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AN INTEGRAL INVOLVING THE ERROR TERM OF THE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION IN THE CRITICAL STRIP

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1. Introduction. Let $s = \sigma + it$ $(1/2 \le \sigma < 1,\ t \ge 0)$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and define $E(T)$ by

$$E(T) = \int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt - T\log T - (2\gamma - 1 - \log 2\pi)T$$

for $T \ge 2$, where γ is the Euler constant. The asymptotic formula

$$\int_2^T E(t)^2\, dt = c_0 T^{\frac{3}{2}} + O(T^{\frac{5}{4}} \log^2 T) \qquad \Big(c_0 = \frac{2\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)}\Big)$$

was proved by Heath-Brown [1] who used Atkinson's formula, and recently the error term was improved to $O(T\log^4 T)$ by Preissmann [7] and, independently, by Ivić [3]. Ivić [2] proved another asymptotic formula involving the square of $E(t)$:

$$\int_0^T E(t)^2 \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt = c_0\Big(\log\Big(\frac{T}{2\pi}\Big) + 2\gamma - \frac{2}{3}\Big)T^{\frac{3}{2}} + O(T\log^6 T), \tag{1}$$

and the error term was improved to $O(T\log^5 T)$ by Ivić [3].

We define $E_\sigma(T)$ by

$$E_\sigma(T) = \int_0^T |\zeta(\sigma + it)|^2\, dt - c_1 T - c_2 T^{2-2\sigma} \qquad \Big(\frac{1}{2} < \sigma < 1\Big), \tag{2}$$

where

$$c_1 = \zeta(2\sigma) \quad \text{and} \quad c_2 = (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}.$$

Matsumoto [4] proved that

$$\int_2^T E_\sigma(t)^2\, dt = c_3 T^{\frac{5}{2}-2\sigma} + F_\sigma(T) \qquad \Big(\frac{1}{2} < \sigma < \frac{3}{4}\Big) \tag{3}$$

with

$$F_\sigma(T) = O(T^{\frac{7}{4}-\sigma}\log T)$$

and

$$c_3 = \frac{2}{5-4\sigma}(2\pi)^{2\sigma-\frac{3}{2}}\frac{\zeta^2(3/2)}{\zeta(3)}\zeta\Big(\frac{5}{2}-2\sigma\Big)\zeta\Big(\frac{1}{2}+2\sigma\Big).$$

Recently, Matsumoto and Meurman [5] improved this to

$$F_\sigma(T) = O(T). \tag{4}$$

Moreover, they [6] proved that

$$\int_2^T E_{\frac{3}{4}}(t)^2\,dt = c_4 T\log T + O(T(\log T)^{\frac{1}{2}}) \tag{5}$$

and

$$\int_2^T E_\sigma(t)^2\,dt = O(T) \qquad \Big(\frac{3}{4}<\sigma<1\Big), \tag{6}$$

where

$$c_4 = \frac{\zeta^2(3/2)\zeta(2)}{\zeta(3)}.$$

It should be stressed that their results imply that the line $\sigma = 3/4$ has a kind of critical property for the Riemann zeta-function, or at least for the function $E_\sigma(t)$.

The aim of this note is to evaluate the analogue of (1) in the critical strip. We prove the following:

Theorem. *We have*

$$\int_2^T E_\sigma(t)^2\,|\zeta(\sigma+it)|^2\,dt$$

$$= \begin{cases} c_1c_3T^{\frac{5}{2}-2\sigma} + c_5T^{\frac{7}{2}-4\sigma} + O(T) & \Big(\frac{1}{2}<\sigma<\frac{5}{8}\Big), \quad (7)\\ c_1c_3T^{\frac{5}{2}-2\sigma} + O(T) & \Big(\frac{5}{8}\le\sigma<\frac{3}{4}\Big), \quad (8)\\ c_4\zeta\Big(\frac{3}{2}\Big)T\log T + O(T(\log T)^{\frac{1}{2}}) & \Big(\sigma=\frac{3}{4}\Big), \quad (9)\\ O(T) & \Big(\frac{3}{4}<\sigma<1\Big), \quad (10)\end{cases}$$

where c_1, c_3 *and* c_4 *are as above, and*

$$c_5 = \frac{2}{7-8\sigma}(2\pi)^{4\sigma-\frac{5}{2}}\frac{\zeta^2(3/2)}{\zeta(3)}\zeta(2-2\sigma)\zeta\Big(\frac{5}{2}-2\sigma\Big)\zeta\Big(\frac{1}{2}+2\sigma\Big).$$

The formula (7) and (8) suggests that the line $\sigma = 5/8$ may also have a kind of critical property. In fact, if the estimate (4) would be best-possible, as Matsumoto and Meurman have suggested in [5], the error estimates $O(T)$ in (7) and (8) would be best-possible. To clarify the situation on the line $\sigma = 7/8$ may be difficult, but we remark that the constant c_5 has a singularity at $\sigma = 7/8$.

2. Proof of (7)–(10). We define I_σ by

$$I_\sigma = \int_T^{T+H} f(E_\sigma(t))\,|\zeta(\sigma+it)|^2\,dt \qquad \Big(\frac{1}{2}<\sigma<1\Big),$$

where $T \geq 2$, $0 \leq H \leq T$, and $f(t)$ is a given function which is continuous in $[T, T+H]$. If $F' = f$, then from (2) it follows that

$$\begin{aligned} I_\sigma &= F(E_\sigma(T+H)) - F(E_\sigma(T)) \\ &\quad + \int_T^{T+H} f(E_\sigma(t))\{c_1 + 2(1-\sigma)c_2t^{1-2\sigma}\}\,dt. \end{aligned} \tag{11}$$

Applying (11) with $H = T$, $f(t) = t^2$, $F(t) = t^3/3$, we have

$$\begin{aligned} I_\sigma &= \frac{1}{3}E_\sigma(2T)^3 - \frac{1}{3}E_\sigma(T)^3 \\ &\quad + \int_T^{2T} E_\sigma(t)^2\{c_1 + 2(1-\sigma)c_2t^{1-2\sigma}\}\,dt. \end{aligned}$$

By using $E_\sigma(T) = O(T^{1/(1+4\sigma)}(\log T)^{(4\sigma-1)/(4\sigma+1)})$ $(1/2 < \sigma < 1)$ (see [3, p.88]) and integrating by parts, we have

$$\begin{aligned} &\int_T^{2T} E_\sigma(t)^2\,|\zeta(\sigma+it)|^2\,dt \\ &= \Big(\int_0^t E_\sigma(u)^2\,du\Big)(c_1 + 2(1-\sigma)c_2\,t^{1-2\sigma})\Big|_T^{2T} \\ &\quad - 2c_2(1-\sigma)(1-2\sigma)\int_T^{2T}\Big(\int_0^t E_\sigma(u)^2\,du\Big)t^{-2\sigma}\,dt + O(T). \end{aligned} \tag{12}$$

Substituting (3) and (4) into (12), we have

$$\begin{aligned}
&\int_T^{2T} E_\sigma(t)^2 \,|\zeta(\sigma+it)|^2\, dt \\
&= c_1c_3\{(2T)^{\frac{5}{2}-2\sigma} - T^{\frac{5}{2}-2\sigma}\} \\
&\quad + \begin{cases} \dfrac{2(1-\sigma)(5-4\sigma)}{7-8\sigma} c_2c_3\{(2T)^{\frac{7}{2}-4\sigma} - T^{\frac{7}{2}-4\sigma}\} + O(T) & \left(\dfrac{1}{2} < \sigma < \dfrac{5}{8}\right), \\ O(T) & \left(\dfrac{5}{8} \le \sigma < \dfrac{3}{4}\right). \end{cases}
\end{aligned}$$

Hence, we obtain the assertions of (7) and (8). Similarly, from (5) and (12), we have

$$\int_T^{2T} E_{\frac{3}{4}}(t)^2 \left|\zeta\left(\frac{3}{4}+it\right)\right|^2 dt = c_4\,\zeta\left(\frac{3}{2}\right)\{2T\log(2T) - T\log T\} + O(T(\log T)^{\frac{1}{2}}).$$

Finally, from (6) and (12), we obtain, for $3/4 < \sigma < 1$,

$$\int_T^{2T} E_\sigma(t)^2\, |\zeta(\sigma+it)|^2\, dt = O(T).$$

The proofs of (9) and (10) are now completed as before.

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