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THETA FUNCTIONS. I

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In this series of the papers, we discuss in detail the properties of the function

$$\theta(x,q)=\sum_{n\in\mathbf{Z}}x^nq^{n^2}$$

and related functions. These functions are related by a labyrinth of formulae [6]. We try not to be puzzled in this labyrinth. We also try to construct the theory of elliptic functions by our function $\theta(x,q)$.

We refer mainly to [3, Appendix] several formulae for this function $\theta(x,q)$ and the related functions. We quote from this appendix simply by [T10], for example.

1. Jacobi's theta functions and Dedekind's eta function. For a complex number τ with $\operatorname{Im}\tau>0$, we put

$$q=e^{\pi i\tau} \tag{1.1}$$

then $|q|<1$ and the function

$$\theta(x,q)=\sum x^nq^{n^2} \tag{1.2}$$

is convergent for any $x\neq 0$.

In this paper, we denote by λ the 24-th root of unity $e^{2\pi i/24}$;

$$\lambda=\frac{1}{4}\Big((\sqrt{6}+\sqrt{2})+i\,(\sqrt{6}-\sqrt{2})\Big) \tag{1.3}$$

we have

$$\lambda^2=\frac{1}{2}(\sqrt{3}+i),\qquad \lambda^3=\frac{1}{\sqrt{2}}(1+i)=\sqrt{i},\qquad \lambda^4=\tfrac{1}{2}(1+\sqrt{3}\,i)$$
$$\lambda^5=\frac{1}{4}\Big((\sqrt{6}-\sqrt{2})+i\,(\sqrt{6}+\sqrt{2})\Big),\qquad \lambda^6=i,$$

and so on.

Jacobi's theta functions (theta zeros) are described in the following way.

$$\begin{cases}\theta_3(\tau)=\sum q^{n^2} = \theta(1,q)\\ \theta_4(\tau)=\sum(-1)^nq^{n^2}=\theta(-1,q)\\ \theta_2(\tau)=\sum q^{(n+1/2)^2}=q^{1/4}\,\theta(q,q)\end{cases} \tag{1.4}$$

Note that we denote $q^{\alpha} = e^{\pi i \tau \alpha}$ for a real number α.

Dedekind's eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) \tag{1.5}$$

This function has the expression

$$\eta(\tau) = q^{1/12}\, \theta(-q, q^3) \tag{1.6}$$

[T23]. Between these functions stand the following fundamental formulae:

$$\theta_3(\tau)^4 = \theta_2(\tau)^4 + \theta_4(\tau)^4 \tag{1.7}$$

$$\theta_2(\tau)\,\theta_3(\tau)\,\theta_4(\tau) = 2\eta(\tau)^3 \tag{1.8}$$

[T11 and A22]. We also consider the following functions

$$\begin{cases} \chi_0(\tau) = q^{1/8}\, \theta(q, q^2) \\ \chi_1(\tau) = q^{1/8}\, \theta(-q, q^2) \end{cases} \tag{1.9}$$

These functions are $\chi_0(\tau) = \rho_0(2\tau)$ and $\chi_1(\tau) = \rho_1(2\tau)$ in the notation of [3, A8]. These functions are related to $\theta_p(\tau)$ in the following way. That is, $\theta_2(\tau) = 2\,\chi_0(2\tau)$ [T3] and

$$\begin{cases} 2\chi_0(\tau)^2 = \theta_2(\tau)\,\theta_3(\tau) \\ 2\chi_1(\tau)^2 = \theta_2(\tau)\,\theta_4(\tau) \\ 2\chi_0(\tau)\,\chi_1(\tau) = \theta_2(\tau)\,\theta_4(2\tau) \end{cases} \tag{1.10}$$

[T8–10]. Now it is easy to see that

$$\begin{cases} \theta_3(\tau+1) = \theta_4(\tau), & \theta_4(\tau+1) = \theta_3(\tau) \\ \theta_2(\tau+1) = \lambda^3\, \theta_2(\tau), & \eta(\tau+1) = \lambda\, \eta(\tau) \\ \chi_0(\tau+1) = \lambda^{3/2}\, \chi_1(\tau), & \chi_1(\tau+1) = \lambda^{3/2}\, \chi_0(\tau) \end{cases} \tag{1.11}$$

where $\lambda^{1/2} = e^{2\pi i/48}$. Note that $\theta(-x, -q) = \theta(x, q)$.

The theta formula for $\theta(x, q)$ is described in the following way;

$$\theta(e^{\gamma}, e^{\delta}) = \kappa\, \theta(e^{\alpha}, e^{\beta}) \tag{1.12}$$

with

$$\kappa = \lambda^{-3} e^{\alpha^2/4\beta} \sqrt{\frac{\beta}{\pi i}}$$

where α and β are complex numbers with $\mathrm{Re}(\beta) < 0$, and

$$\beta\delta = \pi^2, \qquad \alpha^2\delta + \gamma^2\beta = 0$$

and $\sqrt{\beta/\pi i}$ is assumed to be in the first quadrant of the complex plane. [A23]

From this theta formula, it follows that

$$(1.13) \quad \begin{cases} \theta_3(-1/\tau) = \lambda^{-3}\sqrt{\tau}\,\theta_3(\tau), & \eta(-1/\tau) = \lambda^{-3}\sqrt{\tau}\,\eta(\tau) \\ \theta_4(-1/\tau) = \lambda^{-3}\sqrt{\tau}\,\theta_2(\tau), & \theta_2(-1/\tau) = \lambda^{-3}\sqrt{\tau}\,\theta_4(\tau) \\ \chi_0(-1/\tau) = \lambda^{-3}\sqrt{\tau/2}\,\theta_4(2\tau), & \chi_1(-1/\tau) = \lambda^{-3}\sqrt{\tau}\,\chi_1(\tau) \end{cases}$$

Note that the theta formulae for $\theta_p(\tau)$ are obtained by direct use of the formula (12), but for the functions $\eta(\tau)$ and $\chi_p(\tau)$, we need the following lemma.

Lemma. *It holds that*

$$\theta(ix, q) = \theta(-x^2, q^4) + ixq\,\theta(-x^2q^4, q^4)$$
$$\theta(\zeta x, q) = \theta(-x^3, q^9) + \zeta xq\,\theta(-x^3q^6, q^9) + q(\zeta x)^{-1}\theta(-x^3q^{-6}, q^9)$$

where $\zeta^3 = -1$.

Proof. A simple calculus shows this lemma.

For the even unimodular lattice E_8 of 8-dimension, its theta function $E_4(\tau)$ has the expression

$$(1.14) \qquad E_4(\tau) = \frac{1}{2}\Big(\theta_3(\tau)^8 + \theta_2(\tau)^8 + \theta_4(\tau)^8\Big)$$

which is a modular form of weight 4 as easily seen from (11) and (13). Also the discriminant $\Delta(\tau)$ of elliptic curve has the expression

$$(1.15) \qquad \Delta(\tau) = \eta(\tau)^{24} = q^2\prod(1 - q^{2n})^{24}$$

which is a modular form of weight 12. From (8), it follows

$$(1.16) \qquad \Delta(\tau) = \Big(\frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)\Big)^8$$

These functions $E_4(\tau)$ and $\Delta(\tau)$ generate the ring of all modular forms for the group $SL(2, \mathbf{Z})$ in the sense of [1, p.164]. Putting

$$(1.17) \qquad \Delta_4(\tau) = \frac{1}{16}\Big(\theta_3(\tau)^8 - E_4(\tau)\Big) = 2^{-4}\,\theta_2(\tau)^4\,\theta_4(\tau)^4$$

we see that $\Delta_4(\tau)$ is a modular form of weight 4 for the group

$$\Gamma_\theta = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

This can be expressed as $\Delta_4(\tau) = \chi_1(\tau)^8$ by (10). Also $\theta_3(\tau)^2$ is a "modular form" of weight 1 for Γ_θ. The ring of all "modular forms" of integral weight for Γ_θ in the sense of [1, p.164] is generated by $\theta_3(\tau)^2$ and $\Delta_4(\tau)$. By infinite product expansions of $\theta_p(\tau)$ [A19–21], or by that of $\chi_1(\tau)$, we have

$$\Delta_4(\tau) = \chi_1(\tau)^8 = q\prod\Big((1-q^{4n})(1-q^{2n-1})\Big)^8 \tag{1.18}$$

The group Γ_θ is of index 3 in the group $SL(2,\mathbf{Z})$ having the representatives

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. The modular functions of theta type. From the functions $\theta_p(\tau)$ and $\eta(\tau)$, we define

$$\begin{cases} \alpha(\tau) = \theta_3(\tau)/\eta(\tau) \\ \beta(\tau) = \theta_4(\tau)/\theta_3(\tau) \\ \gamma(\tau) = \theta_2(\tau)/\theta_3(\tau) \end{cases} \tag{2.1}$$

These functions are real-valued and positive on the imaginary axis of the upper half plane. If we put $\tau = it$ with real t such that $t > 0$, then $q = e^{-\pi t}$ and we can evaluate $\theta_p(it)$ and $\eta(it)$ approximately. The function $\beta(it)$ is monotone increasing with t and the function $\gamma(it)$ is monotone decreasing with t. We have $\beta(i\infty) = 1$, $\gamma(i\infty) = 0$, $\beta(i0) = 0$ and $\gamma(i0) = 1$, and also $\alpha(i\infty) = +\infty$ and $\alpha(i0) = +\infty$. These can be read off from the following fundamental formulae

$$\beta(\tau)^4 + \gamma(\tau)^4 = 1 \tag{2.2}$$

$$\alpha(\tau)^3\beta(\tau)\gamma(\tau) = 2 \tag{2.3}$$

which follow from (1.7) and (1.8) of the preceding section. From (3), it follows that

$$\beta(\tau)^4\gamma(\tau)^4 = 2^4/\alpha(\tau)^{12}$$

Thus we have

$$\beta^4,\ \gamma^4 = \frac{1}{2\alpha^6}\left(\alpha^6 \pm \sqrt{\alpha^{12} - 64}\right)$$

and

$$\beta^2,\ \gamma^2 = \frac{1}{2\alpha^3}\left(\sqrt{\alpha^6 + 8} \pm \sqrt{\alpha^6 - 8}\right)$$

By the preceding remark, we have

$$\beta(\tau)^2 = \frac{1}{2\alpha^3}\left(\sqrt{\alpha^6 + 8} + \sqrt{\alpha^6 - 8}\right) \tag{2.4}$$

$$\gamma(\tau)^2 = \frac{1}{2\alpha^3}\left(\sqrt{\alpha^6 + 8} - \sqrt{\alpha^6 - 8}\right) \tag{2.5}$$

where $\sqrt{\alpha(\tau)^6 \pm 8}$ are assumed to be positive on the imaginary axis with $t \gg 1$.

From (1.11), it follows that

$$\begin{cases} \alpha(\tau+1) = \lambda^{-1}\alpha(\tau)\beta(\tau) \\ \beta(\tau+1) = 1/\beta(\tau) \\ \gamma(\tau+1) = \lambda^3\gamma(\tau)/\beta(\tau) \end{cases} \tag{2.6}$$

and that

$$\begin{cases} \alpha(\tau+2) = \lambda^{-2}\alpha(\tau) \\ \beta(\tau+2) = \beta(\tau) \\ \gamma(\tau+2) = \lambda^6\gamma(\tau) \end{cases} \tag{2.7}$$

From (1.13), it follows that

$$\begin{cases} \alpha(-1/\tau) = \alpha(\tau) \\ \beta(-1/\tau) = \gamma(\tau) \\ \gamma(-1/\tau) = \beta(\tau) \end{cases} \tag{2.8}$$

From these, we can deduce that $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ are modular functions belonging to the respective congruence subgroups of $SL(2, \mathbf{Z})$. We will call these functions the modular functions of theta type.

For $\tau = i$, we have $\beta(i) = \gamma(i)$ by (8). So

$$\beta(i) = \gamma(i) = 2^{-1/4} \tag{2.9}$$

by (2), and also $\alpha(i)^3 = 2^{3/2}$. Thus

$$\alpha(i) = 2^{1/2} \tag{2.10}$$

So $\tau = i$ is the zero point of $\sqrt{\alpha(\tau)^6 - 8}$, and the sign of this function should be changed to negative one for $t < 1$.

From (10), it follows that

$$\eta(i) = 2^{-1/2}\,\theta_3(i)$$

and the value $\theta_3(i)$ is calculated from the complete elliptic integral for the modulus $k(i) = \gamma(i)^2$. For the meaning of $k(\tau) = \gamma(\tau)^2$, see below (the formula (13)). As $k(i)^2 = 1/2$, the value $(\pi/2)\theta_3(i)^2$ is equal to the integral

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}}$$

The fact that $K = (\pi/2)\theta_3(\tau)^2$ is shown in the standard book of the elliptic function theory [6], or will be explained in the subsequent paper. The above mentioned integral K is related to

$$\omega = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\Gamma(1/4)^2$$

in the following way. Putting $\tau = i + 1$, from (6), it follows that

$$\alpha(i+1) = \lambda^{-1}\,2^{1/4}, \quad \beta(i+1) = 2^{1/4}, \quad \gamma(i+1) = \lambda^3$$

As $\gamma(i+1)^4 = -1$, we have $\omega = (\pi/2)\theta_3(i+1)^2 = (\pi/2)\theta_4(i)^2$. Thus $\omega = (\pi/2)\beta(i)^2\theta_3(i)^2 = (1/\sqrt{2})K$, and we have

$$\eta(i) = \Gamma(1/4)/(2\,\pi^{3/4}). \tag{2.11}$$

We discuss the relations between the classical modular functions and our functions $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$.

Weber defined the functions

$$\begin{cases} f(\tau) = q^{-1/24}\prod(1+q^{2n-1}) \\ f_1(\tau) = q^{-1/24}\prod(1-q^{2n-1}) \\ f_2(\tau) = \sqrt{2}\,q^{1/12}\prod(1+q^{2n}) \end{cases} \tag{2.12}$$

[5]. Seeing the infinite product expansion of $\theta_p(\tau)$ [A18–21], we have $\alpha(\tau) = f(\tau)^2$, $\alpha_1(\tau) = \alpha(\tau)\beta(\tau) = f_1(\tau)^2$ and $\alpha_2(\tau) = \alpha(\tau)\gamma(\tau) = f_2(\tau)^2$. Note that $\alpha_1(\tau) = \theta_4(\tau)/\eta(\tau)$ and $\alpha_2(\tau) = \theta_2(\tau)/\eta(\tau)$. Thus our functions are the squares of Weber's functions. The merits of our functions would

be that we can apply the formulae for the theta functions to our functions directly.

For the modulus $k = k(\tau)$ of the elliptic integral of the first kind

$$\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

we have

$$k(\tau) = \theta_2(\tau)^2/\theta_3(\tau)^2 = \gamma(\tau)^2 \tag{2.13}$$

[6]. For the complementary modulus $k'(\tau) = \sqrt{1-k(\tau)^2}$, we have

$$k'(\tau) = \theta_4(\tau)^2/\theta_3(\tau)^2 = \beta(\tau)^2 \tag{2.14}$$

We will discuss these relations by constructing the Jacobi's elliptic functions newly in the subsequent paper.

Weierstrass' lambda function $\Lambda(\tau)$ is defined as

$$\Lambda(\tau) = k(\tau)^2 = \left(f_2(\tau)/f(\tau)\right)^8$$

This function is a modular function for the group $\Gamma(2)$, where $\Gamma(2) = \{A \in SL(2,\mathbf{Z}): A \equiv E \pmod 2\}$. From the above one, it follows

$$\Lambda(\tau) = \gamma(\tau)^4 = 1 - \beta(\tau)^4 \tag{2.15}$$

and $\Lambda(\tau)(1-\Lambda(\tau)) = \beta(\tau)^4\gamma(\tau)^4 = 16/\alpha(\tau)^{12}$. Thus

$$\Lambda(\tau)^2 - \Lambda(\tau) + 1 = \left(\alpha(\tau)^{12} - 16\right)/\alpha(\tau)^{12} \tag{2.16}$$

On the other hand,

$$\gamma^8 - \gamma^4 + 1 = 1 - \beta^4\gamma^4 = \frac{1}{2}(1 + \beta^8 + \gamma^8) = E_4(\tau)/\theta_3(\tau)^8$$

so we have

$$\Lambda(\tau)^2 - \Lambda(\tau) + 1 = E_4(\tau)/\theta_3(\tau)^8 \tag{2.17}$$

The absolute invariant $J(\tau)$ which is a modular function for $SL(2,\mathbf{Z})$ is expressed as

$$J(\tau) = \frac{4}{27}(\Lambda^2 - \Lambda + 1)^3/(\Lambda(1-\Lambda))^2$$

Putting $j(\tau) = 12^3 J(\tau) = 1728\, J(\tau)$, it follows that

$$(2.18) \qquad j(\tau) = \left(\alpha(\tau)^{12} - 16\right)^3 / \alpha(\tau)^{12} = E_4(\tau)^3/\Delta(\tau)$$

where $\Delta(\tau) = \eta(\tau)^{24}$. This function has the following q-expansion;

$$(2.19) \qquad j(\tau) = q^{-2} + 744 + \sum c_n q^{2n}$$

by (18), with $c_1 = 196884$, $c_2 = 21493760, \ldots$.

From the theory of $\wp$-function, the invariant $J(\tau)$ is defined as

$$(2.20) \qquad J(\tau) = g_2^3/(g_2^3 - 27\, g_3^2)$$

where the function $\wp$ satisfies the equation

$$\wp'^{\,2} = 4\wp^3 - g_2\wp - g_3$$

We will explain the meaning of this definition from our view point in the subsequent paper.

Weber also defined the function $\gamma_2(\tau)$ and $\gamma_3(\tau)$ by

$$\gamma_2(\tau) = j(\tau)^{1/3}, \qquad \gamma_3(\tau) = \sqrt{j(\tau) - 12^3}$$

where these functions should be positive on the upper part of the imaginary axis [5]. For these functions, we have

$$(2.21) \qquad \gamma_2(\tau) = \left(\alpha(\tau)^{12} - 16\right)/\alpha(\tau)^4 = E_4(\tau)/\eta(\tau)^8$$

$$(2.22) \qquad \gamma_3(\tau) = \left(\alpha(\tau)^{12} + 8\right)\left(\beta(\tau)^4 - \gamma(\tau)^4\right)$$

Proof. The formula (21) is clear. By the calculus

$$\begin{aligned} \frac{\alpha^{12}}{12^3}(j - 12^3) &= \left(\frac{\alpha^{12} - 16}{12}\right)^3 - 12\left(\frac{\alpha^{12} - 16}{12}\right) - 16 \\ &= (X + 2)^2(X - 4) \end{aligned}$$

with $X = (\alpha^{12} - 16)/12$, we have

$$\frac{\alpha^6}{\sqrt{12^3}}\gamma_3(\tau) = \frac{\alpha^{12} + 8}{12}\sqrt{\frac{\alpha^{12} - 64}{12}}$$

From (4) and (5), it follows that $\beta^4 - \gamma^4 = \sqrt{\alpha^{12} - 64}/\alpha^6$.

On the analogy of (18), we define

$$j_1(\tau) = \theta_3(\tau)^8/\Delta_4(\tau) = \Big(\theta_3(\tau)/\chi_1(\tau)\Big)^8 \tag{2.23}$$

which is a modular function for Γ_θ by (1.11) and (1.13). This function has the following q-expansion;

$$j_1(\tau) = q^{-1} + 24 + \sum b_n q^n \tag{2.24}$$

with $b_1 = 276$, $b_2 = 2048, \ldots$.

As $\theta_3(\tau)^8 = E_4(\tau) + 16\chi_1(\tau)^8$, we have $j_1(\tau) = \alpha(\tau)^{12}$. This can be written in the following very interesting formula;

$$\Big(\theta_3(\tau)/\chi_1(\tau)\Big)^8 = \Big(\theta_3(\tau)/\eta(\tau)\Big)^{12}. \tag{2.25}$$

But this is a puzzling formula. The appearance of $\theta_4(2\tau)$ in (1.10) and (1.13) would explain this situation.

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