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TTF-THEORY OVER A SEMIPERFECT RING

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In his paper [4], Storrer has studied hereditary torsion theories over a perfect ring. He proved that every hereditary torsion class is a TTF-class and determined the smallest element of the Gabriel topology corresponding to a hereditary torsion theory by using the notion of corresponding idempotents to modules. In this paper, we study hereditary 3-fold torsion theories over a semiperfect ring by means of the concept of corresponding idempotents. In this case, every module does not necessarily have a simple submodule. Thus we must admit 0 as a corresponding idempotent to a module M such that $Soc(M) = 0$. Let (T_1, T_2, T_3) be a 3-fold torsion theory. First we give equivalent conditions for which the 3-fold torsion theory over a right perfect ring R has length 2 (Theorem 1.7). Next we prove that if (T_1, T_2, T_3) is hereditary, then $\mathscr{F}(t_1)$ has the smallest element ReR, where t_1 is the cotorsion radical corresponding to a heretitary torsion theory (T_1, T_2) and e is an idempotent of R (Theorem 2.2). Also let e be an idempotent element of R with Re two-sided ideal of R. Finally, we give necessary and sufficient condition for e which is the idempotent corresponding to Re (Theorem 2.4).

Throughout this note, R means a semiperfect ring with Jacobson radical $J(R)$ and modules mean unitary left R-modules. We denote the injective hull (resp. socle) of a module M by $E(M)$ (resp. $Soc(M)$). Let e and f be idempotents of R. We call e is isomorphic to f if $Re \cong Rf$.

We consider a fixed representation of the identity 1 of R as sum of orthogonal primitive idempotents

$$1 = e_{11} + \cdots + e_{1k_1} + \cdots + e_{n1} + \cdots + e_{nk_n}$$

where e_{ij} is isomorphic to e_{kl} if and only if $i = k$. We also put $e_i = e_{i1}$. For each simple module S_i, there is a unique primitive idempotent e_i such that $e_i S_i \neq 0$. We shall say that e_i corresponds to S_i. Also for each module M, we put e the sum of e_i corresponds to the simple submodules of M. Again we say that e corresponds to M. If $Soc(M) = 0$, then we shall say that 0 is the corresponding idempotent to M.

As for terminologies and basic properties concerning of torsion theories and preradicals, we refer to [1] and [3]. For each preradical t, we denote the *t-torsion* (resp. *t-torsionfree*) class by $T(t)$ (resp. $F(t)$). Also we denote the left linear topology corresponding to a left exact preradical t by $\mathscr{L}(t)$.

1. We shall begin with useful lemmas.

Lemma 1.1. *Let I be a left ideal of R. Then $I = Rf + X$ for some idempotent f of R and submodule X of $J(R)$.*

Proof. Let $0 \to K \to P \xrightarrow{\alpha} R/I \to 0$ be the projective cover of R/I. We consider a diagram

$$\begin{array}{ccccccccc} & & & & & & R & & \\ & & & & & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{\alpha} & R/I & \longrightarrow & 0 \end{array}$$

where π is the canonical map. Then there exists an R-homomorphism $h: R \to P$ such that $\alpha \circ h = \pi$. Since $P = Im(h) + K$ and K is small in P, h is epic. Thus there exists an R-monomorphism $g: P \to R$ such that $h \circ g = 1_P$ and so $R = Im(g) + Ker(h)$. We put $Re = Im(g)$ and $Rf = Ker(h)$. Since $Rf \subset I$, we have the exact sequence

$$0 \longrightarrow Re \cap I \longrightarrow Re \xrightarrow{\pi'} R/I \longrightarrow 0$$

where π' is the restriction of π to Re. Also we have $Re \cap I = g(K)$, namely, $Re \cap I$ is small in Re. Thus $Re \cap I \subset J(R)$. Hence $I = Rf + X$ for some submodule X of $J(R)$.

Lemma 1.2. *Let S_i be a simple module with corresponding idempotent e_i and I a two-sided ideal of R. Then $IS_i = S_i$ if and only if $e_i \in I$.*

Proof. It is sufficient to prove the "only if" part. Suppose that $IS_i = S_i$. By Lemma 1.1, $I = Rf + X$, where f is an idempotent of R and $X \subseteq J(R)$. Then $IS_i = (Rf + X)S_i = RfS_i + XS_i = RfS_i = S_i$. Thus $fS_i \neq 0$ and so there exists a primitive idempotent f_i such that $f_iS_i \neq 0$. Since f_i is isomorphic to e_i and $f_i \in I$, $e_i \in I$.

Lemma 1.3. *Let M be a module with corresponding idempotent e. If $m \in Soc(M)$ and $eRm = 0$, then $m = 0$.*

Proof. Let m be a nonzero element of $Soc(M)$. Then $Rm = \sum_{\lambda\in\Lambda} \oplus S_\lambda$ for some family of simple modules $\{S_\lambda\}_{\lambda\in\Lambda}$. Thus $eRm \neq 0$. This is a contradiction.

Proposition 1.4. *Let M be a module with corresponding idempotent e and I a two-sided ideal of R. Then $ISoc(M) = Soc(M)$ if and only if $e \in I$.*

Proof. Suppose that $ISoc(M) = Soc(M)$. We may assume that $Soc(M) \neq 0$. Let S_i be a simple submodule of M with corresponding idempotent e_i. Then $Soc(M) = S_i \oplus X$ for some submodule X of M. Thus $Soc(M) = IS_i \oplus IX$. If $IS_i = 0$, then $IX = Soc(M)$. Hence $X \supset Soc(M)$ and so $S_i = 0$. Therefore $IS_i = S_i$. By Lemma 1.2, $e_i \in I$. Hence $e \in I$. Conversely suppose that $e \in I$. Let S_i be a simple submodule of M. Then $IS_i \neq 0$, namely, $IS_i = S_i$. Since $Soc(M) = \sum_{\lambda\in\Lambda} \oplus S_\lambda$ for some family of simple modules $\{S_\lambda\}_{\lambda\in\Lambda'}$, $ISoc(M) = \sum_{\lambda\in\Lambda} \oplus IS_\lambda = \sum_{\lambda\in\Lambda} \oplus S_\lambda = Soc(M)$.

Let (T_1, T_2, T_3) be the 3-fold torsion theory corresponding to an idempotent two-sided ideal I of R and M a module with corresponding idempotent e. Now we consider the following conditions:

(C_1): *If $M \in T_1$, then $e \in I$.*

(C_2): *If $e \in I$, then $M \in T_1$.*

The following examples show that both (C_1) and (C_2) do not hold in general.

Example 1.5. Let R be the ring of 2×2 upper triangular matrices over a field K, and set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}.$$

(1) Let (T_1, T_2, T_3) be the 3-fold torsion theory corresponding to an idempotent two-sided ideal $I = A$ of R. We put $M = A$. Then $M \in T_1$ and e_1 is the corresponding idempotent to M, namely, e_1 does not belong to I.

(2) Let (T_1, T_2, T_3) be the 3-fold torsion theory corresponding to an idempotent two-sided ideal $I = B$ of R. We put $M = A$. Then e_1 is the corresponding idempotent to M. However

$$IM = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \neq M,$$

that is, M is not in T_1.

Moreover we consider the following conditions:

(D_1): *If $A \in T_1$, then every simple submodule of A is in T_1.*
(D_2): *If every simple submodule of A is in T_1, then $A \in T_1$.*

Lemma 1.6. *If T_1 satisfies (D_1), then it satisfies (C_1).*

Proof. Let M be a module in T_1. If $Soc(M) = 0$, then the corresponding idempotent to M is 0. Thus we assume that $Soc(M) \neq 0$. Since T_1 satisfies (D_1), $Soc(M) \in T_1$ and $ISoc(M) = Soc(M)$. By Proposition 1.4, $e \in I$.

We call a class of modules *$\mathscr{E}$ stable* if it is closed under essential extensions. Also we shall say that a 3-fold torsion theory (T_1, T_2, T_3) has length 2 if $T_1 = T_3$.

Theorem 1.7. *Let R be a right perfect ring. Then the following conditions are equivalent.*
(1) *T_1 satisfies (D_1) and is stable.*
(2) *T_1 satisfies (D_1) and (D_2).*
(3) *T_1 satisfies (C_1) and (C_2).*
(4) *(T_1, T_2, T_3) is a 3-fold torsion theory with length 2.*

Proof. (1) $\Longrightarrow$ (2). Let M be a module with $Soc(M) \in T_1$. Since T_1 is stable and $Soc(M)$ is essential in M, M is in T_1. Hence T_1 satisfies (D_2).

(2) $\Longrightarrow$ (4). Let M be a module in T_1 and N its submodule. Since every simple submodule of N is in T_1, $Soc(N)$ is in T_1. Thus N is in T_1 by assumption. Hence T_1 is hereditary and so T_1 is a *TTF*-class by [4, Proposition 2.3]. Also since $Soc(M) = Soc(E(M))$, $E(M)$ is in T_1 and so T_1 is stable. Let (T_0, T_1, T_2, T_3) be a 4-fold torsion theory. Since T_1 is stable, T_0 is hereditary and so T_0 is a TTF-class. Thus (T_1, T_2, T_3) has length 2.

(4) $\Longrightarrow$ (3). Let M be a module in T_1 with corresponding idempotent e. Since T_1 is hereditary, $Soc(M)$ is in T_1. Thus $ISoc(M) = Soc(M)$ and so e is in I by Lemma 1.3. Thus T_1 satisfies (C_1). Let N be a module with corresponding idempotent f in I. Then f is the corresponding idempotent to $Soc(N)$. Since $e \in I$, $ISoc(N) = Soc(N)$. Also since T_1 is stable, $M \in T_1$. Thus T_1 satisfies (C_2).

(3) $\Longrightarrow$ (1). Let M be a module in T_1 with corresponding idempotent e and S its simple submodule with corresponding idempotent e_i. By assumption, $e \in I$ and so $e_i = e_i e \in I$. By (C_2), $S \in T_1$. Let N be a module in T_1 with corresponding idempotent e. By (C_1), $e \in I$. Let L be an essential extension of N. Since $Soc(L) = Soc(N)$, e is the corresponding idempotent to L. Thus by (C_2), L is in T_1. Thus T_1 is stable.

2. In this section, we shall treat hereditary 3-fold torsion theories. Let (T_1, T_2, T_3) be a 3-fold torsion theory and t_i the torsion functor corresponding to torsion theories (T_i, T_{i+1}) $(i = 1,2)$. Let $\mathscr{S} = \{S_1, S_2, \cdots, S_m\}$ be the complete set of non-isomorphic simple modules in T_2. We put $S = S_1 \oplus S_2 \oplus \cdots \oplus S_m$ and $E = E(S)$. For each module M, k_M denotes the largest one of those preradicals r such that $r(M) = 0$. As is well-known, k_M is a radical and is left exact if M is injective.

Proposition 2.1 ([2, Proposition 2]). *Let (T_1, T_2, T_3) be a hereditary 3-fold torsion theory. Then $t_1 = k_E$.*

Proof. Let M be a module in T_1. Then $Hom_R(M, E) = 0$, namely, $k_E(M) = M$. Thus $T_1 \subseteq T(k_E)$ and so $t_1 \leq k_E$. Suppose that $t_1 \neq k_E$. Then there exists a module N such that $k_E(N) = N$ and $t_1(N) \neq N$. Since $N/t_1(N) \in T_2$, $R\overline{m} \in T_2$ for some $\overline{m} = m + t_1(N)$. Also $R\overline{m}$ has a maximal submodule X. Since $R\overline{m}/X$ is simple and T_2 is closed under homomorphic images, $R\overline{m}/X \in T_2$. Thus $Hom_R(R\overline{m}/X, E) \neq 0$. Therefore $Hom_R(R\overline{m}, E) \neq 0$. Since E is injective, $Hom_R(N/t_1(N), E) \neq 0$. Hence $Hom_R(N, E) \neq 0$, namely, $k_E(N) \neq N$. This is a contradiction.

Theorem 2.2. *Let (T_1, T_2, T_3) be a hereditary 3-fold torsion theory. We put $e = e_1 + e_2 + \cdots + e_m$, where e_i is the corresponding idempotent of S_i $(i = 1, 2, \cdots, m)$. Then $\mathscr{L}(t_1) = \{{}_RJ \leq R \mid J \supseteq ReR\}$.*

Proof. As is well-known, $\mathscr{L}(t_1) = \{{}_RJ \leq R \mid R = t_1(R) + J\}$. Let

J be left ideal of R in $\mathscr{F}(t_1)$. We put $K = \{a \in R \mid aR \subseteq J\}$. Then K is the largest two-sided ideal of R contained in J. If K is in $\mathscr{F}(t_1)$, then $S = SR = t_1(R)S + KS = KS$. By Lemma 1.2, $e \in K$ and so $K \supseteq ReR$. Hence it is sufficient to prove that K is in $\mathscr{F}(t_1)$. Now let $0 \to N \to P \to S \to 0$ be the projective cover of S and let M be a module. If $Hom_R(P,M) = 0$, then $Hom_R(M,E) = 0$. In fact, we assume that $Hom_R(M,E) \neq 0$. Then $Rf(x) \cap S \neq 0$ for some $f \in Hom_R(M,E)$ and $x \in M$. Thus $Rf(x) \cap C \neq 0$ for some simple module C which is isomorphic to some S_i $(1 \leq i \leq m)$ namely, $Rf(x) \cap C = C$. Hence $Rf(x) \supseteq C$, and so $Hom_R(P,C) \subseteq Hom_R(P,Rf(x))$. Therefore we have the following commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & {}^{h}\swarrow & \downarrow{\scriptstyle g\neq 0} & & \\ Rx & \underset{\pi}{\longrightarrow} & Rf(x) & \longrightarrow & 0 \end{array}$$

where $g(\neq 0) \in Hom_R(P,Rf(x))$ and $\pi(rx) = rf(x)$ for all $r \in R$. Thus $Hom_R(P,M) \neq 0$. This is a contradiction. Next assume that R/K is not in T_1. Then $Hom_R(R/K,E) \neq 0$, namely, $Hom_R(P,R/K) \neq 0$. Since R/K is embeded in $\prod_{a\in R} R\bar{a}$, $Hom_R(P,R/K) \subseteq Hom_R(P,\prod_{a\in R} R\bar{a}) \cong \prod_{a\in R} Hom_R(P,R\bar{a})$ where $\bar{a} = a + J$. We show that $Hom_R(P,R\bar{a}) = 0$ for all $a \in R$. We assume that there exists $0 \neq \alpha \in Hom_R(P,R\bar{a})$. Since P is finitely generated, $\alpha(P)$ is finitely generated. Thus $\alpha(P)$ has a maximal submodule L. We put $S_n = \alpha(P)/L$. Since $\alpha(P)$ is in T_1, S_n is in T_1. Thus there exists an epimoprphism $\rho\circ\alpha\colon P \to S_n$, where ρ is the canonical map $\alpha(P) \to S_n$. Let $0 \to X \to Q \to S_n \to 0$ be the projective cover of S_n. Then $Q/(t_1(Q) + X)$ is a homomorphic image of $Q/t_1(Q) \in T_2$ and $Q/X \in T_1$. Since both T_1 and T_2 are closed under homomorphic images, $Q/(t_1(Q)+X) \in T_1 \cap T_2 = \{0\}$. Thus $Q = t_1(Q) + X$. Since X is small in Q, $Q = t_1(Q)$. Also Q is isomorphic to a direct summand of P. However P is the projective cover of S. Since $Hom_R(T_1,T_2) = 0$, Rf_n is in N, that is, $J(R)$ contains an idempotent element f_n of R. This is a contradiction. Hence $\prod_{a\in R} Hom_R(P,R\bar{a}) = 0$ and so $Hom_R(P,R/K) = 0$. Therefore R/K is in T_1, that is K is in $\mathscr{F}(t_1)$. Hence $K \supseteq ReR$. Conversely, let J be a left ideal of R with $J \supseteq ReR$. We assume that $Hom_R(R/J,E) \neq 0$. Then $Jx = 0$ for some $x(\neq 0) \in E$. Since $Rx \cap S \neq 0$, there exists an element a of R such that $ax(\neq 0) \in S$. Therefore $ReRax \subseteq ReRx \subseteq Jx = 0$, namely $eRax = 0$. By Lemma 1.3, $ax = 0$. This is a contradiction. Hence J is in $\mathscr{F}(t_1)$.

Corollary 2.3. *Let (T_1, T_2, T_3) be a hereditary 3-fold torsion theory. Then T_1 is a TTF-class.*

Let R be the ring of 2×2 upper triangular matrices over a field K. We put

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $I = Re_2$. Then I is a two-sided ideal of R and e_1 is the corresponding idempotent.

Let e be an idempotent of R with Re two-sided ideal of R. It is an interesting problem when e corresponds to Re.

Theorem 2.4. *Let (T_1, T_2, T_3, T_4) be a 4-fold torsion theory and $\{S_1, S_2, \cdots, S_m\}$ the complete set of non-isomorphic simple modules in T_3. We put $e = e_1 + e_2 + \cdots + e_m$, where e_i is the corresponding idempotent of S_i $(i = 1, 2, \cdots, m)$. Then e is the corresponding idempotent to Re if and only if every simple submodule of Re is in T_1 and there exists a simple submodule of Re which is isomorphic to each simple module in T_1.*

Proof. Let S' be a simple submodule of Re. Then $ReS' = S'$. By Theorem 2.2, $ReR = t_1(R)$. Thus S' is in T_1. Let S be a simple module in T_1. Then $ReS = S$ and so $e_iS \neq 0$ for some $e_i \in \{e_1, e_2, \cdots, e_m\}$. Hence $S \cong Re_i/J(R)e_i$. Since e corresponds to Re, there exists a simple submodule S_i of Re with $e_iS_i \neq 0$. Therefore $S \cong S_i$. Conversely, $eS_i = 0$, $ReS_i = S_i$ for all $i = 1, 2, \cdots, m$. Thus S_i is in T_1 for all $i = 1, 2, \cdots, m$. By assumption there exists a simple submodule S_i' of Re such that $S_i' \cong S_i$ for all $i = 1, 2, \cdots, m$. Therefore e corresponds to Re.

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