Math. J. Okayama Univ. 36(1994), 29–34

NOTE ON THE GENERALIZED EULER CONSTANTS

TAKASHI TASAKA

1. Introduction. In this paper, we consider the zeta functions of Hurwitz type and the related constants which may be called the generalized Euler constants. That is, for a real number a $(0 < a \leq 1)$, we put

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}. \tag{1}$$

This function of complex variable s is analytic for $\operatorname{Re} s > 1$, and if $a = 1$, then $\zeta(s, 1) = \zeta(s)$ is the usual Riemann zeta function. Now we define a constant C_a as

$$C_a = \lim_{\ell \to \infty} \Big\{ \sum_{n=0}^{\ell} \frac{1}{n + a} - \log \ell \Big\} \tag{2}$$

whose existence can be proved in similar way as in the case of the usual Euler constant $C = C_1$. Specifically we are interested in the generalized Euler constants C_a for rational number $a = m/q$ $(1 \leq m \leq q)$.

From the following calculation

$$\begin{aligned} C_{1/2} + C_1 &= \lim_{\ell \to \infty} \Big\{ \sum_{n=0}^{\ell} \frac{2}{2n+1} - \log \ell + \sum_{n=1}^{\ell+1} \frac{2}{2n} - \log \ell \Big\} \\ &= \lim \Big\{ 2\Big(\sum_{n=1}^{2\ell+2} \frac{1}{n} - \log 2\ell \Big) - 2 \log \ell + 2 \log 2\ell \Big\} \\ &= 2C + 2 \log 2 \end{aligned}$$

it follows that $C_{1/2} = C + 2\log 2 = 1.9634 \cdots$.

In the similar way, we have

$$\sum_{m=1}^{q} C_{m/q} = qC + q \log q \tag{3}$$

for a natural number q.

The function

$$f_a(s) = \zeta(s, a) - \frac{1}{s-1} \tag{4}$$

is extended to an entire function of s, and

$$f_a(1) = C_a, \qquad f_a(0) = \frac{3}{2} - a. \tag{5}$$

Indeed, by Euler's summation formula [1, I, p.76], we have

$$\sum_{n=k}^{\ell} \frac{1}{(n+a)^s}$$
$$= \frac{1}{2}\Big(\frac{1}{(k+a)^s} + \frac{1}{(\ell+a)^s}\Big) + \int_k^{\ell} \frac{dx}{(x+a)^s} - s\int_k^{\ell} \frac{B(x)}{(x+a)^{s+1}}\,dx$$

with

$$B(x) = x - [x] - \frac{1}{2}. \tag{6}$$

If we put $s = 1$ and $k = 0$, then we have

$$\sum_{n=0}^{\ell} \frac{1}{n+a} - \log(\ell + a) = \frac{1}{2}\Big(\frac{1}{a} + \frac{1}{\ell+a}\Big) - \log a - \int_0^{\ell} \frac{B(x)}{(x+a)^2}\,dx$$

letting ℓ to the infinity, it follows

$$C_a = \frac{1}{2a} - \log a - \int_0^{\infty} \frac{B(x)}{(x+a)^2}\,dx.$$

Now we put $s > 1$, $k = 0$ and $\ell = \infty$. Then

$$\zeta(s,a) - \frac{1}{s-1} = \frac{1}{s-1}(a^{1-s} - 1) + \frac{1}{2a^s} - s\int_0^{\infty} \frac{B(x)}{(x+a)^{s+1}}\,dx.$$

The function of right hand side can be proved that it extends to an entire function of s (For example see [2, p.139]).

In [3, p.267], for $\nu \geq 0$, $\nu \in \mathbf{Z}$, it is shown that

$$\zeta(-\nu, a) = -\frac{\phi'_{\nu+2}(a)}{(\nu+1)(\nu+2)}$$

where $\phi_n(x)$ are the n-th Bernoulli polynomial. Thus

$$f_a(-\nu) = \frac{\nu + 2 - \phi'_{\nu+2}(a)}{(\nu+1)(\nu+2)}.$$

Especially $f_a(0) = 3/2 - a$, because $\phi_2(x) = x^2 - x + 1/6$.

Also in [3, p.271], it is shown that

$$\zeta(0,a) = \frac{1}{2} - a, \qquad \zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$$

and that

$$\lim_{s\to 1}\Big(\zeta(s,a) - \frac{1}{s-1}\Big) = -\frac{\Gamma'(a)}{\Gamma(a)}.$$

Thus we have

Proposition 1.

$$C_a = -\frac{\Gamma'(a)}{\Gamma(a)}. \tag{7}$$

2. Dirichlet's L-functions. Let q be a natural number, we consider the set of Dirichlet characters $\bmod q$ (primitive or non-primitive). We denote by χ_0 the trivial character $\bmod q$.

For a character $\chi \bmod q$, the Dirichlet L-function $L(s,\chi)$ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}. \tag{8}$$

This can be written

$$L(s,\chi) = \sum_{m=1}^{q} \chi(m) \sum_{n=1}^{\infty} (qn+m)^{-s} = q^{-s} \sum_{m=1}^{q} \chi(m)\,\zeta(s, m/q)$$

using the zeta function of Hurwitz type. For a non-trivial character χ, as $\sum \chi(m) = 0$, so

$$L(s,\chi) = q^{-s} \sum_{m=1}^{q} \chi(m) f_{m/q}(s) \tag{9}$$

is an entire function. Letting $s = 1$, 0 or $-\nu$, we have

$$L(1,\chi) = q^{-1} \sum_{m=1}^{q} \chi(m)\, C_{m/q}, \tag{10}$$

$$L(0,\chi) = \sum \chi(m)\Big(\frac{3}{2} - m/q\Big) = -q^{-1} \sum_{m=1}^{q} \chi(m)\, m, \tag{11}$$

(12) $$L(-\nu,\chi)=-\frac{q^{\nu}}{(\nu+1)(\nu+2)}\sum\chi(m)\,\phi'_{\nu+2}(m/q).$$

Also we have

(13) $$L(2,\chi)=q^{-2}\sum\chi(m)\,\zeta(2,m/q).$$

For the trivial character $\chi_0 \bmod q$,

$$L(s,\chi_0)=q^{-s}\sum\chi_0(m)\,\zeta(s,m/q)=q^{-s}\sum_{(m,q)=1}\zeta(s,m/q)$$

from the definition, and also

$$L(s,\chi_0)=Q_q(s)\,\zeta(s)$$

with $Q_q(s)=\prod_{p|q}(1-p^{-s})$ and $\zeta(s)$ is the Riemann zeta function.

The value $L(1,\chi)$ is well-known for a primitive character χ. That is,

$$L(1,\chi)=\begin{cases}-\dfrac{\tau(\chi)}{q}\displaystyle\sum_{m=1}^{q}\bar{\chi}(m)\log\Big(\sin\frac{m}{q}\pi\Big), & \chi(-1)=1\\[2ex] \dfrac{\pi i\tau(\chi)}{q^2}\displaystyle\sum_{m=1}^{q}\bar{\chi}(m)\,m, & \chi(-1)=-1\end{cases}$$

[1, II, p.140], where $\bar{\chi}$ denotes the conjugate of χ or the inverse of χ, and

$$\tau(\chi)=\sum_{m=1}^{q}\chi(m)\,e^{\frac{2\pi i m}{q}}$$

is the character sum for χ. If χ is the derived character of the primitive character $\psi \bmod f$, then

$$L(1,\chi)=Q_\chi(1)L(1,\psi)$$

[1, II, p.110], where

$$Q_\chi(s)=\prod_{p|q}(1-\psi(p)\,p^{-s})\quad\text{and}\quad Q_\chi(1)=\prod_{p|q}(1-\psi(p)\,p^{-1}).$$

We denote by $P=P_q$ the character table of the multiplicative group $(\mathbf{Z}/q\mathbf{Z})^\times$. If we consider P the square matrix of size $\varphi(q)$, then the orthogonal relation of the characters are written as

$${}^t\bar{P}\cdot P=\varphi(q)E,\qquad \varphi(q)=\left|(\mathbf{Z}/q\mathbf{Z})^\times\right|$$

If we put $\boldsymbol{C}$ = the column vector ${}^t(C_{m/q}:(m,q)=1)$, then the components of $q^{-1}P\boldsymbol{C}$ are $L(1,\chi)$ by the formula (10) except for the trivial character χ_0, to which corresponds the value

$$\tilde{L}(1,\chi_0) = q^{-1}\sum_{(m,q)=1} C_{m/q}.$$

This value can be derived from (3) inductively. Thus

Proposition 2.

$$\varphi(q)\,q^{-1}\boldsymbol{C} = {}^t\bar{P}\,{}^t(\tilde{L}(1,\chi_0),\ldots,L(1,\chi),\ldots) \tag{14}$$

and $C_{m/q}$, $(1 \leq m \leq q)$ are calculated concretely.

Example. For $q=3$, $\tilde{L}(1,\chi_0) = 2C/3 + \log 3$, $L(1,\chi_3) = \pi/3\sqrt{3}$ and

$$C_{1/3} = C + \frac{3}{2}\log 3 + \frac{\pi}{2\sqrt{3}}$$
$$C_{2/3} = C + \frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}}$$

For $q=4$, $\tilde{L}(1,\chi_0) = (C + 3\log 2)/2$, $L(1,\chi_4) = \pi/4$ and

$$C_{1/4} = C + 3\log 2 + \frac{\pi}{2}$$
$$C_{3/4} = C + 3\log 2 - \frac{\pi}{2}$$

References

[1] C. L. Siegel: Analytische Zahlentheorie I, II. Göttingen, 1963/64.
[2] T. Tatsuzawa: The Function Theory (in Japanese). Kyoritsu, 1980.
[3] E. T. Whittaker and G. N. Watoson: Modern Analysis. Cambridge Univ. Press, 1969.

Department of Mathematical Science
University of Tokyo
3–8–1 Komaba, Meguro, Tokyo 153, Japan

(*Received July 7, 1993*)

Current Address:
Department of Mathematics
Faculty of Science
Okayama University
Okayama 700, Japan