of characteristic 2 with a non-trivial automorphism σ such that $(|K|-1)/(|K^\sigma|-1)$ divides $n/2$.*

In preparation for proving our theorems, we state the next lemma.

Lemma 2.3. *Let R be a ring with 1. Suppose that, for each $x,y \in R$, $\tilde{E}(x,y)$ contains $m_1,\cdots,m_r$ such that $[x,y]$ is $(m_1(m_1-1)/2,\cdots,m_r(m_r-1)/2)$-torsion free. Then there hold the following:*

(1) *$D \subseteq N$.*

(2) *$2[N,R] = 0$, namely $2N \subseteq C$.*

(3) *R satisfies (S).*

(4) *R is completely reflexive, namely $xy = 0$ implies $yx = 0$ for any $x,y \in R$.*

(5) *Let $a \in N$, and $x \in R$. If $n \in \tilde{E}(1+a,x)$ then $[a,x^{e(n)}] = 0$.*

Proof. In preparation for proving (1), we state three claims.

Claim 1. If $x,y \in R$ and $n \in \tilde{E}(x,y)$, then $y[x^n,y^{n-1}]y = 0$.

Proof. Actually, $y[x^n,y^{n-1}]y = yx^ny^n - y^nx^ny = y(xy)^n - (yx)^ny = 0$.

Claim 2. For each $x,y \in R$, and for each positive integer k, there exists a positive integer m and $f(X,Y) \in \boldsymbol{K}_k$ such that $m[x,y] = f(x,y)$

and $[x,y]$ is m-torsion free.

Proof. There exist positive integers $m_1, \cdots, m_r$ in $\widetilde{E}(1+x, 1+y)$ such that $[x,y] = [1+x, 1+y]$ is m-torsion free, where $m = (m_1(m_1-1)/2, \cdots, m_r(m_r-1)/2)$. Then we can easily see that there exist $f_i(X,Y) \in K_3$ such that

$$\begin{aligned} 0 &= (1+x)^{m_i}(1+y)^{m_i} - \{(1+x)(1+y)\}^{m_i} \\ &= \frac{m_i(m_i-1)}{2}[x,y] + f_i(x,y). \end{aligned}$$

Hence we obtain $m[x,y] = f(x,y)$ with some $f(X,Y) \in K_3$. Claim 2 is an easy consequence of this fact.

Claim 3. N is a commutative ideal of R.

Proof. Let $a \in N$, and $x \in R$. Obviously, $a^{2^s} = 0$ for some positive integer s. Now, choose $n_1 \in \widetilde{E}(a,x)$, and inductively $n_{i+1} \in \widetilde{E}(a^{n_1 \cdots n_i}, x^{n_1 \cdots n_i})$ $(i = 1, \cdots, s-1)$. Then $n_1 \cdots n_s$ is in $\widetilde{E}(a,x)$ and $a^{n_1 \cdots n_s} = 0$, and so both ax and xa are in N. Now let $c \in N$, and suppose that $ac = 0$. Then $cxa = [c, xa] = 0$. If not, Claim 2 shows that there exists a positive integer m such that $0 \neq m[c,xa] = f(c,xa)$ with some $f(X,Y) \in K_{2n}$, where n is a positive integer such that $c^n = 0 = (xa)^n$. But this forces a contradiction that $f(c,xa) = 0$. We have thus seen that if $c \in N$ and $ac = 0$ then $cRa = 0$. Now, let $b \in N$, and $a^\nu = 0 = b^\nu$. Then, by the above, we see that $a^i R a^j = 0$, provided $i + j \geq \nu$. Further, we see that $a^i R a^j R a^k = 0$, provided $i + j + k \geq \nu$. Continuing this procedure, we obtain eventually $(a)^\nu = 0$; similarly, $(b)^\nu = 0$. If $[a,b] \neq 0$ then Claim 2 shows that there exists a positive integer μ such that $0 \neq \mu[a,b] = g(a,b)$ with some $g(X,Y) \in K_{2\nu}$. But, as is easily seen, $g(a,b) = 0$. This contradiction shows that N is commutative, and therefore N forms a commutative ideal.

(1) Note that N is an ideal by Claim 3. It suffices to show that R/N is commutative. Let $x, y \in R$. Then, by Claim 1, $y[x^n, y^{n-1}]y = 0$ for some $n > 1$. Since $y[x^n, y^{n-1}]$ and $[x^n, y^{n-1}]y$ belong to N, we see that

$$[x^n, y^{2(n-1)}] = y^{n-1}[x^n, y^{n-1}] + [x^n, y^{n-1}]y^{n-1} \in N.$$

Hence R/N is commutative, by [3, Theorem].

Claim 4. Let $a \in N$, $x \in R$, and n a positive integer. If $x^n[a,x] = 0 = [a,x]x^n$, then $[a,x] = 0$.

Proof. It is easy to see that $[a, \langle x\rangle^{2n}] = 0$. Suppose, to the contrary, that $[a,x] \neq 0$. Then Claim 2 shows that there exists a positive integer m and $f(X,Y) \in K_{2n}$ such that $0 \neq m[a,x] = f(a,x)$. Since $\langle x\rangle^{2n} \subseteq C_R(a)$, N is commutative and $N^2 \subseteq C$ (Claim 3), we can easily see that $f(a,x) = 0$. But this is a contradiction.

Claim 5. If $a \in N$, $x \in R$, and $n \in \tilde{E}(1+a,x)$, then $nx[a,x^{n-1}]x = 0$, $(n-1)[a,x^n] = 0$ and $n(n-1)[a,x] = 0$.

Proof. Noting that $N^2 \subseteq C$ by Claim 3, we see that

$$nx[a,x^{n-1}]x = x[(1+a)^n, x^{n-1}]x = 0$$

by Claim 1, and

$$\begin{aligned}(n-1)[a,x^n] &= -(1+a)^{-1}\{(1+a)[x^n,(1+a)^{n-1}](1+a)\}(1+a)^{-1}\\ &= 0.\end{aligned}$$

From those above, we obtain

$$n(n-1)[a,x]x^n = n(n-1)[a,x^n]x - n(n-1)x[a,x^{n-1}]x = 0\,;$$

and similarly, $n(n-1)x^n[a,x] = 0$. Hence $n(n-1)[a,x] = 0$, by Claim 4.

(2) Let $a \in N$, and $x \in R$. Then $\tilde{E}(1+a,x)$ contains $m_1,\cdots,m_r$ such that $[a,x] = [1+a,x]$ is m-torsion free, where $m = (m_1(m_1-1)/2, \cdots, m_r(m_r-1)/2)$. By Claim 5, $m_i(m_i-1)[a,x] = 0$ $(i = 1,\cdots,r)$; and so $2m[a,x] = 0$. Hence $2[a,x] = 0$.

(3) Let $x,y \in R$, and choose n in $\tilde{E}(x,y)$. Since $2D \subseteq 2N \subseteq C$ by (1) and (2), we see that

$$0 = 2y[x^n, y^{n-1}]y = 2(n-1)[x^n,y]y^n = 2n(n-1)x^{n-1}[x,y]y^n$$

by Claim 1. Similarly, for $n' \in \tilde{E}(x,1+y)$, we obtain

$$2n'(n'-1)x^{n'-1}[x,y](1+y)^{n'} = 0.$$

Then, as is easily seen, $lx^k[x,y] = 0$ for some positive integers k,l. Repeating the above process, we see that $l'[x,y] = 0$ for some positive integer l'. Now, by Claim 2, there exists a positive integer m and $f(X,Y) \in K_3$ such that $m[x,y] = f(x,y)$ and $[x,y]$ is m-torsion free. Since the additive order of $[x,y]$ is finite, there exists an integer m' such that $[x,y] = m'm[x,y] = m'f(x,y)$.

(4) Obviously,

$$\{e_{11}(e_{12}+e_{22})\}^n - e_{11}^n(e_{12}+e_{22})^n = -e_{12} \neq 0$$

for any integer $n > 1$. Hence R has no factorsubring of type a)^1. On the other hand, R satisfies (S) by (3), and so R has no factorsubring of type e)^1. Hence, by Lemma 1.2, R is completely reflexive.

(5) By Claim 5, we see that $nx[a,x^{n-1}]x = 0 = (n-1)[a,x^n]$. If n is even then $[a,x^n] = 0$ by (2). On the other hand, if n is odd then $x[a,x^{n-1}]x = 0$ again by (2). Further, $x^2[a,x^{n-1}] = 0 = [a,x^{n-1}]x^2$, by (4). Now, Claim 4 shows that $[a,x^{n-1}] = 0$.

Proof of Theorem 2.1. By Lemma 2.3(1) and (3), $D \subseteq N$ and R satisfies (S). Hence $C_R(N)$ is commutative by Lemma 1.3(2). Now, let $a \in N$, and $x \in R$. Then $\tilde{E}(1+a,x)$ contains $n_1,\cdots,n_s$ such that $(e(n_1),\cdots,e(n_s)) \equiv 0 \pmod 4$ and $[a,x] = [1+a,x]$ is $(n_1(n_1-1)/2,\cdots,n_s(n_s-1)/2)$-torsion free. As is easily seen, $(e(n_1),\cdots,e(n_s)) \equiv 0 \pmod 4$ if and only if $(n_1(n_1-1)/2,\cdots,n_s(n_s-1)/2)$ is even. Hence, in view of Lemma 2.3(2), we obtain $[a,x] = 0$. We have thus seen that $R = C_R(N)$, which is commutative.

Proof of Theorem 2.2. Obviously, 1) implies 4), and Theorem 1.7 shows that 2), 3) and 5) are equivalent. It suffices therefore to show that 4) $\Rightarrow$ 5) $\Rightarrow$ 1).

4) $\Longrightarrow$ 5). Suppose that R satisfies 4). Then, by Lemma 2.3(1) and (3), $D \subseteq N$ and R satisfies (S). Let $a \in N$, and $x \in R$. Then $\tilde{E}(1+a,x)$ contains $n_1,\cdots,n_s$ such that $(e(n_1),\cdots,e(n_s)) = n$. Without loss of generality, we may assume that $n = -m_1e(n_1) - \cdots - m_te(n_t) + m_{t+1}e(n_{t+1}) + \cdots + m_se(n_s)$ with some $m_i \geq 0$. Put $k = m_1e(n_1) + \cdots + m_te(n_t)$ and $l = m_{t+1}e(n_{t+1}) + \cdots + m_se(n_s)$. By Lemma 2.3(5), $[a,x^{e(n_i)}] = 0$ $(i = 1,\cdots,s)$. Then we see that

$$[a,x^n]x^k = [a,x^{n+k}] = [a,x^l] = 0,$$

and similarly $x^k[a,x^n] = 0$. Hence $[a,x^n] = 0$, by Lemma 1.3(5)(i). Further, $C_R(N)$ is commutative by Lemma 1.3(2). Hence R satisfies the identity $[X^n,Y^n] = 0$. Now, we can apply Theorem 1.6 to see that R is a subdirect sum of a commutative ring and rings each of which is isomorphic to some $\mathrm{M}_\sigma(K)$, where $(|K|-1)/(|K^\sigma|-1)$ divides n. Furthermore, by Lemma 2.3(2), $2R \subseteq C_R(N)$, which is commutative. Hence $4[R,R] = 0$, and so we can easily see that $2K = 0$. Then $(|K| - 1)/(|K^\sigma| - 1)$ is odd and divides $n/2$.

5) $\Longrightarrow$ 1). It is easy to see that R satisfies $Q(n(n-1)/2)$. Suppose that $M_\sigma(K)$ is of type ii). Then, $\alpha^n \in K^\sigma$ for all $\alpha \in K$. Since, n is even and K is of characteristic 2, we can easily see that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix}^n = \begin{pmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{pmatrix}$$

for all $\alpha, \beta \in K$, by (1.1) in §1. Hence, $M_\sigma(K)$ satisfies the identity $(XY)^n - X^nY^n = 0$.

3. Commutativity theorems for rings satisfying the polynomial identities of the form $(XY)^n - X^nY^n = 0$ on certain subsets. In this section, we shall generalize some known commutativity theorems for a ring R satisfying the polynomial identities of the form $(XY)^n - X^nY^n = 0$ by assuming that the identities hold merely for the elements of a certain subset of R rather than for all elements of R.

Let k be a positive integer, and A a subset of R. We consider the following conditions:

$P_0(k, A)$ $(xy)^k = x^ky^k$ for all $x, y \in A$.

$P_0^*(k, A)$ $(xy)^k = y^kx^k$ for all $x, y \in A$.

The statements in the following theorem are included in [15, Theorem 2] and [18, Theorem 4], respectively.

Theorem B. *Suppose that a ring R with 1 satisfies $P_0(k, R)$ $(k = n, n+2, n+4)$.*

(1) *If n is even, then R is commutative.*

(2) *If $x^4 \in C$ for all x in R, then R is commutative.*

More recently, Komatsu and Tominaga proved [13, Theorems 2.4 and 2.7] which encompass several results of Grosen [2]. From [13, Theorems 2.4 and 2.7], we readily obtain

Theorem C. (1) *Suppose that a ring R with 1 satisfies $P_0(k, R\backslash Q)$ $(k = m, m+1, n, n+1)$. If R satisfies $Q((m, n))$, then R is commutative.*

(2) *Suppose that a ring R with 1 satisfies $P_0(n+1, R\backslash Q)$ (or $P_0^*(n, R\backslash Q)$). If R satisfies $Q(n(n+1))$, then R is commutative.*

Obviously, Theorem C(2) includes [10, Theorem 1(b) and Theorem 2(b)]. The first main theorem of this section is stated as follows:

Theorem 3.1. *Let R be a ring with 1. Let $n_1, \cdots, n_r$ be positive integers such that $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2) = 1$. If R satisfies $P_0(n_i, R \backslash J)$ $(i = 1, \cdots, r)$, then R is commutative.*

In preparation for proving Theorem 3.1, we state the following two lemmas.

Lemma 3.2. *Let R be a ring with 1. Let k, m, n be non-negative integers, and $f: R \to R$ a function such that $f(x) = f(x+1)$ for all $x \in R$. If $f(x)(x+k)^m x^n = 0$ (or $x^n(x+k)^m f(x) = 0$) for all $x \in R$, then $(k+1)^{mn} f(x) = 0$. In particular, if $f(x)x^n = 0$ (or $x^n f(x) = 0$) for all $x \in R$, then $f(x) = 0$.*

Proof. Obviously,

$$\begin{aligned} 0 &= f(x)(x+k+1)^m (x+1)^n (x+k)^m x^{n-1} \\ &= (k+1)^m f(x)(x+k)^m x^{n-1}. \end{aligned}$$

Continuing this process, we get $(k+1)^{mn} f(x)(x+k)^m = 0$. Next, we obtain

$$\begin{aligned} 0 &= (k+1)^{mn} f(x)((x+k)+1)^m (x+k)^{m-1} \\ &= (k+1)^{mn} f(x)(x+k)^{m-1}. \end{aligned}$$

Continuing this process, we conclude that $(k+1)^{mn} f(x) = 0$.

Lemma 3.3. *Let R be a ring with 1. Suppose that R satisfies $P_0(n, R \backslash Q)$ $(n > 1)$. Then, for each $u \in U, u^{n(n-1)} \in C$, and $D \subseteq N$. In particular, if R satisfies $P_0(k, R \backslash Q)$ $(k = n\ (\geq 1), n+2, n+4)$, then $u^2 \in C$ for each $u \in U$.*

Proof. Let $u, v \in U$, and $x \in R \backslash Q$. Then

$$u[x^n, u^{n-1}]u = ux^n u^n - u^n x^n u = u(xu)^n - (ux)^n u = 0,$$

and so $[x^n, u^{n-1}] = 0$. In particular, $[v^n, u^{n-1}] = 0 = [u^n, v^{n-1}]$. Accordingly, $[v^n, u^{n(n-1)}] = 0 = [u^{n(n-1)}, v^{n-1}]$, whence $[u^{n(n-1)}, v] = 0$ follows.

Now, let $u \in U$, and $x \in R$. In case $u^{n-1}x \in Q$, by the above,

$$0 = [u^{n(n-1)}, 1 - u^{n-1}x] = -u^{n-1}[u^{n(n-1)}, x],$$

and so $[u^{n(n-1)}, x] = 0$. Similarly, in case $x \in Q$, we obtain $[u^{n(n-1)}, x] = 0$. Finally, we consider the case that $u^{n-1}x \notin Q$ and $x \notin Q$. Obviously, $x^{n-1} \notin Q$. Recalling that $[x^n, u^{n-1}] = 0$, we see that

$$\begin{aligned}(x^{n-1}u^{n-1}x)^n &= x^{n(n-1)}(u^{n-1}x)^n = x^{n(n-1)}u^{n(n-1)}x^n \\ &= x^{n^2}u^{n(n-1)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(x^{n-1}u^{n-1}x)^n &= x^{n-1}(u^{n-1}x^n)^{n-1}u^{n-1}x \\ &= x^{n-1}x^{n(n-1)}u^{(n-1)^2}u^{n-1}x \\ &= x^{n^2-1}u^{n(n-1)}x.\end{aligned}$$

Hence $x^{n^2-1}[u^{n(n-1)}, x] = 0$, and so $[u^{n(n-1)}, x] = 0$ by Lemma 3.2.

Now, it is easy to see that R satisfies the polynomial identity

$$\{(XY)^n - X^nY^n\}Z[(1-X)^{n(n-1)}, (1-Y)^{n(n-1)}] = 0.$$

But, no $M_2(GF(p))$, p a prime, satisfies the above identity, as a consideration of the following elements shows: $X = e_{11}$, $Y = e_{12} + e_{22}$, $Z = e_{21}$. Hence, $D \subseteq N$ by [1, Theorem 1] (or [4, Proposition 2]).

Noting that $(n(n-1), (n+2)(n+1), (n+4)(n+3)) = 2$, we can easily see the latter assertion.

Proof of Theorem 3.1. First, we shall show that J is commutative. Let $x, y \in J$. Then

$$\begin{aligned}0 &= (1+x)^{n_i}(1+y)^{n_i} - \{(1+x)(1+y)\}^{n_i} \\ &= \frac{n_i(n_i-1)}{2}[x, y] + f_i(x, y),\end{aligned}$$

where $f_i(X,Y) \in K_3$. Since $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2) = 1$, we see that J satisfies the condition (S) (and $D \subseteq N$ by Lemma 3.3). By Lemma 3.3, $u^{n_i(n_i-1)} \in C$ for each $u \in U$ $(i = 1, \cdots, r)$, so that $u^2 \in C$. Now, let $a \in J$, and $d \in N^*$. Then $2d = 1 - (1-d)^2 \in C$, and $[d, a^2] = [d, (1-a)^2] + [2d, a] = 0$. Now, by Lemma 1.3(5)(iii) shows that $[d, a] = 0$. Hence $[N^*, J] = 0$. By Lemma 1.3(2), this implies that J is commutative.

Noting that J is a commutative ideal, we readily see that $[J,R]J = [J,RJ] = 0$. This enables us to see that if a is in J and x in R then

$$(xa)^{n_i} - x^{n_i}a^{n_i} = x\{(ax)^{n_i-1} - x^{n_i-1}a^{n_i-1}\}a \in x([a,x])a = 0.$$

Similarly, $(ax)^{n_i} - a^{n_i}x^{n_i} = 0$. We have thus seen that R satisfies $P_0(n_i, R)$. Repeating the argument employed at the opening of this proof, we see that R satisfies (S) (and $D \subseteq N$). Now, by Lemma 1.4, R is a subdirect sum of commutative rings and some $\mathrm{M}_\sigma(K)$'s. Suppose, to the contrary, that $\mathrm{M}_\sigma(K)$ appears as a factor of the subdirect sum. Then, by Lemma 1.5, $(|K|-1)/(|K^\sigma|-1)$ divides $n_i(n_i-1)$ $(i = 1, \cdots, r)$, and so does 2. But this is impossible.

Corollary 3.4. *Let R be a ring with 1, and n a positive integer. If R satisfies $P_0(k, R\backslash J)$ $(k = n, n+2, n+4)$, then R is commutative.*

Proof. It suffices to note that $(n(n-1)/2, (n+2)(n+1)/2, (n+4)(n+3)/2) = 1$.

Theorem 3.1′. *Let R be a ring with 1. Let $n_1, \cdots, n_r$ be positive integers such that $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2) = 1$. If R satisfies $P_0(n_i, R\backslash N)$ $(i = 1, \cdots, r)$, then R is commutative.*

Proof. Since $D \subseteq N$ by Lemma 3.3, N forms an ideal of R. Then, careful scrutiny of the proof of Theorem 3.1 shows that R is commutative.

Corollary 3.4′. *Let R be a ring with 1, and n a positive integer. If R satisfies $P_0(k, R\backslash N)$ $(k = n, n+2, n+4)$, then R is commutative.*

Corollary 3.5. *Let R be a ring with 1.*

(1) *If there exist positive integers $n_1, \cdots, n_r$ with $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2) = 1$ such that R satisfies $P_0(n_i, R)$ $(i = 1, \cdots, r)$, then R is commutative.*

(2) *If there exist positive integers m, n with $(m, n) = 1$ or 2 such that R satisfies $P_0(k, R)$ $(k = m, m+1, n, n+1)$, then R is commutative.*

(3) *If there exists a positive integer n such that R satisfies $P_0(k, R)$ $(k = n, n+2, n+4)$, then R is commutative.*

Needless to say, Theorem B is included in Corollary 3.5(3).

Now, by making use of Corollary 3.5, we shall prove the following two theorems, which are related with Theorem C.

Theorem 3.6. *Let R be a ring with 1. Suppose that R satisfies $P_0(k, R\backslash Q)$ $(k = n, n+2, n+4)$.*

(1) *If n is even, then R is commutative.*

(2) *If $2[x, a] = 0$ implies $[x, a] = 0$ for each $a \in Q$ and $x \in R$, then R is commutative.*

Theorem 3.7. *Let R be a ring with 1. Suppose that R satisfies $P_0^*(k, R\backslash Q)$ $(k = n, n+2, n+4)$. Then R is commutative.*

Proof of Theorem 3.6. In view of Corollary 3.5, it suffices to show that R satisfies $P_0(k, R)$ $(k = n, n+2, n+4)$.

Let $u, v \in U$, and $x \in R$. Then $(vu)^2 \in C$, by Lemma 3.2. Hence

$$\begin{aligned} 0 &= (vu)^2 v^n u^n - v^{n+2} u^{n+2} = v^n (vu)^2 u^n - v^{n+2} u^{n+2} \\ &= v^{n+1}[u, v] u^{n+1}, \end{aligned}$$

and so $[u, v] = 0$. In particular, $[Q, u] = [u, 1 - Q] = 0$. From this we see that

$$(3.1) \qquad (xu)^k = x^k u^k \qquad (k = n, n+2, n+4).$$

(1) Noting that n is even and $[x, u^2] = 0$ by Lemma 3.3, from (3.1) we get

$$u^n x^n (xu)^2 = x^n u^n (xu)^2 = x^{n+2} u^{n+2} = u^n x^n x^2 u^2,$$

whence $x^{n+1}[x, u] = 0$ follows. By Lemma 3.2, $[x, u] = 0$. This proves that $Q \subseteq C$. Hence R satisfies $P_0(k, R)$.

(2) In view of (1), we may assume that n is odd. Then, noting that $[x, u^2] = 0$ by Lemma 3.3, from (3.1) we get

$$(xu)^2 x^n u^n = (xu)^{n+2} = x^{n+2} u^2 u^n = x u^2 x^{n+1} u^n,$$

whence $xu[x, u]x^n = 0$ follows. In this equation, replacing x by $x + 1$ and multiplying x^n from right, we obtain $[x, u](x+1)^n x^n = 0$. By Lemma 3.2, $2^{n^2}[x, u] = 0$. Thus $[x, u] = 0$, and so $[x, Q] = 0$ by the hypothesis. Hence R satisfies $P_0(k, R)$.

Proof of Theorem 3.7. Obviously, R satisfies $P_0(k+1, R\backslash Q)$ $(k = n, n+2, n+4)$. By Lemma 3.3, $u^2 \in C$ for each $u \in U$. In case n is odd, Theorem 3.6(1) guarantees the commutativity of R. Thus, henceforth, we may restrict our attention to the case that n is even. Let $u \in U$, and $x \in R$. Then, as was shown in the proof of Theorem 3.6, $[Q, u] = 0$. From this and $P_0^*(k, R\backslash Q)$, we see that

$$(3.2) \qquad (xu)^k = u^k x^k \qquad (k = n, n+2, n+4).$$

Noting that n is even and $[x, u^2] = 0$, from (3.2) we get

$$u^n x^n (xu)^2 = u^{n+2} x^{n+2} = u^n x^{n+2} u^2,$$

whence $x^{n+1}[x, u] = 0$ follows. By Lemma 3.2, $[x, u] = 0$. This proves that R satisfies $P_0^*(k, R)$, and also $P_0(k+1, R)$. Hence R is commutative, by Corollary 3.5.

4. On Conjecture 0.2. In [8, Theorem 2], Y. Kobayashi proved the following theorem which gives an affirmative answer to Conjecture 0.2 in a somewhat weak form.

Theorem D. *Let R be a ring with 1. If for each $x, y \in R$, $\tilde{E}(x, y)$ contains integers $n_1, \cdots, n_r$ such that $(n_1(n_1-1)/2, \cdots, n_r(n_r-1)/2) = 1$ and some of n_i's is even, then R is commutative.*

In this section, we shall prove a generaliation of Theorem D. We consider the following conditions:

($*$) For each $x, y \in R$, there exist integers $k \geq 0$, $n > 1$ and words $w(x, y)$, $w'(x, y) \in \boldsymbol{W}$ such that
$$w(x, y)\{(xy)^n - x^n y^n\} w'(x, y) = 0 = y^k \{(yx)^n - y^n x^n\} x^k.$$

($\sharp$) For each $x, y \in R$, there exist non-negative integers $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5 \leq r_6 \leq r_7 \leq r_8$ with $1 < r_8$, positive integers n_i $(1 \leq i \leq r_8)$, m_i $(r_2 + 1 \leq i \leq r_8)$, l_i $(r_4 + 1 \leq i \leq r_8)$, and words $w_i(x, y)$, $w_i'(x, y) \in \boldsymbol{W}$ $(1 \leq i \leq r_8)$ such that

$(\sharp)_0$ $(n_1(n_1+1)/2, \cdots, n_{r_2}(n_{r_2}+1)/2, m_{r_2+1}n_{r_2+1}, \cdots, m_{r_4}n_{r_4}, l_{r_4+1}m_{r_4+1}n_{r_4+1}, \cdots, l_{r_8}m_{r_8}n_{r_8}) = 1$,

$(\sharp)_1$ $w_i(x, y)\{(xy)^{n_i} - y^{n_i}x^{n_i}\} w_i'(x, y) = 0 \qquad (1 \leq i \leq r_1)$,

$(\sharp)_2$ $w_i(x, y)\{(yx)^{n_i} - x^{n_i}y^{n_i}\} w_i'(x, y) = 0 \qquad (r_1 + 1 \leq i \leq r_2)$,

$(\sharp)_3$ $w_i(x, y)\{(x^{m_i}y^{m_i})^{n_i} - (y^{m_i}x^{m_i})^{n_i}\} w_i'(x, y) = 0 \quad (r_2 + 1 \leq i \leq r_3)$,

$(\sharp)_4$ $w_i(x,y)[x^{m_i}, y^{n_i}]w'_i(x,y) = 0$ $(r_3 + 1 \leq i \leq r_4)$,

$(\sharp)_5$ $w_i(x,y)[x^{l_i}, (x^{m_i}y^{m_i})^{n_i}]w'_i(x,y) = 0$ $(r_4 + 1 \leq i \leq r_5)$,

$(\sharp)_6$ $w_i(x,y)[x^{l_i}, (y^{m_i}x^{m_i})^{n_i}]w'_i(x,y) = 0$ $(r_5 + 1 \leq i \leq r_6)$,

$(\sharp)_7$ $w_i(x,y)[y^{l_i}, (x^{m_i}y^{m_i})^{n_i}]w'_i(x,y) = 0$ $(r_6 + 1 \leq i \leq r_7)$,

$(\sharp)_8$ $w_i(x,y)[y^{l_i}, (y^{m_i}x^{m_i})^{n_i}]w'_i(x,y) = 0$ $(r_7 + 1 \leq i \leq r_8)$.

Now, the main theorem of this section is stated as follows:

Theorem 4.1. *Let R be a ring with 1. If R satisfies $(*)$ and $(\sharp)$, then R is commutative.*

According to Theorem 1.1, in order to complete the proof of Theorem 4.1, it suffices to prove the following two lemmas.

Lemma 4.2. *If R is of type* a)[1], c) *or* d)[1], *then R does not satisfy $(*)$.*

Lemma 4.3. *If R is of type* b) *or* e)[1], *then R does not satisfy $(\sharp)$.*

Proof of Lemma 4.2. First, assume that R is of type a)[1]. Then, for any integers $k \geq 0$ and $n > 1$, we see that

$$e_{11}^k\{(e_{11}(e_{12}+e_{22}))^n - e_{11}^n(e_{12}+e_{22})^n\}(e_{12}+e_{22})^k = -e_{12} \neq 0.$$

Hence R does not satisfy $(*)$.

Next, assume that R is of type c) or d)[1]. Suppose, to the contrary, that R satisfies $(*)$. Now, let $x, y \in R$, and choose $k \geq 0$, $n > 1$ and $w(x,y)$, $w'(x,y) \in \boldsymbol{W}$ such that

$$w(x,y)\{(xy)^n - x^ny^n\}w'(x,y) = 0 = y^k\{(yx)^n - y^nx^n\}x^k.$$

Since R is a domain, we see that $(xy)^n = x^ny^n$ and $(yx)^n = y^nx^n$. Therefore,

$$y[x^n, y^{n-1}]y = yx^ny^n - y^nx^ny = y(xy)^n - (yx)^ny = 0,$$

and hence $[x^n, y^{n-1}] = 0$. Now, [3, Theorem] forces a contradiction that R is commutative.

Proof of Lemma 4.3. First, assume that $R = \mathrm{M}_\sigma(K)$. Let γ be a generating element of the multiplicative group of K, and put

$$x = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \gamma & 1 \\ 0 & \sigma(\gamma) \end{pmatrix}.$$

Suppose, to the contrary, that R satisfies ($\sharp$). Then there exist non-negative integers $r_1 \le r_2 \le r_3 \le r_4 \le r_5 \le r_6 \le r_7 \le r_8$ with $1 < r_8$, positive integers n_i $(1 \le i \le r_8)$, m_i $(r_2+1 \le i \le r_8)$, l_i $(r_4+1 \le i \le r_8)$, and words $w_i(x,y)$, $w_i'(x,y) \in \boldsymbol{W}$ $(1 \le i \le r_8)$ such that $(\sharp)_0$–$(\sharp)_8$ hold good. Since x and y are units in R, $(\sharp)_1$–$(\sharp)_8$ become

$$\begin{aligned}
&(xy)^{n_i} - y^{n_i}x^{n_i} = 0 && (1 \le i \le r_1),\\
&(yx)^{n_i} - x^{n_i}y^{n_i} = 0 && (r_1+1 \le i \le r_2),\\
&(x^{m_i}y^{m_i})^{n_i} - (y^{m_i}x^{m_i})^{n_i} = 0 && (r_2+1 \le i \le r_3),\\
&[x^{m_i}, y^{n_i}] = 0 && (r_3+1 \le i \le r_4),\\
&[x^{l_i}, (x^{m_i}y^{m_i})^{n_i}] = 0 && (r_4+1 \le i \le r_5),\\
&[x^{l_i}, (y^{m_i}x^{m_i})^{n_i}] = 0 && (r_5+1 \le i \le r_6),\\
&[y^{l_i}, (x^{m_i}y^{m_i})^{n_i}] = 0 && (r_6+1 \le i \le r_7),\\
&[y^{l_i}, (y^{m_i}x^{m_i})^{n_i}] = 0 && (r_7+1 \le i \le r_8).
\end{aligned}$$

Further, by making use of (1.1) in §1, we can easily see that for arbitrary positive integers l, m, n,

$$\begin{aligned}
&(xy)^n - y^nx^n = -\{(yx)^n - x^ny^n\}\\
&\quad = (\sigma(\gamma^n) - \gamma^n)(\sigma(\gamma^{n+1}) - \gamma^{n+1})(\gamma^2 - \sigma(\gamma^2))^{-1}e_{12},\\
&(x^my^m)^n - (y^mx^m)^n\\
&\quad = (\sigma(\gamma^{2mn}) - \gamma^{2mn})(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma^m) + \gamma^m)^{-1}(\gamma - \sigma(\gamma))^{-1}e_{12},\\
&\quad \text{provided} \quad \gamma^{2m} \notin K^\sigma,\\
&[x^m, y^n] = (\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma^n) - \gamma^n)(\gamma - \sigma(\gamma))^{-1}e_{12},\\
&[x^l, (x^my^m)^n] = -[y^l, (y^mx^m)^n]\\
&\quad = (\sigma(\gamma^{2mn}) - \gamma^{2mn})(\sigma(\gamma^l) - \gamma^l)(\sigma(\gamma^m) + \gamma^m)^{-1}(\gamma - \sigma(\gamma))^{-1}\gamma^m e_{12},\\
&\quad \text{provided} \quad \gamma^{2m} \notin K^\sigma,\\
&[x^l, (y^mx^m)^n] = -[y^l, (x^my^m)^n]\\
&\quad = (\sigma(\gamma^{2mn}) - \gamma^{2mn})(\sigma(\gamma^l) - \gamma^l)(\sigma(\gamma^m) + \gamma^m)^{-1}(\gamma - \sigma(\gamma))^{-1}\sigma(\gamma^m)e_{12},\\
&\quad \text{provided} \quad \gamma^{2m} \notin K^\sigma.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\gamma^{n_i(n_i+1)} \in K^\sigma && (1 \le i \le r_2),\\
&\gamma^{2m_in_i} \in K^\sigma && (r_2+1 \le i \le r_4),\\
&\gamma^{2l_im_in_i} \in K^\sigma && (r_4+1 \le i \le r_8).
\end{aligned}$$

Hence, by $(\sharp)_0$, we get $\gamma^2 \in K^\sigma$. But this is impossible.

Next, assume that R is of type e)[1]. Then $\mathrm{Ann}(D)$ contains a, b with $[a,b] \neq 0$. Put $x = 1 + a$ and $y = 1 + b$. Now, suppose, to the contrary, that R satisfies $(\sharp)$. Then there exist non-negative integes $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5 \leq r_6 \leq r_7 \leq r_8$ with $1 < r_8$, positive integers n_i $(1 \leq i \leq r_8)$, m_i $(r_2 + 1 \leq i \leq r_8)$, l_i $(r_4 + 1 \leq i \leq r_8)$, and words $w_i(x,y)$, $w_i'(x,y) \in \boldsymbol{W}$ $(1 \leq i \leq r_8)$ such that $(\sharp)_0$–$(\sharp)_8$ hold good. For each positive integers l, m, n, we can easily see that

$$\begin{aligned}
&(xy)^n - y^n x^n = (yx)^n - x^n y^n = \frac{n(n+1)}{2}[a,b],\\
&(x^m y^m)^n - (y^m x^m)^n = m^2 n[a,b],\\
&[x^m, y^n] = mn[a,b],\\
&[x^l, (x^m y^m)^n] = [x^l, (y^m x^m)^n]\\
&\quad = -[y^l, (x^m y^m)^n] = -[y^l, (y^m x^m)^n] = lmn[a,b].
\end{aligned}$$

Recalling here that $\langle a,b\rangle[a,b] = 0 = [a,b]\langle a,b\rangle$, we readily obtain

$$\begin{aligned}
\frac{n_i(n_i+1)}{2}[a,b] &= 0 \qquad (1 \leq i \leq r_2),\\
m_i^2 n_i[a,b] &= 0 \qquad (r_2 + 1 \leq i \leq r_4),\\
l_i m_i n_i[a,b] &= 0 \qquad (r_4 + 1 \leq i \leq r_8).
\end{aligned}$$

Hence $[a,b] = 0$ by $(\sharp)_0$, which is a contradiction.

Noting that $(xy)^n - x^n y^n = x\{(yx)^{n-1} - x^{n-1}y^{n-1}\}y$ for any positive integer n, we obtain the next as an immediate consequence of Theorem 4.1.

Corollary 4.4. *Let R be a ring with 1. Suppose that for each $x, y \in R$, there exist positive integers $r \leq s$ and $n_i > 1$ $(i = 1, \cdots, s)$ such that*

1) $(n_1(n_1-1)/2, \cdots, n_s(n_s-1)/2) = 1$,
2) $(xy)^{n_i} = x^{n_i} y^{n_i}$ $\quad (i = 1, \cdots, r)$,
3) $(yx)^{n_i} = y^{n_i} x^{n_i}$ $\quad (i = r, \cdots, s)$.

Then R is commutative.

Needless to say, Corollary 3.5 is a direct consequence of Corollary 4.4.

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(*Received December 17, 1992*)