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ANTI-INTEGRAL EXTENSIONS AND UNRAMIFIED EXTENSIONS

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All rings considered in this paper are assumed to be commutative and have an identity.

We will study some properties of anti-integral extensions and unramified extensions.

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\Psi: R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$

Let $I_\alpha^* = \bigcap_{i=1}^d (R:_R \eta_i)$. For $f(X) \in R[X]$, let $c(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_\alpha = c(I_\alpha^* \varphi_\alpha(X))$, which is an ideal of R and contains I_α^*. The element α is called an *anti-integral element* of degree d over R if $\operatorname{Ker}\Psi = I_\alpha^* \varphi_\alpha(X) R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. In the case $K(\alpha) = K$, an anti-integral element α is the same as an anti-integral element (i.e., $R = R[\alpha] \cap R[1/\alpha]$) defined in [5]. The element α is called a *super-primitive element* of degree d over R if $J_\alpha \not\subset \wp$ for all primes $\wp$ of depth one.

1. Birational case. Let R be a Noetherian integral domain with the quotient field K and α be a non-zero element of K.

Write $A_1 = R[\alpha]$ $(= A)$, $A_2 = R[\alpha^{-1}]$, $I_\alpha = \{r \in R \mid r\alpha \in R\}$ and $J_\alpha = I_\alpha + \alpha I_\alpha$. We recall that α is an anti-integral element or that α is anti-integral over R if $A_1 \cap A_2 = R$. When α is anti-integral over R, $R[\alpha]$ is said to be an anti-integral extension of R.

From now on, we assume that α is anti-integral over R.

Let $\varphi_{A_i/R}: \operatorname{Spec}(A_i) \to \operatorname{Spec}(R)$ be maps which satisfy the condition $\varphi_{A_i/R}(P) = P \cap R$ for any $P \in \operatorname{Spec}(A_i)$. For any ideal N of R, we define

$$V(N) = \{P \in \operatorname{Spec}(R) \mid P \supset N\}.$$

We write $D(N)$ to denote the set $\mathrm{Spec}(R)\backslash V(N)$. Let $\Delta_{A/R} = \{\wp \in \mathrm{Spec}(R) \mid \wp A = A\}$ and $\Gamma_{J_\alpha} = \{\wp \in \mathrm{Spec}(R) \mid \wp + J_\alpha = R\}$.

Proposition 1. *The following results are satisfied.*

(1) $\Delta_{A/R} = V(I_\alpha) \cap \Gamma_{J_\alpha}$.

(2) $\mathrm{Im}\varphi_{A/R} = D(I_\alpha) \cup V(J_\alpha)$, *where* $\varphi = \varphi_{A/R}: \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is a map such that* $\varphi_{A/R}(P) = P \cap R$ *for any* $P \in \mathrm{Spec}(A)$.

(3) $\Delta_{A_1/R} \cap \Delta_{A_2/R} = \phi$.

(4) $\mathrm{Im}\varphi_{A_1/R} \cup \mathrm{Im}\varphi_{A_2/R} = \mathrm{Spec}(R)$.

Proof. (2) Suppose that $\wp \in D(I_\alpha) \cup V(J_\alpha)$. When $\wp \in V(J_\alpha)$, we claim that $\wp \in \mathrm{Im}\varphi$. By [2, Theorem 1.4(3)], we have that $A/\wp A \cong (R/\wp)[T]$ where T is an indeterminate. Thus $\wp A$ is a prime ideal of A and $\wp A \cap R = \wp$. Hence we have $\wp \in \mathrm{Im}\varphi$. In the case that $\wp \in D(I_\alpha)$, it follows that $A_\wp = R_\wp$. Write $\wp R_\wp \cap A = P$. Then $P \in \mathrm{Spec}(A)$ and $P \cap R = \wp$. Therefore we have that $\wp \in \mathrm{Im}\varphi$. Hence we proved that $D(I_\alpha) \cup V(J_\alpha) \subset \mathrm{Im}\varphi$. To prove the opposite inclusion, assume that $\wp \notin D(I_\alpha) \cup V(J_\alpha)$. We claim that $\wp \notin \mathrm{Im}\varphi$. Suppose that $\wp \in \mathrm{Im}\varphi$. Then there exists a prime ideal P of A such that $P \cap R = \wp$. Since $\wp \supset I_\alpha$ and $\wp \not\supset J_\alpha$, it follows that $\wp \not\supset \alpha I_\alpha$. Since $P \supset \alpha I_\alpha$ and $R \supset \alpha I_\alpha$, we have that $\wp \supset \alpha I_\alpha$. This is a contradiction.

(1) Let $\wp \in V(I_\alpha) \cap \Gamma_{J_\alpha}$. Suppose that $\wp \notin \Delta_{A/R}$. Since $\wp A \neq A$, there exists a prime ideal P of A such that $P \cap R \supset \wp$. Put $q = P \cap R$. Then $q \in \mathrm{Im}\varphi$. Since $\wp \supset I_\alpha$, we have that $q = P \cap R \supset I_\alpha + \alpha I_\alpha = J_\alpha$. Since $\wp \in \Gamma_{J_\alpha}$, it follows that $\wp + J_\alpha = R$. Therefore $q = R$. This is a contradiction. Hence $\wp \in \Delta_{A/R}$, that is, $V(I_\alpha) \cap \Gamma_{J_\alpha} \subset \Delta_{A/R}$.

To prove the opposite inclusion, assume that $\wp \in \Delta_{A/R}$. Suppose that $\wp \notin V(I_\alpha)$. Then $\wp \in \mathrm{Im}\varphi$ by (2). This contradicts the fact that $\wp \in \Delta_{A/R}$. Thus we proved that $\wp \in V(I_\alpha)$.

Next, suppose that $\wp \notin \Gamma_{J_\alpha}$, that is, $\wp + J_\alpha \neq R$. Then, there exists a prime ideal q of R such that $\wp + J_\alpha \subset q$. So $q \in \mathrm{Im}\varphi$ by (2). Therefore there exists a prime ideal P of A such that $P \cap R = q$ and so $P \supset qA \supset \wp A = A$. This is a contradiction. Hence $\wp \in \Gamma_{J_\alpha}$. This completes the proof of (1).

(3) By (1), we have that

$$\Delta_{A_1/R} \cap \Delta_{A_2/R} = (V(I_\alpha) \cap \Gamma_{J_\alpha}) \cap (V(I_{\alpha^{-1}}) \cap \Gamma_{J_{\alpha^{-1}}}).$$

Since $\alpha I_\alpha = I_{\alpha^{-1}}$, it follows that $J_{\alpha^{-1}} = J_\alpha$. Hence $\Gamma_{J_\alpha} = \Gamma_{J_{\alpha^{-1}}}$. Thus $\Delta_{A_1/R} \cap \Delta_{A_2/R} = (V(I_\alpha) \cap V(\alpha I_\alpha)) \cap \Gamma_{J_\alpha} = \phi$.

(4) By (2), we have that

$$\begin{aligned}\operatorname{Im}\varphi_{A_1/R} \cup \operatorname{Im}\varphi_{A_2/R} &= (D(I_\alpha) \cup V(J_\alpha)) \cup (D(I_{\alpha^{-1}}) \cup V(J_{\alpha^{-1}}))\\ &= D(I_\alpha) \cup D(I_{\alpha^{-1}}) \cup V(J_\alpha)\\ &= D(I_\alpha + I_{\alpha^{-1}}) \cup V(J_\alpha) = D(J_\alpha) \cup V(J_\alpha)\\ &= \operatorname{Spec}(R).\end{aligned}$$

Proposition 2. *It holds that* $\operatorname{Im}\varphi_{A_1/R} \cap \operatorname{Im}\varphi_{A_2/R} = D(\alpha I_\alpha^2) \cup V(J_\alpha)$.

Proof. By Proposition 1(2), we have that

$$\begin{aligned}\operatorname{Im}\varphi_{A_1/R} \cap \operatorname{Im}\varphi_{A_2/R} &= (D(I_\alpha) \cup V(J_\alpha)) \cap (D(I_{\alpha^{-1}}) \cup V(J_{\alpha^{-1}}))\\ &= (D(I_\alpha) \cap D(I_{\alpha^{-1}})) \cup (D(I_\alpha) \cap V(J_\alpha))\\ &\quad \cup ((V(J_\alpha) \cap D(I_{\alpha^{-1}})) \cup V(J_\alpha).\end{aligned}$$

It holds that

$$D(I_\alpha) \cap D(I_{\alpha^{-1}}) = D(I_\alpha I_{\alpha^{-1}}) = D(\alpha I_\alpha^2).$$

Since $J_\alpha \supset I_\alpha$, we have that $D(I_\alpha) \cap V(J_\alpha) = \phi$. Since $D(I_{\alpha^{-1}}) \cap V(J_\alpha) = \phi$, we get $\operatorname{Im}\varphi_{A_1/R} \cap \operatorname{Im}\varphi_{A_2/R} = D(\alpha I_\alpha^2) \cup V(J_\alpha)$.

Proposition 3. *It holds that* $\Delta_{A_1/R} \cup \Delta_{A_2/R} = V(\alpha I_\alpha^2) \cap \Gamma_{J_\alpha}$.

Proof. It follows that

$$\begin{aligned}\Delta_{A_1/R} \cup \Delta_{A_2/R} &= (V(I_\alpha) \cap \Gamma_{J_\alpha}) \cup (V(I_{\alpha^{-1}}) \cap \Gamma_{J_{\alpha^{-1}}})\\ &= (V(I_\alpha) \cup V(I_{\alpha^{-1}})) \cap \Gamma_{J_\alpha}\\ &= (V(I_\alpha) \cup V(\alpha I_\alpha)) \cap \Gamma_{J_\alpha} = V(\alpha I_\alpha^2) \cap \Gamma_{J_\alpha}.\end{aligned}$$

Proposition 4. *Let* $C = R[\alpha, \alpha^{-1}]$ *and* $\psi\colon \operatorname{Spec}(C) \to \operatorname{Spec}(R)$ *be a restriction mapping. Then we have that*

$$\begin{aligned}\operatorname{Im}\psi &= (D(I_\alpha) \cap D(I_{\alpha^{-1}})) \cup V(J_\alpha) = D(\alpha I_\alpha^2) \cup V(J_\alpha)\\ &= \operatorname{Im}\varphi_{A_1/R} \cap \operatorname{Im}\varphi_{A_2/R}.\end{aligned}$$

Proof. By Proposition 2, we enough prove that

$$\operatorname{Im}\psi = (D(I_\alpha) \cap D(I_{\alpha^{-1}})) \cup V(J_\alpha).$$

Let $\wp$ be an element of $D(I_\alpha)\cap D(I_{\alpha^{-1}})$. Then it follows that $I_\alpha \not\subset \wp$ and $\alpha I_\alpha \not\subset \wp$. Hence $C_\wp = R_\wp$. Put $\wp C_\wp \cap C = P$. Then $P\cap R = \wp$. Hence $\wp \in \operatorname{Im}\psi$. Hence we proved that $D(I_\alpha)\cap D(I_{\alpha^{-1}}) \subset \operatorname{Im}\psi$. Next, we claim that

$$V(J_\alpha)\subset \operatorname{Im}\psi.$$

Let $\wp \in V(J_\alpha)$. Since $\wp \supset J_\alpha$, it follows that $\wp \supset I_\alpha$ and $\wp \supset I_{\alpha^{-1}} = \alpha I_\alpha$. Suppose that $\alpha \in P = \wp A$. As the same as the proof of Proposition 1(2), we have that $P\neq A$ and $A/\wp A \cong (R/\wp)[T]$ where T is an indeterminate. Write $\alpha = a_0+a_1\alpha+\cdots+a_n\alpha^n$ (each $a_i\in\wp$). Since $P\cap R = \wp$ and $\wp\supset \alpha A\cap R$, we have that $\bar{\alpha} = \bar{0}$ where $\bar{\alpha}$ denotes the image of α in $A/\wp A$. On the other hand, we have that $\bar{\alpha} = T$. This is a contradiction. Therefore $P \not\ni \alpha$. Thus PC is a prime ideal of C and $PC\cap R = \wp$. Hence $\wp\in\operatorname{Im}\psi$. Thus we proved that $(D(I_\alpha)\cap D(I_{\alpha^{-1}}))\cup V(J_\alpha)\subset \operatorname{Im}\psi$.

To prove the opposite inclusion, assume $\wp\in\operatorname{Im}\psi$. Suppose that $\wp\notin V(J_\alpha)$. By

$$\operatorname{Spec}(C)\longrightarrow \operatorname{Spec}(A_1)\longrightarrow \operatorname{Spec}(R),$$

we have that $\wp\in \operatorname{Im}\varphi_{A_1/R} = D(I_\alpha)\cup V(J_\alpha)$. On the other hand, by

$$\operatorname{Spec}(C)\longrightarrow \operatorname{Spec}(A_2)\longrightarrow \operatorname{Spec}(R),$$

we have that $\wp\in\operatorname{Im}\varphi_{A_2/R} = D(I_{\alpha^{-1}})\cup V(J_{\alpha^{-1}}) = D(I_{\alpha^{-1}})\cup V(J_\alpha)$. Since $\wp\notin V(J_\alpha)$, we get that $\wp\in D(I_\alpha)\cap D(I_{\alpha^{-1}})$. Thus the proof is complete.

Theorem 5. *Assume that α is an anti-integral element over R. Write $C = R[\alpha,\alpha^{-1}]$, $A_1 = R[\alpha]$ and $A_2 = R[\alpha^{-1}]$. It holds that $\Delta_{C/R} = \Delta_{A_1/R}\cup\Delta_{A_2/R} = V(\alpha I_\alpha^2)\cap \Gamma_{J_\alpha}$.*

Proof. It is sufficient to prove the first equality. We recall that $\varphi_i\colon \operatorname{Spec}(A_i)\to\operatorname{Spec}(R)$ are restriction mappings. Evidently, $\Delta_{C/R}\supset \Delta_{A_1/R}\cup\Delta_{A_2/R}$. Let $\wp\in\Delta_{C/R}$. Suppose that $\wp\notin \Delta_{A_1/R}\cup\Delta_{A_2/R}$. By Proposition 3, we have that $\alpha I_\alpha^2 = (I_\alpha)(\alpha I_\alpha)\not\subset\wp$ or $\wp + J_\alpha \neq R$.

Case 1). $\wp\not\supset I_\alpha$ and $\wp\not\supset \alpha I_\alpha$. Then $C_\wp = R_\wp$. Since $\wp C = C$, we have that $\wp R_\wp = R_\wp$. This is a contradiction.

Case 2). $\wp + J_\alpha\neq R$.

There exists a prime ideal q of R such that $q\supset \wp + J_\alpha$. Then qA is a prime ideal of A and $\alpha\notin qA$ from the proof of Proposition 4. We recall that $C = A[\alpha^{-1}]$. Therefore $(qA)C = qC$ is a prime ideal of C and $C/qC \cong (R/q)[T, T^{-1}]$. But, since $\wp \in \Delta_{C/R}$, we have that $C = \wp C \subset qC \neq C$. This is a contradiction. Hence $\wp \in \Delta_{A_1/R} \cup \Delta_{A_2/R}$.

Remark. $\wp C = C$ $(\wp \in \mathrm{Spec}(R))$ if and only if $\wp + J_\alpha = R$ and $I_\alpha \subset \wp$ or $I_{\alpha^{-1}} \subset \wp$.

2. High degree extensions of rings. Let R be a Noetherian domain with the quotient field K and L be a finite algebraic extension field of K such that $[L: K] = d$. Let $\alpha \in L$ and $A = R[\alpha]$. Let L be the quotient field of A.

When $A = R[\alpha]$ is an unramified extension of R, we call α an unramified element over R. Let S be an R-algebra of finite type. We recall that S is an unramified over R if and only if the differential module $\Omega_R(S) = (0)$ (cf. [1]).

Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d \in K[X]$ be the monic minimal polynomial of α over K.

Write $I_\alpha^* = \bigcap_1^d I_{\eta_i}$, $J_\alpha = c(\varphi_\alpha(X) I_\alpha^*)$, $\tilde{I}_\alpha = I_\alpha^*(1, \eta_1, \ldots, \eta_{d-1})$ and $\tilde{J}_\alpha = \eta_d I_\alpha^*$, where $c(\varphi_\alpha(X) I_\alpha^*)$ denotes the ideal generated by the coefficients of all polynomials in $\varphi_\alpha(X) I_\alpha^*$. Note that $J_\alpha = \tilde{I}_\alpha + \tilde{J}_\alpha$.

Let $\psi: R[X] \to R[\alpha]$ be the canonical homomorphism such that $\psi(X) = \alpha$ and $\psi(a) = a$ for any $a \in R$. α is called an *anti-integral element* of degree d over R if $\mathrm{Ker}\,\psi = I_\alpha^* \varphi_\alpha(X) R[X]$. When α is an anti-integral element, we say that $R[\alpha]$ is an *anti-integral extension* of R.

In the rest of this section, we assume that α is an anti-integral element of degree d over R.

Proposition 6. *$\tilde{I}_\alpha = R$ if and only if the following two conditions hold.*

(1) *A is a flat R-module.*

(2) *$\varphi: \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is surjective.*

Proof. ($\Longleftarrow$) Suppose that $\tilde{I}_\alpha \neq R$. Then there exists a prime ideal $\wp$ of R such that $\wp \supset \tilde{I}_\alpha$. Since φ is surjective, there exists a prime ideal P of A such that $P \cap R = \wp$. Thus $\tilde{I}_\alpha A \subset P$.

We claim that $\tilde{I}_\alpha A = A$. Let a be any element of I_α^*. Since $\varphi_\alpha(\alpha) = 0$, we have that $a\alpha^d + (a\eta_1)\alpha^{d-1} + \cdots + (a\eta_{d-1})\alpha = -a\eta_d \in \tilde{I}_\alpha A$. Hence

$\tilde{I}_\alpha A \supset \tilde{I}_\alpha A + I_\alpha^* \eta_d \supset \tilde{I}_\alpha + \tilde{J}_\alpha = J_\alpha$. Since A is R-flat, we have that $J_\alpha = R$ by [4, Theorem 1.8 or Theorem 3.4] and so $\tilde{I}_\alpha A \supset J_\alpha A = A$. Therefore $\tilde{I}_\alpha A = A$. Hence $P = A$. This is a contradiction. Thus $\tilde{I}_\alpha = R$.

($\Longrightarrow$) If $\tilde{I}_\alpha = R$, then $J_\alpha = R$. By [4, Theorem 1.8 or Theorem 3.4], we have that A is R-flat. Now, by the following Theorem 7(1), $\mathrm{Im}\varphi = D(\tilde{I}_\alpha) \cup V(J_\alpha)$, we have that $\mathrm{Im}\varphi = \mathrm{Spec}(R)$. Hence φ is surjective.

Theorem 7. *Assume that α is an anti-integral element of degree d over R. Then we have the following results.*

(1) $\mathrm{Im}\varphi = D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha) = D(\tilde{I}_\alpha) \cup V(J_\alpha)$.

(2) $\Delta_{A/R} = V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha} = V(\tilde{I}_\alpha) \cap \Gamma_{J_\alpha}$.

Proof. First, we shall prove (1). Let $\wp \in D(\tilde{I}_\alpha)$ and $\Psi: R[X] \to A = R[\alpha]$ be the canonical map. Then $A_\wp = R_\wp[X]/(\mathrm{Ker}\,\Psi)R_\wp[X]$ and $A_\wp/\wp A_\wp \cong (R_\wp/\wp R_\wp)[X]/(\overline{\mathrm{Ker}\,\Psi})(R_\wp/\wp R_\wp)[X]$, where $\overline{\mathrm{Ker}\,\Psi}$ denotes the image of $\mathrm{Ker}\,\Psi$ in $(R_\wp/\wp R_\wp)[X]$. Since $\wp \not\supset \tilde{I}_\alpha$, $\overline{(\mathrm{Ker}\,\Psi)}$ is not a constant. Thus $\wp A_\wp \neq A_\wp$. Let $\tilde{P}$ be a prime divisor of $\wp A_\wp$. Put $\tilde{P} \cap A = P$. Then $\tilde{P} \cap R_\wp = \wp R_\wp$ and so $P \cap R = \wp$. Hence $\wp \in \mathrm{Im}\varphi$.

Next, let $\wp \supset \tilde{J}_\alpha = \eta_d I_\alpha^*$. We recall that if $\wp \not\supset \tilde{I}_\alpha$ then $\wp \in \mathrm{Im}\varphi$. So, we suppose that $\wp \supset \tilde{I}_\alpha$. Then $\wp \supset J_\alpha$. It follows that $A/\wp A \cong (R/\wp)[T]$, where T is an indeterminate. Since $\wp R[X] \supset \mathrm{Ker}\,\Psi$, $P = \wp A \in \mathrm{Spec}(A)$ and $P \cap R = \wp$, we have that $\wp \in \mathrm{Im}\varphi$. Thus we proved that $D(\tilde{I}_\alpha) \cup V(J_\alpha) \subset D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha) \subset \mathrm{Im}\varphi$. To prove the opposite inclusion, assume that $\wp \notin D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha)$. We claim that $\wp \notin \mathrm{Im}\varphi$. Suppose that $\wp \in \mathrm{Im}\varphi$. There exists a prime ideal P of A such that $P \cap R = \wp$. Since $\wp \not\supset \tilde{J}_\alpha = \eta_d I_\alpha^*$, there exists some element $a \in I_\alpha^*$ such that $a\eta_d \notin \wp$. Since $\wp \supset \tilde{I}_\alpha \ni a, a\eta_1, \dots, a\eta_{d-1}$, it follows that $P \ni (a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha) = -a\eta_d$. So $\wp \ni -a\eta_d$. This is a contradiction. We have that $\wp \notin \mathrm{Im}\varphi$. Hence we have that $\mathrm{Im}\varphi = D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha)$.

We claim that $D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha) = D(\tilde{I}_\alpha) \cup V(J_\alpha)$. Clearly, it follows that $D(\tilde{I}_\alpha) \cup V(\tilde{J}_\alpha) \supset D(\tilde{I}_\alpha) \cup V(J_\alpha)$. To prove the opposite inclusion, assume that $\wp \notin D(\tilde{I}_\alpha) \cup V(J_\alpha)$. Then $\wp \supset \tilde{I}_\alpha$ and $\wp \not\supset J_\alpha$. Suppose that $\wp \in V(\tilde{J}_\alpha) \cup D(\tilde{I}_\alpha)$. Then $\wp \supset \tilde{J}_\alpha + \tilde{I}_\alpha = J_\alpha$. This is a contradiction. Therefore we have that $\wp \notin V(\tilde{J}_\alpha) \cup D(\tilde{I}_\alpha)$. Thus we complete the proof of the claim.

(2). Let $\wp \in V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Suppose that $\wp \notin \Delta_{A/R}$. Then there exists a prime ideal P of A such that $P \supset \wp A$. Put $P \cap R = q$. Then we have that $q \supset \wp$ and $q \in \mathrm{Im}\varphi$. Since $a, a\eta_1, \dots, a\eta_{d-1} \in \tilde{I}_\alpha \subset \wp \subset P$, for any

$a \in I_\alpha^*$, we have that

$$a\alpha^d + a\eta_1\alpha^{d-1} + \cdots + a\eta_{d-1}\alpha = -a\eta_d \in P \cap R = q.$$

Consequently, we have that $\tilde{J}_\alpha \subset q$. Since $\wp \in \Gamma_{\tilde{J}_\alpha}$, it follows that $q \supset \tilde{J}_\alpha + \wp = R$. This is a contradiction. Therefore we proved that $V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha} \subset \Delta_{A/R}$.

Conversely, we shall prove that $\Delta_{A/R} \subset V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Let $\wp \notin V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Then $\wp \not\supset \tilde{I}_\alpha$ or $\wp + \tilde{J}_\alpha \neq R$. If $\wp \not\supset \tilde{I}_\alpha$, then $\wp \in \operatorname{Im}\varphi$ by Theorem 7(1). Therefore there exists a prime ideal P of A such that $P \cap R = \wp$. Suppose that $\wp \in \Delta_{A/R}$. Then $P \supset \wp A = A$. This is a contradiction. Thus $\wp \notin \Delta_{A/R}$.

Next, if $\wp + \tilde{J}_\alpha \neq R$, then there exists a prime ideal q of R such that $\wp + \tilde{J}_\alpha \subset q$. Since $q \in \operatorname{Im}\varphi$ by (1), it follows that $qA \neq A$. Since $\wp \subset q$, we have that $\wp A \neq A$. So we get $\wp \notin \Delta_{A/R}$. Hence we complete the proof of $\Delta_{A/R} = V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. At last, we shall prove that $V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha} = V(\tilde{I}_\alpha) \cap \Gamma_{J_\alpha}$. Clearly, we have that $V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha} \subset V(\tilde{I}_\alpha) \cap \Gamma_{J_\alpha}$. So we shall prove that $V(\tilde{I}_\alpha) \cap \Gamma_{J_\alpha} \subset V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Let $\wp \in V(\tilde{I}_\alpha) \cap \Gamma_{J_\alpha}$. Suppose that $\wp \notin V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Then $\wp + \tilde{J}_\alpha \neq R$. Also there exists a prime ideal q of R such that $q \supset \wp + \tilde{J}_\alpha$. Since $q \supset J_\alpha$ and $\wp + J_\alpha = R$, we have that $q \supset \wp + J_\alpha = R$. This is a contradiction. Hence $\wp \in V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. This completes the proof.

We consider unramified extensions of rings.

Theorem 8. *Let α be an anti-integral element of degree d over R. Put $A = R[\alpha]$. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the monic minimal polynomial of α over K. Put $I_\alpha^* = \bigcap_{i=1}^d I_{\eta_i}$. Then the following conditions are equivalent.*

(1) $\Omega_R(A) = (0)$.

(2) $I_\alpha^* \varphi'_\alpha(\alpha) A = A$.

Proof. Let $0 \to N \to R[X] \to A = R[X]/N \to 0$ be an exact sequence. From this exact sequence, we get the following exact sequence:

$$N/N^2 \longrightarrow A \otimes_{R[X]} R[X]dX \cong A \longrightarrow \Omega_R(A) \longrightarrow 0,$$

where $\rho\colon N/N^2 \to A \otimes_{R[X]} R[X]dX$ is a map such that $\rho(x \ mod \ N^2) = d_{R[X]/R}x \otimes 1$ for $x \in N$. Note that $A \otimes_{R[X]} \Omega_R(R[X]) \cong A$. Since $N =$

$I_\alpha^* \varphi_\alpha(X)R[X]$, it follows that $\mathrm{Im}\,\rho = I_\alpha^* \varphi'_\alpha(\alpha)A$. Hence $\Omega_R(A) = (0)$ if and only if $I_\alpha^* \varphi'_\alpha(\alpha)A = A$.

Example. Let k be a field with characteristic $\neq 2$ and let $R = k[X^2, 1/X^2]$ be a subring of $k[X, 1/X]$. Put $A = R[X]$. Then $K = k(X^2)$ is the quotient field of R and $L = k(X)$ is the quotient field of A. Put $\alpha = X$. Then $\varphi_\alpha(Y) = Y^2 - X^2 \in R[Y]$ is the monic minimal polynomial of α over K. Since $\varphi'_\alpha(Y) = 2Y$, $\varphi'_\alpha(X) = 2X$ is a unit of $A = k[X, 1/X]$. Hence A is an unramified over R (cf. [3, Theorem 10.0.1]).

Remark. If A is unramified over R, then $\tilde{I}_\alpha A = I_\alpha^* A = A$. In fact, since $I_\alpha^* \varphi'_\alpha(\alpha)A \subset I_\alpha^* A$ and $I_\alpha^* \subset \tilde{I}_\alpha$, it follows that $I_\alpha^* A = \tilde{I}_\alpha A = A$.

Theorem 9. *Under the assumption in Theorem 8, if $\Omega_R(A) = (0)$, then A is a flat R-module.*

Proof. Suppose that A is not a flat R-module. Then there exists a prime ideal $\wp$ of R such that $\wp \supset c(I_\alpha^* \varphi_\alpha(X))$ by [4, Theorem 1.8]. Since $\wp R[X] \supset I_\alpha^* \varphi_\alpha(X)R[X] = N$, we have that $A/\wp A \cong (R/\wp)[X]$. Hence $\wp A \in \mathrm{Spec}(A)$. From an exact sequence $0 \to \wp A \to A \to A/\wp A \to 0$, we have the following exact sequence. $\wp A/\wp^2 A \to \Omega_R(A) \otimes_A A/\wp A \to \Omega_{R/\wp}(A/\wp A) \to 0$. But $\Omega_R(A) = (0)$, and so $\Omega_{R/\wp}(A/\wp A) = (0)$. Also, since $A/\wp A \cong (R/\wp)[X]$, we have that $\Omega_{R/\wp}(A/\wp A) = ((R/\wp)[X])dX \neq (0)$. This is a contradiction. Hence A is a flat R-module.

Remark. Although A is a flat R-module, A is not necessarily unramified over R. For example, let $R = k[X^2]$ be a subring of $k[X]$ where k denotes a field with characteristic $\neq 2$. Set $A = R[X] = k[X]$. Since $\Omega_R(A) = A/XA \neq (0)$, we have that A is not an unramified extension over R.

Proposition 10. *Under the assumption in Theorem 8, it follows that $V(\mathrm{Ann}_R(\Omega_R(A))) \supset V(J_\alpha)$, where $\mathrm{Ann}_R(\Omega_R(A))$ denotes the annihilator ideal of $\Omega_R(A)$.*

Proof. Assume that $\wp \notin V(\mathrm{Ann}_R(\Omega_R(A)))$. So there exists an element $s \in \mathrm{Ann}_R(\Omega_R(A))$ such that $s \notin \wp$. Therefore $s\Omega_R(A) = (0)$ and so $\Omega_{R_\wp}(A_\wp) = (0)$. Note that $A_\wp = R_\wp[\alpha]$. Using Theorem 9, it follows that

$A_\wp$ is a flat $R_\wp$-module. From this fact and [4, Theorem 1.8], we obtain that $\wp \not\supset J_\alpha$. Therefore $\wp \notin V(J_\alpha)$.

Proposition 11. *The element α is an anti-integral element of degree d over R if and only if α is an anti-integral element of degree d over $R_\wp$ for any $\wp \in \mathrm{Spec}(R)$.*

Proof. Let $0 \to N \to R[X] \to A = R[\alpha] \to 0$ be an exact sequence.

($\Longrightarrow$) N is an ideal generated by some polynomials of degree d, where $d = [L : K]$. Also,

$$0 \longrightarrow N_\wp \longrightarrow R_\wp[X] \longrightarrow A_\wp = R_\wp[\alpha] \longrightarrow 0$$

is an exact sequence. Hence $N_\wp$ is also an ideal generated by some polynomials of degree d. Thus α is an anti-integral element of degree d over $R_\wp$.

($\Longleftarrow$) Put $B = J_\alpha \varphi_\alpha(X) R[X]$. Then $B \subset N$. And $N = B$ if and only if α is anti-integral over R. By the assumption, we have that $N_\wp = B_\wp$ for any $\wp \in \mathrm{Spec}(R)$. Hence we get $N = B$.

Theorem 12. *In special case, if α be an element of K then the following* (i)-(iv) *are equivalent.*

(i) $I_\alpha A = A$.

(ii) *A is a flat R-module.*

(iii) $J_\alpha A = A$.

(iv) *A is an unramified extension of R.*

Proof. Using the fact that $I_\alpha A = J_\alpha A$ and [5, Proposition 2.7], the equivalence of (i), (ii) and (iii) are already proved. Also, it proved that (iv) implies (ii) from Theorem 9. Now, we claim that (ii) implies (iv). Since $d = 1$, we have that $J_\alpha \varphi'_\alpha(\alpha) A = J_\alpha A$ where $\varphi_\alpha(X) = X - \alpha$. Since A is a flat R-module, it follows that $J_\alpha = R$. So we get that $J_\alpha A = A$. Therefore we have that $I_\alpha \varphi'_\alpha(\alpha) A = J_\alpha \varphi'_\alpha(\alpha) A = A$. Using Theorem 8, we see that $\Omega_R(A) = (0)$. Hence A is an unramified extension of R.

Remark. In Theorem 12, a simple birational anti-integral extension is flat if and only if it is an unramified extension, but, in case of a non-birational extension, flatness is not equivalent to unramifiedness (cf. Remark of Theorem 9).

Remark. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the monic minimal polynomial of α over K, where $d = [L:K] > 1$. Then $I_\alpha^* \varphi'_\alpha(\alpha)A \subset \tilde{I}_\alpha A$. For, $I_\alpha^*\varphi'_\alpha(\alpha)A = I_\alpha^*(d\alpha^{d-1} + (d-1)\eta_1\alpha^{d-2} + \cdots + \eta_{d-1})A \subset I_\alpha^*(1, \eta_1, \ldots, \eta_{d-1})A \subset \tilde{I}_\alpha A$. From Theorem 8, if $\Omega_R(A) = (0)$, then $\tilde{I}_\alpha A = A$. As an example of $I_\alpha^*\varphi'_\alpha(\alpha) \neq \tilde{I}_\alpha A$, we give the following example. Put $\varphi_\alpha(X) = X^2 - a \in R[X]$ where a is not a unit of R. Then $I_\alpha^*\varphi'_\alpha(\alpha)A = 2\alpha A \neq A$ and $\tilde{I}_\alpha A = A$ (We assume that 2 is not a unit of A).

Propositipn 13. *Let $A = R[\alpha]$ be an unramified extension of R and let φ: Spec$(A) \to$ Spec(R) be a restriction map. Then φ is surjective if and only if $\tilde{I}_\alpha = R$.*

Proof. ($\Longleftarrow$) From Theorem 7, we know that $\mathrm{Im}\varphi = D(\tilde{I}_\alpha) \cup V(J_\alpha)$. So $\mathrm{Im}\varphi = \mathrm{Spec}(R)$ by assumption. Hence φ is surjective.

($\Longrightarrow$) Suppose that $\tilde{I}_\alpha \neq R$. There exists a prime ideal $\wp$ of R such that $\tilde{I}_\alpha \subset \wp$. Since φ is surjective, there exists a prime ideal P of A such that $P \cap R = \wp$. Thus $\tilde{I}_\alpha A \subset P$. But A is an unramified extension over R, $\tilde{I}_\alpha A = A$ from Theorem 8 and Remark. Hence $A = P$, contradicting. Therefore $\tilde{I}_\alpha = R$.

Remark. When φ is surjective, $NA = A$ if and only if $N = R$ for an ideal N of R.

Lemma 14. *Let A be a ring extension of R and N be an ideal of R. Let $N = q_1 \cap q_2 \cap \ldots \cap q_n$ be a primary decomposition of N, where $\sqrt{q_i} = \wp_i$ for $1 \le i \le n$. Then $NA = A$ if and only if $\wp_i A = A$ for $1 \le i \le n$.*

Proof. ($\Longrightarrow$) Since $NA \subset \wp_i A$, we have $\wp_i A = A$.

($\Longleftarrow$) From $\wp_i A = A$, we have that $1 = \sum a_{ij}\alpha_{ij}$ ($a_{ij} \in \wp_i$, $\alpha_{ij} \in A$). Clearly, it can be assumed that $a_{ij} \in q_i$. Then $1 = \prod_i(\sum a_{ij}\alpha_{ij}) \in NA$ and so $NA = A$.

Theorem 15. *Let α be an anti-integral element of degree d over R. A is a flat R-module if and only if $\tilde{I}_\alpha A = A$.*

Proof. Let $\tilde{I}_\alpha = q_1 \cap q_2 \cap \cdots \cap q_n$ ($\sqrt{q_i} = \wp_i$) be a primary decomposition of $\tilde{I}_\alpha$.

($\Longrightarrow$) From Theorem 7(2), $\Delta_{A/R} = V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$, where $\tilde{J}_\alpha = I_\alpha^*\eta_d$

and $J_\alpha = \tilde{I}_\alpha + \tilde{J}_\alpha$. Since A is R-flat, it follows that $J_\alpha = R$ from [4, Theorem 1.8 or Proposition 2.6]. Since $\tilde{I}_\alpha \subset \wp_i$, we get that $\wp_i \in V(\tilde{I}_\alpha)$ and $\wp_i \in \Gamma_{\tilde{J}_\alpha}$. Therefore $\wp_i \in V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha} = \Delta_{A/R}$ and so $\wp_i A = A$. From Lemma 14, we have $\tilde{I}_\alpha A = A$.

($\Longleftarrow$) Since $\tilde{I}_\alpha A = A$, it follows that $\wp_i A = A$ from Proposition 14. So $\wp_i \in \Delta_{A/R} = V(\tilde{I}_\alpha) \cap \Gamma_{\tilde{J}_\alpha}$. Since $\wp_i + \tilde{J}_\alpha = R$, we have that $J_\alpha = \tilde{J}_\alpha + \tilde{I}_\alpha = R$. Hence A is a flat R-module.

We recall that α is an unramified element over R if $R[\alpha]$ is an unramified extension of R.

Theorem 16. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be unramified elements over R. Then $R[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is an unramified extension over R.*

Proof. Clearly, it can be assumed that $n = 2$. Let $B = R[\alpha_1]$, $C = R[\alpha_2]$ and $A = B[\alpha_2] = R[\alpha_1, \alpha_2]$. Then we have the following exact sequence:

$$\Omega_R(B) \otimes_B A \longrightarrow \Omega_R(A) \longrightarrow \Omega_B(A) \longrightarrow 0.$$

Since B and C are unramified extensions over R, we see that $\Omega_R(B) = (0)$ and $\Omega_R(C) = (0)$. So it suffices to show that $\Omega_B(A) = (0)$. Since C is an unramified over R, it follows that $B \otimes_R C$ is an unramified over B, that is, $\Omega_B(B \otimes_R C) = (0)$. From the exact sequence $0 \to N \to B \otimes_R C \to A \to 0$, and the fact that $\Omega_B(A) = \Omega_B(B \otimes_R C)/\Omega(N)$ where $\Omega(N)$ denotes the submodule generated by $\{da \mid a \in N\}$, we get that $\Omega_B(A) = (0)$. Thus we have that $\Omega_R(A) = (0)$. The proof is complete.

Remark. Let α be an element of the quotient field of R. Let $A = R[\alpha]$ be an anti-integral extension of R. Then A is a flat R-module if and only if $A_\wp = R_\wp$ or $A_\wp = R_\wp[1/a] (\forall \wp \in \mathrm{Spec}(R))$ for some element $a \in R$.

Proof. ($\Longleftarrow$) It is trivial.

($\Longrightarrow$) In the case $\wp \not\supset I_\alpha$, it follows that $A_\wp = R_\wp$. In the remaining case, since $J_\alpha = I_\alpha + \alpha I_\alpha = R$, we have that $\wp \not\supset \alpha I_\alpha$. And so there exists an element a of I_α such that $a\alpha \notin \wp$. Put $b = a\alpha$. Then $A_\wp = R_\wp[\alpha] = R_\wp[1/a]$.

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