Math. J. Okayama Univ. 35(1993), 253-275

SUBMANIFOLDS IN A MANIFOLD WITH GENERAL CONNECTIONS

NAOTO ABE, HIROAKI NEMOTO and SEIICHI YAMAGUCHI

0. Introduction. The main purpose of this paper is to study submanifold geometry in terms of O-derivative operator, which was defined by N. Abe in [2]. An O-derivative operators is a first order differential operator, from a vector bundle over a manifold to another vector bundle over the manifold, whose properties are similar to those of a covariant derivative. When these vector bundles are the tangent bundle of a manifold, the operator is the covariant derivative of a general connection in the sense of T. Otsuki. The notion of general connections was defined by T. Otsuki in [10] as a generalization of usual ones. He defined the general connections on the tangent tensor bundles of a manifold and defined associating geometrical objects analogous to those of usual ones, for example, their curvature and torsion forms [20]. In his papers [10]-[20], many results about general connections were obtained. N. Abe defined general connections on arbitrary vector bundles and studied some fundamental properties in [1]. H. Nemoto [9] applied the theory to the normal bundle of a submanifold and developed the submanifold geometry initiated by T. Otsuki and C. -S. Houh [7], [19] and [22].

In §1, we will prepare notations used in this paper and define the O-derivative operator and review some algebraic properties of the space of these operators. In §2, the definitions and fundamental properties of the curvature and torsion forms of O-derivative operators will be reviewed. In §3, we will study geometry of immersed submanifolds in a manifold with a general connection. The second fundamental form, the shape operator and the transversal connection will be defined and we will have fundamental formulae on submanifold geometry. In §4, the case where the ambient manifold have a metric will be treated. We will study totally geodesic submanifolds and totally umbilical submanifolds in §5 and §6.

1. Preliminaries. We assume that all objects are smooth and all vector bundles are real throughout this paper. Let M be an n-dimensional manifold, $T(M)$ its tangent bundle and $C(M)$ the ring of real-valued functions on M. We will generally use letters V, W and these with superscript and prime to denote vector bundles over M. The fibre of a vector bundle V at $x \in M$ is denoted by V_x and the dual bundle of V by V^*. The space of cross-sections of V is denoted by $\Gamma(V)$, which has a canonical $C(M)$-module structure. Let $Hom(V,W)$ be the vector bundle of which fibre $Hom(V,W)_x$ at x is the vector space $Hom(V_x,W_x)$ of linear mappings from V_x to W_x. Especially $Hom(V,V)$ is denoted by $End(V)$. We note that $Hom(V,W)$ can be naturally identified with the tensor product $W \otimes V^*$. The space of vector bundle homomorphisms from V to W is denoted by $HOM(V,W)$. We denote zero homomorphism from V to W by $0_{V,W}$ or simply 0. Especially $HOM(V,V)$, the ring of endomorphisms on V, is denoted by $END(V)$. We denote the identity (resp. zero) endomorphism of V by I_V (resp. 0_V or simply 0). We note thatb $HOM(V,W)$ can be naturally identified with the space $\Gamma(Hom(V,W))$. For a non-negative integer r, we denote $HOM(\Lambda^r(T(M)),W)$ by $A^r(M,W)$ or simply $A^r(W)$, which consists of W-valued r-forms on M, and $A^r(M,M \times \mathbb{R})$ by $A^r(M)$ or simply A^r. We will use the same symbol to denote a vector bundle homomorphism and the induced linear mapping on the cross-sections.

Definition. For $P \in HOM(V,W)$, a bilinear mapping $\nabla : \Gamma(T(M)) \times\Gamma(V) \ni (X,s) \to \nabla_X s \in \Gamma(W)$ is called an O-derivative operator from V to W with the principal homomorphism P, if ∇ satisfies

$$\nabla_{fX} s = f\nabla_X s \quad \text{and} \quad \nabla_X fs = (Xf)Ps + f\nabla_X s$$

for each $X \in \Gamma(T(M))$, $s \in \Gamma(V)$ and $f \in C(M)$. Let $O(V,W;P)$ be the set of O-derivative operators from V to W with the principal homomorphism P. Put $O(V,W) := \cup\{O(V,W;P)|P \in HOM(V,W)\}$. Especially we denote $O(V,V;P)$ by $O(V;P)$ and $O(V,V)$ by $O(V)$. An element of $O(V;P)$ is called the covariant derivative of a general connection, or simply a general connection on V with the principal endomorphism $P \in END(V)$.

For special examples, we see that $O(V,W;0_{V,W}) = HOM(V,T(M)^* \otimes W) = A^1(Hom(V,W))$ and $O(V;I_V)$ is the set of covariant derivatives of usual connections on V.

Definition. If $X \in \Gamma(T(M))$, then we define a linear mapping ∇_X : $\Gamma(V) \to \Gamma(W)$ by $\nabla_X s := \nabla_X s$ for $s \in \Gamma(V)$. We call ∇_X the O-derivative operator along X. Similarly if $x \in M$ and $v \in T(M)_x$, then we can define a linear mapping $\nabla_v : \Gamma(V) \to W_x$ by $\nabla_v s := (\nabla_X s)(x)$ for $s \in \Gamma(V)$ and $X \in \Gamma(T(M))$ such that $X(x) = v$.

We see also that ∇_v or ∇_X uniquely determines ∇.

At first we consider addition in $O(V,W)$ and multiplication (composition) by elements of $HOM(W,W')$ and $HOM(V',V)$ as follows.

Definition. If $\nabla^i \in O(V,W;P^i)$ $(i = 1,2)$, then we define the sum $\nabla^1 + \nabla^2$ by

$$(\nabla^1 + \nabla^2)_X s := \nabla^1_X s + \nabla^2_X s$$

for $s \in \Gamma(V)$ and $X \in \Gamma(T(M))$. If $\nabla \in O(V,W;P)$, $L \in HOM(W,W')$ and $R \in HOM(V',V)$, then we define the products $L\nabla$ and ∇R by

$$(L\nabla)_X s := L(\nabla_X s) \text{ and } (\nabla R)_X t := \nabla_X(Rt)$$

for $s \in \Gamma(V)$, $t \in \Gamma(V')$ and $X \in \Gamma(T(M))$.

Proposition 1.1([2]). *$\nabla^1 + \nabla^2 \in O(V,W;P^1 + P^2)$, $L\nabla \in O(V,W';LP)$ and $\nabla R \in O(V',W;PR)$. The mappings $HOM(W,W') \times O(V,W) \to O(V,W')$ and $O(V,W) \times HOM(V',V)) \to O(V',W)$ are bilinear. Moreover, the set $O(V,W)$ has a right $END(V)$- and left $END(W)$-module structure with respect to these addition and multiplication (composition).*

Since $(L\nabla)R = L(\nabla R)$, we denote $(L\nabla)R$ by $L\nabla R$. We will define O-derivative operators on dual and tensor product bundles. If $P \in HOM(V,W)$, then $P^* \in HOM(W^*,V^*)$ is defined by $(P^*\eta)(s) := \eta(Ps)$ for $\eta \in \Gamma(W^*)$ and $s \in \Gamma(V)$. Let $\nabla \in O(V,W;P)$.

Definition. The dual ∇^* of ∇ is defined by

$$(\nabla^*_X \eta)(s) := X(\eta(Ps)) - \eta(\nabla_X s)$$

for $\eta \in \Gamma(W^*)$, $X \in \Gamma(T(M))$ and $s \in \Gamma(V)$.

We see that $\nabla^*_X \eta \in \Gamma(V^*)$, $\nabla^* \in O(W^*,V^*;P^*)$ and the mapping $* : O(V,W) \to O(W^*,V^*)$ is linear.

If $P^i \in HOM(V^i,W^i)$ $(i = 1,2)$, then we define $P^1 \otimes P^2 \in HOM(V^1 \otimes V^2, W^1 \otimes W^2)$ by requiring $(P^1 \otimes P^2)(s_1 \otimes s_2) = (P^1 s_1) \otimes (P^2 s_2)$ for $s_i \in \Gamma(V^i)$. Let $\nabla^i \in O(V^i,W^i;P^i)$ $(i = 1,2)$.

Definition. The tensor product $\nabla^1 \otimes \nabla^2$ of ∇^1 and ∇^2 is defined by requiring

$$(\nabla^1 \otimes \nabla^2)_X(s_1 \otimes s_2) = (\nabla^1_X s_1) \otimes (P^2 s_2) + (P^1 s_1) \otimes (\nabla^2_X s_2)$$

for $s_i \in \Gamma(V^i)$ and $X \in \Gamma(T(M))$.

We see that $\nabla^1 \otimes \nabla^2 \in O(V^1 \otimes V^2, W^1 \otimes W^2; P^1 \otimes P^2)$ and the mapping $\otimes : O(V^1, W^1) \times O(V^2, W^2) \to O(V^1 \otimes V^2, W^1 \otimes W^2)$ is bilinear. When $V \cong V'$ and $W \cong W'$ are isomorphic vector bundles, $O(V, W)$ and $O(V', W')$ are isomorphic as modules. Then we will denote the corresponding O-derivative operators by the same symbol in a natural isomorphism, for example, $(V^1 \otimes V^2) \otimes V^3 \cong V^1 \otimes (V^2 \otimes V^3)$, $W \otimes V^* \cong Hom(V, W)$ and $V^* \otimes V^* \cong (V \otimes V)^*$. For a later section, we prepare

Proposition 1.2 ([2]). *Let $\nabla \in O(V, W; P)$. Under the canonical isomorphism $V^* \otimes V^* \cong (V \otimes V)^*$ and $W^* \otimes W^* \cong (W \otimes W)^*$, the corresponding $\nabla^* \otimes \nabla^* = (\nabla \otimes \nabla)^* \in O((W \otimes W)^*, (V \otimes V)^*; (P \otimes P)^*)$ satisfies*

$$((\nabla^* \otimes \nabla^*)_X g)(s_1, s_2) = X(g(Ps_1, Ps_2)) - g(\nabla_X s_1, Ps_2) - g(Ps_1, \nabla_X s_2)$$

for $g \in \Gamma((W \otimes W)^*)$ and $s_i \in \Gamma(V)$.

Proposition 1.3 ([2]). *Let ${}'\nabla \in O(W, W'; {}'P)$ and $\nabla' \in O(V', V; P')$. Under the canonical isomorphism $W \otimes V^* \cong Hom(V, W)$ and $W' \otimes (V')^* \cong Hom(V', W')$, the corresponding ${}'\nabla \otimes (\nabla')^* \in O(Hom(V, W), Hom(V', W'); {}'P \otimes (P')^*)$ satisfies*

$$(({}'\nabla \otimes (\nabla')^*_X C)t = {}'\nabla_X((CP't) - {}'PC(\nabla'_X t)$$

for $C \in \Gamma(Hom(V, W))$ and $t \in \Gamma(V')$.

2. Curvature and torsion forms. At first we define the following form which generalizes the difference of two covariant derivatives of usual connections. Let $\nabla \in O(V, W; P)$ and $\nabla' \in O(V', V; P')$.

Definition If $X \in \Gamma(T(M))$, then a linear mapping $S(\nabla, \nabla')_X : \Gamma(V') \to \Gamma(W)$ is defined by

$$S(\nabla, \nabla')_X t := \nabla_X(P't) - P(\nabla'_X t) \text{ for } t \in \Gamma(V').$$

We call $S(\nabla, \nabla')$ the difference form of the pair (∇, ∇').

Proposition 2.1 ([2]). *$S(\nabla,\nabla')=(\nabla\otimes(\nabla'))^* I_V \in A^1(Hom(V',W))$ and the mapping $S: O(V,W)\times O(V',V)\to A^1(Hom(V',W))$ is bilinear. Moreover, for $'\nabla\in O(W,W';'P)$, we have*

$$\begin{aligned} &S(L\nabla,\nabla') = LS(\nabla,\nabla') && \textit{for}\quad L\in HOM(W,W'),\\ &S('\nabla,R\nabla') = S('\nabla R,\nabla') && \textit{for}\quad R\in HOM(V,W)\\ \textit{and}\quad &S('\nabla,\nabla R') = S('\nabla,\nabla)R' && \textit{for}\quad R'\in HOM(V',V). \end{aligned}$$

Definition. If $W=T(M)$, then $s(\nabla,\nabla')\in\Gamma(V'^*)$ is defined by

$$s(\nabla,\nabla')t := tr(S(\nabla,\nabla').t) = \textstyle\sum_{i=1}^n \omega^i(S(\nabla,\nabla')_{e_i}t) \quad \text{for} \quad t\in V'_x,$$

where $e_1,\cdots,e_n$ is a base of $T(M)_x$ and $\omega^1,\cdots,\omega^n$ is the dual base. We call $s(\nabla,\nabla')$ the contracting difference form of (∇,∇'). Especially we denote $s(\nabla,\nabla)$ by $s(\nabla)$.

Next we define the following auxiliary operator which can be used in some formulae concerning curvature forms.

Definition. If $X,Y\in\Gamma(T(M))$, then a linear mapping $(\nabla\wedge\nabla')_{X,Y}:\Gamma(V')\to\Gamma(W)$ is defined by

$$(\nabla\wedge\nabla')_{X,Y}t := \nabla_X(\nabla'_Y t) - \nabla_Y(\nabla'_X t) \quad \text{for} \quad t\in\Gamma(V')$$

and denote $(\nabla_{[\,]})_{X,Y} := \nabla_{[X,Y]}:\Gamma(V')\to\Gamma(W)$.

Now we define the curvature form of a triple as follows. Let $'\nabla\in O(W,W';'P)$.

Definition. If $X,Y\in\Gamma(T(M))$, then a linear mapping $K('\nabla,\nabla,\nabla')_{X,Y}:\Gamma(V')\to\Gamma(W')$ is defined by

$$\begin{aligned} K('\nabla,\nabla,\nabla')_{X,Y}t := {}&('\nabla\wedge\nabla P')_{X,Y}t - ('\nabla\wedge P\nabla')_{X,Y}t\\ &+('P\nabla\wedge\nabla')_{X,Y}t - ('P\nabla_{[\,]}P')_{X,Y}t \quad \text{for} \quad t\in\Gamma(V'). \end{aligned}$$

We call $K('\nabla,\nabla,\nabla')$ the curvature form of the triple $('\nabla,\nabla,\nabla')$. Especially we denote $K(\nabla,\nabla,\nabla)$ by $K(\nabla)$ for $\nabla\in O(V;P)$.

We get the following fundamental fact:

Proposition 2.2 ([2]). *$K('\nabla,\nabla,\nabla')\in A^2(Hom(V',W'))$ and the mapping $K: O(W,W')\times O(V,W)\times O(V',V)\to A^2(Hom(V',W'))$ is* *trilinear. Moreover, for* $\nabla^1 \in O(V^1, W)$ *and* $'\nabla^1 \in O(W^1, W')$, *we have*

$$\begin{aligned} &K('L'\nabla,\nabla,\nabla') = {}'LK('\nabla,\nabla,\nabla') && \text{for} \quad 'L \in HOM(W',W^1),\\ &K('\nabla^1, L\nabla,\nabla') = K('\nabla^1 L,\nabla,\nabla') && \text{for} \quad L \in HOM(W,W^1),\\ &K('\nabla,\nabla^1, R\nabla') = K('\nabla,\nabla^1 R,\nabla') && \text{for} \quad R \in HOM(V,V^1)\\ \text{and} \quad &K('\nabla,\nabla,\nabla' R') = K('\nabla,\nabla,\nabla')R' && \text{for} \quad R' \in HOM(V^1,V'). \end{aligned}$$

Definition. If $W' = T(M)$, then $k('\nabla,\nabla,\nabla') \in A^1(V'^*)$ is defined by

$$k('\nabla,\nabla,\nabla')_v t := tr(K('\nabla,\nabla,\nabla')_{\cdot,v} t \quad \text{for} \quad t \in V'_x \quad \text{and} \quad v \in T(M)_x.$$

We call $k('\nabla,\nabla,\nabla')$ the contracted curvature form of the triple $('\nabla,\nabla,\nabla')$. Especially we denote $k(\nabla,\nabla,\nabla)$ by $k(\nabla)$. If $W' = V' = T(M)$ and g is a pseudo-Riemannian metric on M, then $\kappa('\nabla,\nabla,\nabla') \in C(M)$ is defined by

$$\kappa('\nabla,\nabla,\nabla') := tr_g(k('\nabla,\nabla,\nabla')) = \textstyle\sum_{i=1}^n g(e_i,e_i)k('\nabla,\nabla,\nabla')_{e_i}e_i,$$

where $e_1,\cdots,e_n$ is an orthonormal base of $T(M)_x$. Especially we denote $\kappa(\nabla,\nabla,\nabla)$ by $\kappa(\nabla)$.

Remark. If $V = V' = W = W'$ and $\nabla = \nabla' = {}'\nabla$, then $K(\nabla) = K('\nabla,\nabla,\nabla')$ coincides with the curvature form of the general connection $\nabla \in O(V)$, which was defined in [10,1]. If $V' = W' = T(M)$, then $k('\nabla,\nabla,\nabla') \in A^1(T(M)^*)$ is a generalization of the Ricci curvature and $\kappa('\nabla,\nabla,\nabla')$ is a generalization of the scalar curvature.

For a later section, we prepare some formulae in the following special cases:

Proposition 2.3 ([2]).

$$\begin{aligned} &K('\nabla,\nabla,\nabla')\\ &= {}'\nabla \wedge S(\nabla,\nabla') && \text{if} \quad 'P = 0,\\ &= {}'\nabla \wedge \nabla P' + {}'P\nabla \wedge \nabla' - {}'P\nabla_{[\,]}P' && \text{if} \quad P = 0,\\ &= -S('\nabla,\nabla) \wedge \nabla' && \text{if} \quad P' = 0,\\ &= {}'\nabla \wedge \nabla P' && \text{if} \quad 'P = 0 \quad \text{and} \quad P = 0,\\ &= {}'P\nabla \wedge \nabla' && \text{if} \quad P = 0 \quad \text{and} \quad P' = 0,\\ &= -{}'\nabla \wedge P\nabla' && \text{if} \quad 'P = 0 \quad \text{and} \quad P' = 0,\\ &= 0 && \text{if} \quad 'P = 0,\ P = 0 \quad \text{and} \quad P' = 0. \end{aligned}$$

We can generalize the torsion forms of usual connections on the tangent bundle to the case of O-derivative operators from $T(M)$ to another vector bundle over M as follows. Let $\nabla \in O(T(M), W; P)$.

Definition. For X, $Y \in \Gamma(T(M))$, $T(\nabla)_{X,Y} \in \Gamma(W)$ is defined by

$$T(\nabla)_{X,Y} := \nabla_X Y - \nabla_Y X - P([X,Y]).$$

We call $T(\nabla)$ the torsion form of ∇. If $T(\nabla) = 0$, then ∇ is said to be torsion-free.

Remark. In [11], T. Otsuki defined the torsion form of a general connection on the tangent bundle.

Proposition 2.4 **([2]).** *$T(\nabla) \in A^2(W)$ and the mapping $T : O(T(M), W) \to A^2(W)$ is linear. Moreover, we have $T(L\nabla) = LT(\nabla)$ for $L \in HOM(W, W')$.*

Definition. If $W = T(M)$, then $t(\nabla) \in A^1$ is defined by $t(\nabla)_v := tr(T(\nabla)_{\cdot,\, v})$ for $v \in T(M)_x$. We call the 1-form $t(\nabla)$ the contracted torsion form of ∇.

Let $\overline{M}$ be a manifold and $f : M \to \overline{M}$ a mapping. If $\overline{V}$ is a vector bundle over $\overline{M}$, then we denote the induced bundle over M by $f^{\#}\overline{V}$, the bundle map by $f : f^{\#}\overline{V} \to \overline{V}$ and its restriction to the fibre by f_y for $y \in M$. A linear mapping $f^{\#} : \Gamma(\overline{V}) \to \Gamma(f^{\#}\overline{V})$ is defined by $(f^{\#}s)(y) := f_y^{-1}(s(f(y)))$ for $s \in \Gamma(\overline{V})$ and $y \in M$. For $\overline{P} \in HOM(\overline{V}, \overline{W})$, $f^{\#}\overline{P} \in HOM(f^{\#}\overline{V}, f^{\#}\overline{W})$ is defined by requiring $(f^{\#}\overline{P})f^{\#}s = f^{\#}(\overline{P}s)$ for $s \in \Gamma(\overline{V})$. Let $\overline{\nabla} \in O(\overline{V}, \overline{W}; \overline{P})$. We make

Definition. The induced O-derivative operator $f^{\#}\overline{\nabla} \in O(f^{\#}\overline{V}, f^{\#}\overline{W}; f^{\#}\overline{P})$ is defined by requiring $(f^{\#}\overline{\nabla})_v f^{\#}s = f_y^{-1}(\overline{\nabla}_{f_* v} s)$ for each $s \in \Gamma(\overline{V})$, $v \in T(M)_y$ and $y \in M$, where f_* is the differential of the mapping f.

The existence and uniqueness of $f^{\#}\overline{\nabla}$ were proved in [2]. Now we consider the forms S and K of the induced O-derivative operators. Generally, for vector bundle valued forms $\overline{L} \in A^r(\overline{M}, \overline{V})$, we define $f^{\#}\overline{L} \in A^r(M, f^{\#}\overline{V})$ by

$$(f^{\#}\overline{L})_y(v_1, \cdots, v_r) := f_y^{-1}(\overline{L}_{f(y)}(f_* v_1, \cdots, f_* v_r))$$

for $v_i \in T(M)_y$ and $y \in M$. Let $\overline{V}'$ and $\overline{W}'$ be vector bundles over $\overline{M}$, $\overline{\nabla}' \in O(\overline{V}', \overline{V})$ and $'\overline{\nabla} \in O(\overline{W}, \overline{W}')$. Then we have

Proposition 2.5 ([2]).

$$S(f^{\#}\overline{\nabla}, f^{\#}\overline{\nabla}') = f^{\#}(S(\overline{\nabla}, \overline{\nabla}')) \quad in \quad A^1(M, f^{\#}Hom(\overline{V}', \overline{W}))$$

and

$$K(f^{\#\prime}\overline{\nabla}, f^{\#}\overline{\nabla}, f^{\#}\overline{\nabla}') = f^{\#}(K('\overline{\nabla}, \overline{\nabla}, \overline{\nabla}')) \quad in \quad A^2(M, f^{\#}Hom(\overline{V}', \overline{W}')).$$

3. O-derivative operators in submanifold geometry. Let M be an immersed submanifold of a manifold $\overline{M}$ and $f : M \to \overline{M}$ the immersion. Put $\overline{T} = f^{\#}T(\overline{M})$, $T = T(M)$, $\overline{I} = I_{\overline{T}}$, $I = I_T$ and let $i \in HOM(T, \overline{T})$ be the inclusion mapping. A subbundle W will be called a transversal bundle of M in $\overline{T}$ if $\overline{T} = T \oplus W$ (direct sum). Let W be a transversal bundle, $p \in HOM(\overline{T}, T)$, $q \in HOM(\overline{T}, W)$ the projection operators and $j \in HOM(W, \overline{T})$ the inclusion. Put $J = I_W$. Then we see that

$$pi = I, \quad qi = 0, \quad pj = 0 \ \text{ and } \ qj = J.$$

We note that the decomposition of $\overline{T}$ in this section may have no relations with a metric.

For $\overline{\nabla} \in O(T(\overline{M}); \overline{P})$, that is, a covariant derivatives of general connections, we will use the same notations $\overline{P} = f^{\#}\overline{P} \in END(\overline{T})$ and $\overline{\nabla} = f^{\#}\overline{\nabla} \in O(\overline{T}; \overline{P})$ for simplicity. Proposition 2.5 assures us that this convention on notations is also admissible in the study of the forms S and K. Let $\nabla \in O(T; P)$.

Definition. If $\overline{P}i = iP$, we say that (M, ∇) is a submanifold of $(\overline{M}, \overline{\nabla})$ with the extended second fundamental form

$$\overline{B} := \overline{\nabla}i - i\nabla \in O(T, \overline{T}).$$

Choosing a transversal subbundle W of M in $\overline{T}$, we call (M, ∇, W) a submanifold with a transversal bundle in $(\overline{M}, \overline{\nabla})$ if

$$p\overline{B} = 0.$$

Moreover, if $\overline{P}(W) \subset W$, then (M, ∇, W) is said to be adapted in $(\overline{M}, \overline{\nabla})$.

Remark. In the case where $\overline{\nabla}$ and ∇ are usual torsion-free connections, an affine immersion (see [8], for example) is an adapted submanifold with a transversal bundle.

From Proposition 1.1, we have

Proposition 3.1. *If (M, ∇, W) is a submanifold of $(\overline{M}, \overline{\nabla})$, then*

$$\overline{B} = \overline{\nabla} i - i\nabla \in A^1(Hom(T, \overline{T})).$$

Let (M, ∇, W) be a submanifold in $(\overline{M}, \overline{\nabla})$. Note that $\overline{P}(T) \subset T$. For $\overline{P} \in END(\overline{T})$, put

$$E := q\overline{P}i, \quad F := -p\overline{P}j, \quad Q := q\overline{P}j,$$

then we have $E = 0 \in HOM(T, W)$, $F \in HOM(W, T)$, $Q \in END(W)$. For $\overline{\nabla} \in O(\overline{T}; \overline{P})$, put

$$B := q\overline{\nabla}i, \quad A := -p\overline{\nabla}j, \quad D := q\overline{\nabla}j.$$

Then, from Proposition 1.1, we have

Theorem 3.2 ([7,19,9,2]). *If (M, ∇, W) is a submanifold with a transversal bundle, then*

$$\nabla = p\overline{\nabla}i \in O(T; P), \; B = q\overline{B} \in A^1(Hom(T, W)),$$
$$A \in O(W, T; F), \; D \in O(W; Q)$$

and

$$\overline{\nabla}i = i\nabla + \overline{B} = i\nabla + jB \;\; in \;\; O(T, \overline{T}), \; \overline{\nabla}j = -iA + jD \;\; in \;\; O(W, \overline{T}).$$

We call D the transversal connection, B the second fundamental form and A the shape operator of (M, ∇, W) in $(\overline{M}, \overline{\nabla})$. The last two formulae in the above theorem correspond to Gauss' and Weingarten's formulae in usual submanifold geometry.

Corollary 3.3 ([9]). *If (M, ∇, W) is adapted, then*

$$F = 0 \quad and \quad A \in A^1(Hom(W, T)).$$

We denote $A_X\xi$ by $A^\xi X$ for $X \in \Gamma(T)$ and $\xi \in \Gamma(W)$. Then we have $A^\xi \in END(T)$ for $\xi \in \Gamma(W)$.

Definition. If (M, ∇, W) is adapted in $(\overline{M}, \overline{\nabla})$, then the mean curvature covector field $\mu \in \Gamma(W^*)$ is defined to be

$$\mu_x(\xi) := \frac{1}{n} tr(A^\xi) = \frac{1}{n} \sum_{i=1}^n \omega^i(A_{e_i}\xi)$$

for $\xi \in W_x$ and $x \in M$, where $e_1, \cdots, e_n$ is a base of T_x and $\omega^1, \cdots, \omega^n$ is the dual base.

We will study the forms S, K and T defined in §2. Proposition 2.1 and Theorem 3.2 imply

Theorem 3.4.

$$S(\nabla),\ S(A,B) = -FB \in A^1(End(T)),$$
$$S(B,\nabla) = BP,\ S(D,B) = -QB \in A^1(Hom(T,W)),$$
$$S(D),\ S(B,A) = BF \in A^1(End(W)),$$
$$S(A,D),\ S(\nabla,A) \in A^1(Hom(W,T))$$

and

$$pS(\overline{\nabla})i = S(\nabla) - S(A,B) = S(\nabla) + FB,$$
$$qS(\overline{\nabla})i = S(B,\nabla) + S(D,B) = BP - QB,$$
$$qS(\overline{\nabla})j = S(D) - S(B,A) = S(D) - BF,$$
$$pS(\overline{\nabla})j = -S(A,D) - S(\nabla,A).$$

Moreover, from Corollary 3.3, we have

Corollary 3.5. *If (M,∇,W) is adapted, then*

$$S(A,B) = 0,\ S(B,\nabla) = BP,\ S(D,B) = -QB,$$
$$S(B,A) = 0,\ S(A,D) = AQ,\ S(\nabla,A) = -PA$$

and

$$pS(\overline{\nabla})i = S(\nabla),\ qS(\overline{\nabla})i = BP - QB,$$
$$qS(\overline{\nabla})j = S(D),\ pS(\overline{\nabla})j = -AQ + PA.$$

Proposition 2.2, 2.3 and Theorem 3.2 imply

Theorem 3.6.

$$K(\nabla), K(A,B,\nabla) = A \wedge BP + FB \wedge \nabla - FB_{[\ ]}P,$$
$$K(A,D,B) = -S(A,D) \wedge B,\ K(\nabla,A,B)$$
$$= -S(\nabla,A) \wedge B \in A^2(End(T)),$$
$$K(B,\nabla,\nabla) = B \wedge S(\nabla),\ K(D,B,\nabla)$$
$$= D \wedge BP + QB \wedge \nabla - QB_{[\ ]}P,$$

$$K(D,D,B) = -S(D)\wedge B,\ K(B,A,B) = -B\wedge FB \in A^2(Hom(T,W)),$$
$$K(D), K(B,A,D) = B\wedge S(A,D),\ K(B,\nabla,A) = B\wedge S(\nabla,A),$$
$$K(D,B,A) = D\wedge BF + QB\wedge A - QB_{[\,]}F \in A^2(End(W)),$$
$$K(A,D,D),\ K(\nabla,A,D),\ K(\nabla,\nabla,A),$$
$$K(A,B,A) = A\wedge BF + FB\wedge A - FB_{[\,]}F \in A^2(Hom(W,T))$$

and

$$\begin{aligned}
pK(\overline{\nabla})i =& K(\nabla) - K(A,B,\nabla) - K(A,D,B) - K(\nabla,A,B)\\
=& K(\nabla) - A\wedge BP - FB\wedge\nabla + FB_{[\,]}P\\
&+ S(A,D)\wedge B + S(\nabla,A)\wedge B,\\
qK(\overline{\nabla})i =& K(B,\nabla,\nabla) + K(D,B,\nabla) + K(D,D,B) - K(B,A,B)\\
=& B\wedge S(\nabla) + D\wedge BP + QB\wedge\nabla\\
&- QB_{[\,]}P - S(D)\wedge B + B\wedge FB,\\
qK(\overline{\nabla})j =& K(D) - K(B,A,D) - K(B,\nabla,A) - K(D,B,A)\\
=& K(D) - B\wedge S(A,D) - B\wedge S(\nabla,A)\\
&- D\wedge BF - QB\wedge A + QB_{[\,]}F,\\
pK(\overline{\nabla})j =& -K(A,D,D) - K(\nabla,A,D) - K(\nabla,\nabla,A) + K(A,B,A)\\
=& K(A,D,D) - K(\nabla,A,D) - K(\nabla,\nabla,A)\\
&+ A\wedge BF + FB\wedge A - FB_{[\,]}F.
\end{aligned}$$

The last four equations in this theorem correspond to the equations of Gauss, Codazzi and Ricci respectively in usual submanifold geometry. Moreover, from Corollary 3.3, we have

Corollary 3.7 ([2,9]). *If (M,∇,W) is adapted, then*

$$\begin{aligned}
&K(A,B,\nabla) = A\wedge BP, && K(A,D,B) = -AQ\wedge B,\\
&K(\nabla,A,B) = PA\wedge B, && K(B,\nabla,\nabla) = B\wedge S(\nabla),\\
&K(D,B,\nabla) = D\wedge BP + QB\wedge\nabla - QB_{[\,]}P, &&\\
&K(D,D,B) = -S(D)\wedge B, && K(B,A,B) = 0,\\
&K(B,A,D) = B\wedge AQ, && K(B,\nabla,A) = -B\wedge PA,\\
&K(D,B,A) = QB\wedge A, && K(A,D,D) = A\wedge S(D),\\
&K(\nabla,A,D) = \nabla\wedge AQ + PA\wedge D - PA_{[\,]}Q, &&
\end{aligned}$$

$$K(\nabla,\nabla,A) = -S(\nabla)\wedge A,\ \ K(A,B,A) = 0$$

and

$$\begin{aligned}
pK(\overline{\nabla})i &= K(\nabla) - A\wedge BP + AQ\wedge B - PA\wedge B,\\
qK(\overline{\nabla})i &= B\wedge S(\nabla) + D\wedge BP + QB\wedge\nabla - QB_{[\,]}P - S(\nabla)\wedge B,\\
qK(\overline{\nabla})j &= K(D) - B\wedge AQ + B\wedge PA - QB\wedge A,\\
pK(\overline{\nabla})j &= -A\wedge S(\nabla) - \nabla\wedge AQ - PA\wedge D + PA_{[\,]}Q + S(\nabla)\wedge A.
\end{aligned}$$

Hereafter in this paper, the letters X, Y, Z will always denote elements of $\Gamma(T)$, and ξ, η those of $\Gamma(W)$.

Remark. Note that, by Proposition 1.3,

$$D\otimes\nabla^*\otimes\nabla^* : A^1(Hom(T,W)) \to A^1(T^*\otimes Hom(T,W))$$

operates such as

$$((D\otimes\nabla^*\otimes\nabla^*)_X B)_Y Z = D_X B_{PY}PZ - QB_{PY}\nabla_X Z - QB_{\nabla_X Y}PZ.$$

Thus we have

$$\begin{aligned}
&((D\otimes\nabla^*\otimes\nabla^*)_X B)_Y Z - ((D\otimes\nabla^*\otimes\nabla^*)_Y B)_X Z\\
&= (D\wedge B_P P)_{X,Y}Z + (QB_P\wedge\nabla)_{X,Y}Z - QB_{P[X,Y]}PZ - QB_{T(\nabla)_{X,Y}}PZ\\
&= K(D, B_P, \nabla)_{X,Y}Z - QB_{T(\nabla)_{X,Y}}PZ,
\end{aligned}$$

where $B_P \in A^1(Hom(T,W))$ is defined by $(B_P)_X := B_{PX}$.

From Proposition 2.4 and Theorem 3.2 we have the following results on torsion forms.

Proposition 3.8. *$T(\nabla) \in A^2(T)$, $T(\overline{B}) \in A^2(\overline{T})$, $T(B) \in A^2(W)$ and*

$$\begin{aligned}
f^{\#}(T(\overline{\nabla}))_{X,Y} &= T(\overline{\nabla}i)_{X,Y}\\
&= iT(\nabla)_{X,Y} + T(\overline{B})_{X,Y}\\
&= iT(\nabla)_{X,Y} + jT(B)_{X,Y}\\
&= iT(\nabla)_{X,Y} + j(B_X Y - B_Y X).
\end{aligned}$$

Corollary 3.9. *$f^{\#}T(\overline{\nabla}) = iT(\nabla)$ if and only if B is symmetric, that is, $B_X Y = B_Y X$.*

Corollary 3.10 ([9]). *If $\overline{\nabla}$ is torsion-free, then ∇ is torsion-free and B is symmetric.*

4. Submanifold geometry with metrics. At first we make

Definition. If $\nabla \in O(V, W)$ satisfies $(\nabla \otimes \nabla)^* g = 0$ (see Proposition 1.2) for $g \in \Gamma((W \otimes W)^*)$, then ∇ is called a metric O-derivative operator with respect to g. If $g \in \Gamma((W \otimes W)^*)$ is symmetric and non-degenerate on each fibre of W, then g is called a metric on W.

From now on, we denote $(\nabla \otimes \nabla)^*$ by ∇ for $\nabla \in O(V, W)$. For $C \in HOM(V, W)$ and $g \in \Gamma((W \otimes W)^*)$, we define $gC \in \Gamma((V \otimes V)^*)$ by

$$(gC)(s_1, s_2) := g(Cs_1, Cs_2) \text{ for } s_1, s_2 \in \Gamma(V).$$

For $C \in HOM(V, W)$ and metrics $g \in \Gamma((W \otimes W)^*)$ and $h \in \Gamma((V \otimes V)^*)$, we define ${}^tC \in HOM(W, V)$ by requiring

$$h({}^tCs, t) = g(s, Ct) \quad \text{for} \quad s \in \Gamma(W),\ t \in \Gamma(V).$$

Let M be an immersed submanifold of $\overline{M}$ and $f : M \to \overline{M}$ be the immersion. Let (M, ∇) be a submanifold of $(\overline{M}, \overline{\nabla})$ and g be a metric on T.

Definition. The extended mean curvature vector field $\overline{H} \in \Gamma(\overline{T})$ is defined to be

$$\overline{H}_x := \frac{1}{n} tr_g \overline{B} = \frac{1}{n} \sum_{i=1}^{n} g(e_i, e_i) \overline{B}_{e_i} e_i \quad \text{for} \quad x \in M,$$

where $e_1, \cdots, e_n$ is an orthonormal base of T_x.

Let $\overline{g}$ be a metric on $T(\overline{M})$. We use the same notation $\overline{g}$ for the metric $f^{\#}\overline{g}$ on the induced bundle $\overline{T} = f^{\#}T(\overline{M})$. Moreover, we assume that $g = \overline{g}i$ and the take the orthogonal complement $T^{\perp}$ in $\overline{T}$ with respect to $\overline{g}$ as the transversal bundle W and the orthogonal projections as p and q. We call $W = T^{\perp}$ the normal bundle of M.

Definition. If $g = \overline{g}i$ and $p\overline{B} = 0$, we say (M, ∇, g) is a submanifold of $(\overline{M}, \overline{\nabla}, \overline{g})$ and the mean curvature vector field $H \in \Gamma(W)$ is defined to be $H := q\overline{H}$.

Hereafter, in this section, we consider the case where (M, ∇, g) is a submanifold of $(\overline{M}, \overline{\nabla}, \overline{g})$. Put $h := \overline{g}j$, then we have $\overline{g} = gp + hq$.

Theorem 4.1. *If (M, ∇, g) is a submanifold of $(\overline{M}, \overline{\nabla}, \overline{g})$, then we* *have*

$$(\overline{\nabla}_X\overline{g})i = \nabla_X g, \ (\overline{\nabla}_X\overline{g})j = A_X g + D_X h$$

and

$$\begin{aligned}(\overline{\nabla}_X\overline{g})(iY, j\eta) = & -X(g(PY, F\eta)) + g(\nabla_X Y, F\eta) \\ & + g(PY, A_X\eta) - h(B_X Y, Q\eta).\end{aligned}$$

Proof. Proposition 1.2 and Theorem 3.2 imply

$$\begin{aligned}
((\overline{\nabla}_X\overline{g})i)(Y,Z) &= X(\overline{g}(\overline{P}iY, \overline{P}iZ)) - \overline{g}(\overline{\nabla}_X iY, \overline{P}iZ) - \overline{g}(\overline{P}iY, \ \overline{\nabla}_X iZ) \\
&= X(\overline{g}(iPY, iPZ)) - \overline{g}(i\nabla_X Y + jB_X Y, iPZ) \\
&\quad - \overline{g}(iPY, i\nabla_X Z + jB_X Z) \\
&= X(g(PY, PZ)) - g(\nabla_X Y, PZ) - g(PY, \nabla_X Z) \\
&= (\nabla_X g)(Y, Z), \\
((\overline{\nabla}_X\overline{g})j)(\xi,\eta) &= X(\overline{g}(\overline{P}j\xi, \overline{P}j\eta)) - \overline{g}(\overline{\nabla}_X j\xi, \overline{P}j\eta) - \overline{g}(\overline{P}j\xi, \overline{\nabla}_X j\eta) \\
&= X(\overline{g}(-iF\xi, -iF\eta)) + X(\overline{g}(jQ\xi, jQ\eta) \\
&\quad - \overline{g}(-iA_X\xi + jD_X\xi, -iF\eta + jQ\eta) \\
&\quad - \overline{g}(-iF\xi + jQ\xi, -iA_X\eta + jD_X\eta) \\
&= X(g(F\xi, F\eta)) - g(A_X\xi, F\eta) - g(F\xi, A_X\eta) \\
&\quad + X(h(Q\xi, Q\eta)) - h(D_X\xi, Q\eta) - h(Q\xi, D_X\eta) \\
&= (A_X g)(\xi,\eta) + (D_X h)(\xi,\eta), \\
(\overline{\nabla}_X\overline{g})(iY, j\eta) &= X(\overline{g}(\overline{P}iY, \overline{P}j\eta)) - \overline{g}(\overline{\nabla}_X iY, \overline{P}j\eta) - \overline{g}(\overline{P}iY, \overline{\nabla}_X j\eta) \\
&= X(\overline{g}(iPY, -iF\eta)) - \overline{g}(i\nabla_X Y + jB_X Y, -iF\eta + jQ\eta) \\
&\quad - \overline{g}(iPY, -iA_X\eta + jD_X\eta) \\
&= -X(g(PY, F\eta)) + g(\nabla_X Y, F\eta) \\
&\quad + g(PY, A_X\eta) - h(B_X Y, Q\eta).
\end{aligned}$$

Remark. For another proof, see [2].

Let $\overline{\nabla}$ be a metric O-derivative operator with respect to $\overline{g}$.

Corollary 4.2. *If (M, ∇, g) is a submanifold of $(\overline{M}, \overline{\nabla}, \overline{g})$ and $\overline{\nabla}$ is metrical, then we have*

$$\nabla_X g = 0, \ D_X h = -A_X g$$

and $g(PY, A_X\eta) - h(B_X Y, Q\eta) = X(g(PY, F\eta)) - g(\nabla_X Y, F\eta)$.

Moreover, if (M,∇,g) *is an adapted submanifold of* $(\overline{M},\overline{\nabla},\overline{g})$, *then we have*

$$\nabla_X g = 0, \ \ D_X h = 0$$

and $g(PY, A_X\eta) = h(B_XY, Q\eta)$, *that is,* ${}^tPA_X = {}^t B_XQ$.

The last formula in the above corollary corresponds to the well-known one in usual Riemannian submanifold geometry.

Remark. In the case of an adapted submanifold, Corollary 4.2 was partially obtained in [7] and [9].

For the mean curvature vector field $H \in \Gamma(W)$, we have

Corollary 4.3. *If* (M,∇,g) *is an adapted submanifold of* $(\overline{M},\overline{\nabla},\overline{g})$ *and* $\overline{\nabla}$ *is metrical, then we have*

$$h(H,Q\xi) = \frac{1}{n}tr({}^tPA^\xi).$$

Corollary 4.4. *If* (M,∇,g) *is an adapted ubmanifold of* $(\overline{M},\overline{\nabla},\overline{g})$ *and* $\overline{\nabla}$ *is torsion-free and metrical, then we have*

$$g(PY, A^\xi X) = g(PX, A^\xi Y), \ \ \text{that is,} \ \ {}^tPA^\xi = {}^t(A^\xi)P.$$

Remark. For the details of metric general connections, see [14], [3] and [15], where they discussed the existence and uniqueness of metric general connections.

5. Totally $\overline{B}$-umbilical submanifolds. Let (M,∇) be a submanifold in $(\overline{M},\overline{\nabla})$ with the extended second fundamental form $\overline{B}$. Put $\overline{b}_x := dim(Span(\{\overline{B}_vw | v,w \in T_x\}))$ for $x \in M$. If (M,∇,W) is a submanifold with the transversal bundle W, then put $b_x := dim(Span(\{B_vw|v,w \in T_x\}))$ for $x \in M$.

Definition. If $\overline{B} = 0$ at $x \in M$, we say that (M,∇) is $\overline{B}$-geodesic at x. If $\overline{B} = 0$ over M, then (M,∇) is said to be totally $\overline{B}$-geodesic.

Proposition 5.1. *If* (M,∇,W) *is a submanifold with transversal bundle* W, *then* $b_x = \overline{b}_x$ *for* $x \in M$.

Proof. $dim(Span(\{B_vw|v,w \in T_x\})) = dim(jSpan(\{B_vw|v,w \in$

$T_x\}))$ $=$ $dim(Span(j\{B_v w|v, w \in T_x\}))$ $=$ $dim(Span(\{\overline{B}_v w|v, w \in T_x\})) = \overline{b}_x$.

Since the subspace $Span(\{\overline{B}_v w|v, w \in T_x\})$ at $x \in M$ is independent of the choice of the transversal bundle W which satisfies $\nabla = p\overline{\nabla}i$, $dim(Span(\{B_v w|v, w \in T_x\}))$ is also independent of the choice. For totally $\overline{B}$-geodesic submanifold, Theorem 3.4 and 3.6 imply the following theorems:

Theorem 5.2. *If* (M, ∇, W) *is totally* $\overline{B}$*-geodesic, then*

$$S(A,B) = 0,\ S(B,\nabla) = 0,\ S(D,B) = 0,\ S(B,A) = 0$$

and

$$pS(\overline{\nabla})i = S(\nabla),\ qS(\overline{\nabla})i = 0,\ qS(\overline{\nabla})j = S(D).$$

Theorem 5.3. *If* (M, ∇, W) *is totally* $\overline{B}$*-geodesic, then*

$$\begin{aligned} &K(A,B,\nabla)=0,\ K(A,D,B)=0,\ K(\nabla,A,B)=0,\\ &K(B,\nabla,\nabla)=0,\ K(D,B,\nabla)=0,\ K(D,D,B)=0,\\ &K(B,A,B)=0,\ K(B,A,D)=0,\ K(B,\nabla,A)=0,\\ &K(D,B,A)=0,\ K(A,B,A)=0 \end{aligned}$$

and

$$\begin{aligned} pK(\overline{\nabla})i &= K(\nabla),\ qK(\overline{\nabla})i = 0,\ qK(\overline{\nabla})j = K(D),\\ pK(\overline{\nabla})j &= -K(A,D,D) - K(\nabla,A,D) - K(\nabla,\nabla,A). \end{aligned}$$

Let g be a metric on T and $\overline{H}$ the extended mean curvature vector field. We make

Definition. Let (M, ∇) be a submanifold and $\overline{\nabla}$ torsion free. We say that a point $x \in M$ is $\overline{B}$-umbilic point if there exists $\zeta \in \overline{T}_x$ such that

$$\overline{B}_v w = g(v,w)\zeta \text{ forany } v,\ w \in T_x.$$

If (M, ∇) is $\overline{B}$-umbilic at every point, then (M, ∇) is said to be totally $\overline{B}$-umbilic in $(\overline{M}, \overline{\nabla})$.

Proposition 5.4. *If* (M, ∇) *is totally* $\overline{B}$*-umbilical and* $\overline{\nabla}$ *is torsion-free, then* ∇ *is torsion free and*

$$B_X Y = g(X,Y)\overline{H}.$$

From Theorem 3.4, 3.6 and Proposition 5.4, we have the following theorems:

Theorem 5.5. *If (M, ∇, W) is totally $\overline{B}$-umbilical and $\overline{\nabla}$ is torsion-free, then*

$$S(A,B)_X Y = -g(X,Y)FH, \quad S(B,\nabla)_X Y = g(X,PY)H,$$
$$S(D,B)_X Y = -g(X,Y)QH, \quad S(B,A)_X \xi = g(X,F\xi)H,$$

and

$$pS(\overline{\nabla})_X iY = S(\nabla)_X Y + g(X,Y)FH,$$
$$qS(\overline{\nabla})_X iY = g(X,PY)H - g(X,Y)QH,$$
$$qS(\overline{\nabla})_X j\xi = S(D)_X \xi - g(X,F\xi)H.$$

Corollary 5.6. *If (M, ∇, W) is totally $\overline{B}$-umbilical and $\overline{\nabla}$ is torsion-free, then*

$$s(A,B)Y = -g(FH,Y)$$

and

$$s(p\overline{\nabla}, \overline{\nabla}i)Y = s(\nabla)Y + g(FH,Y).$$

Theorem 5.7. *If (M, ∇, W) is totally $\overline{B}$-umbilical and $\overline{\nabla}$ is torsion-free, then*

$$\begin{aligned}
K(A,B,\nabla)_{X,Y}Z &= (Xg(Y,PZ) - Yg(X,PZ) - g(Y,\nabla_X Z) \\
&\quad + g(X,\nabla_Y Z) - g([X,Y],PZ))FH \\
&\quad + g(Y,PZ)A_X H - g(X,PZ)A_Y H, \\
K(A,D,B)_{X,Y}Z &= -g(Y,Z)S(A,D)_X H + g(X,Z)S(A,D)_Y H, \\
K(\nabla,A,B)_{X,Y}Z &= -g(Y,Z)S(\nabla,A)_X H + g(X,Z)S(\nabla,A)_Y H, \\
K(B,\nabla,\nabla)_{X,Y}Z &= (g(X,S(\nabla)_Y Z) - g(Y,S(\nabla)_X Z))H, \\
K(D,B,\nabla)_{X,Y}Z &= (Xg(Y,PZ) - Yg(X,PZ) - g(Y,\nabla_X Z) \\
&\quad + g(X,\nabla_Y Z) - g([X,Y],PZ))QH \\
&\quad + g(Y,PZ)D_X H - g(X,PZ)D_Y H, \\
K(D,D,B)_{X,Y}Z &= -g(Y,Z)S(D)_X H + g(X,Z)S(D)_Y H, \\
K(B,A,B)_{X,Y}Z &= (-g(Y,Z)g(X,FH) + g(X,Z)g(Y,FH))H,
\end{aligned}$$

$$
\begin{aligned}
K(B,A,D)_{X,Y}\xi &= (g(X,S(A,D)_Y\xi) - g(Y,S(A,D)_X\xi))H,\\
K(B,\nabla,A)_{X,Y}\xi &= (g(X,S(\nabla,A)_Y\xi) - g(Y,S(\nabla,A)_X\xi))H,\\
K(D,B,A)_{X,Y}\xi &= (Xg(Y,F\xi) - Yg(X,F\xi) - g(Y,A_X\xi)\\
&\quad + g(X,A_X\xi) - g([X,Y],F\xi))QH\\
&\quad + g(Y,F\xi)D_XH - g(X,F\xi)D_YH,\\
K(A,B,A)_{X,Y}\xi &= (Xg(Y,F\xi) - Yg(X,F\xi) - g(Y,A_X\xi)\\
&\quad + g(X,A_Y\xi) - g([X,Y],F\xi))FH\\
&\quad + g(Y,F\xi)A_XH - g(X,F\xi)A_YH
\end{aligned}
$$

and

$$
\begin{aligned}
pK(\overline{\nabla})_{X,Y}iZ &= K(\nabla)_{X,Y}Z - (Xg(Y,PZ) - Yg(X,PZ)\\
&\quad - g(Y,\nabla_XZ) + g(X,\nabla_YZ) - g([X,Y],PZ))FH\\
&\quad - g(Y,PZ)A_XH + g(X,PZ)A_YH\\
&\quad + g(Y,Z)(S(A,\nabla)_XH + S(\nabla,A)_XH)\\
&\quad - g(X,Z)(S(A,\nabla)_YH + S(\nabla,A)_YH),\\
qK(\overline{\nabla})_{X,Y}iZ &= (g(X,S(\nabla)_YZ) - g(Y,S(\nabla)_XZ)\\
&\quad - g(Y,Z)g(X,FH) + g(X,Z)g(Y,FH))H\\
&\quad + (Xg(Y,PZ) - Yg(X,PZ) + g(Y,\nabla_XZ)\\
&\quad - g(X,\nabla_YZ) - g([X,Y],PZ))QH\\
&\quad + g(Y,PZ)D_XH - g(X,PZ)D_YH\\
&\quad - g(Y,Z)S(D)_XH + g(X,Z)S(D)_YH,\\
qK(\overline{\nabla})_{X,Y}j\xi &= K(D)_{X,Y}\xi - (g(X,S(A,D)_Y\xi) - g(Y,S(A,D)_X\xi)\\
&\quad + g(X,S(\nabla,A)_Y\xi) - g(Y,S(\nabla,A)_X\xi))H\\
&\quad - (X(g(Y,F\xi) - Yg(X,F\xi) - g(Y,A_X\xi)\\
&\quad + g(X,A_Y\xi) - g([X,Y],F\xi))QH\\
&\quad - g(Y,F\xi)D_XH + g(X,F\xi)D_YH,\\
pK(\overline{\nabla})_{X,Y}j\xi &= -K(A,D,D)_{X,Y}\xi - K(\nabla,A,D)_{X,Y}\xi\\
&\quad - K(\nabla,\nabla,A)_{X,Y}\xi + (Xg(Y,F\xi) - Yg(X,F\xi)\\
&\quad - g(Y,A_X\xi) + g(X,A_Y\xi) - g([X,Y],F\xi))FH\\
&\quad + g(Y,F\xi)A_XH - g(X,F\xi)A_YH.
\end{aligned}
$$

Corollary 5.8. *If (M,∇,W) is adapted totally $\overline{B}$-umbilical and $\overline{\nabla}$*

is torsion free, then

$$k(A,B,\nabla)_Y Z = ng(Y,PZ)\mu(H) - g(A_Y H, PZ),$$
$$k(A,D,B)_Y Z = -g(Y,Z)s(A,D)H + g(S(A,D)_Y H, Z),$$
$$k(\nabla,A,B)_Y Z = -g(Y,Z)s(\nabla,A)H + g(S(\nabla,A)_Y H, Z),$$
$$k(A,B,A)_Y \xi = 0$$

and

$$\begin{aligned} k(p\overline{\nabla},\overline{\nabla},\overline{\nabla}i)_Y Z &= k(\nabla)_Y Z - ng(Y,PZ)\mu(H) + g(A_Y H, PZ) \\ &\quad + g(Y,Z)(s(A,\nabla)H + s(\nabla,A)H) \\ &\quad - g(S(A,\nabla)_Y H + S(\nabla,A)_Y H, Z), \\ k(p\overline{\nabla},\overline{\nabla},\overline{\nabla}i)_Y \xi &= -k(A,D,D)_Y\xi - k(\nabla,A,D)_Y\xi - k(\nabla,\nabla,A)_Y\xi, \\ \kappa(p\overline{\nabla},\overline{\nabla},\overline{\nabla}i) &= \kappa(\nabla) - n\mu(H)trP + tr({}^tPA^H) \\ &\quad + (n-1)(s(A,\nabla)H + s(\nabla,A)H). \end{aligned}$$

Let $\overline{g}$ be a metric on $\overline{M}$ and (M,∇,g) be a submanifold of $(\overline{M},\overline{\nabla},\overline{g})$. Corollary 4.2 implies

Proposition 5.9. *If (M,∇,g) is adapted totally $\overline{B}$-umbilical and $\overline{\nabla}$ is torsion-free and metrical, then we have*

$${}^tPA^\xi X = h(H,Q\xi)X.$$

6. Totally A-umbilical submanifolds. Let (M,∇,W) be an adapted submanifold with the transversal bundle W in $(\overline{M},\overline{\nabla})$. We make

Definition. If (M,∇,W) is adapted and $A^\xi v = \mu(\xi)v$ (resp. $A^\xi v = 0$) for any $v \in T_x$, $\xi \in W_x$, we say that (M,∇,W) is A-umbilical (resp. A-geodesic) at x. If (M,∇,W) is A-umbilical (resp. A-geodesic) at every point, then (M,∇,W) is said to be totally A-umbilical (resp. totally A-geodesic) in $(\overline{M},\overline{\nabla})$.

For a totally A-umbilical submanifold, Proposition 2.3, Corollaries 3.5 and 3.7 imply the following theorems:

Theorem 6.1. *If (M,∇,W) is totally A-umbilical, then*

$$S(A,D)_X\xi = A_X Q\xi = \mu(Q\xi)X, \quad S(\nabla,A)_X\xi = -PA_X\xi = -\mu(\xi)PX$$

and

$$pS(\overline{\nabla})_X j\xi = -\mu(Q\xi)X + \mu(\xi)PX.$$

Corollary 6.2. *If (M, ∇, W) is totally A-umbilical, then*

$$s(A, D)\xi = n\mu(Q\xi),\ s(\nabla, A)_X\xi = -\mu(\xi)trP$$

and

$$s(p\overline{\nabla}, \overline{\nabla}j)\xi = -n\mu(Q\xi) + \mu(\xi)trP.$$

Theorem 6.3. *If (M, ∇, W) is totally A-umbilical, then*

$$\begin{aligned}
K(A,B,\nabla)_{X,Y}Z &= (A \wedge BP)_{X,Y}Z = \mu(B_Y PZ)X - \mu(B_X PZ)Y,\\
K(A,D,B)_{X,Y}Z &= -(AQ \wedge B)_{X,Y}Z = -\mu(QB_Y Z)X + \mu(QB_X Z)Y,\\
K(\nabla,A,B)_{X,Y}Z &= (PA \wedge B)_{X,Y}Z = \mu(B_Y Z)PX - \mu(B_X Z)PY,\\
K(B,A,D)_{X,Y}\xi &= (B \wedge AQ)_{X,Y}\xi = \mu(Q\xi)T(B)_{X,Y},\\
K(B,\nabla,A)_{X,Y}\xi &= -(B \wedge PA)_{X,Y}\xi = -\mu(\xi)T(BP)_{X,Y},\\
K(D,B,A)_{X,Y}\xi &= (QB \wedge A)_{X,Y}\xi = \mu(\xi)QT(B)_{X,Y},\\
K(A,D,D)_{X,Y}\xi &= (A \wedge S(D))_{X,Y}\xi = \mu(S(D)_Y\xi)X - \mu(S(D)_X\xi)Y,\\
K(\nabla,A,D)_{X,Y}\xi &= (D^*_X\mu)(\xi)PY - (D^*_Y\mu)(\xi)PX + \mu(Q\xi)T(\nabla)_{X,Y},\\
K(\nabla,\nabla,A)_{X,Y}\xi &= -\mu(\xi)(T(\nabla P)_{X,Y} - PT(\nabla)_{X,Y}),\\
K(A,B,A) &= 0
\end{aligned}$$

and

$$\begin{aligned}
pK(\overline{\nabla})_{X,Y}iZ = &\ K(\nabla)_{X,Y}Z - \mu(B_Y PZ)X + \mu(B_X PZ)Y\\
&+ \mu(QB_Y Z)X - \mu(QB_X Z)Y\\
&- \mu(B_Y Z)PX + \mu(B_X Z)PY,\\
qK(\overline{\nabla})_{X,Y}j\xi = &\ K(D)_{X,Y}\xi - \mu(Q\xi)T(B)_{X,Y}\\
&+ \mu(\xi)(T(BP)_{X,Y} - QT(B)_{X,Y}),\\
pK(\overline{\nabla})_{X,Y}j\xi = &\ -\mu(S(D)_Y\xi)X + \mu(S(D)_X\xi)Y - (D^*_X\mu)(\xi)PY\\
&+ (D^*_Y)\mu(\xi)PX - \mu(Q\xi)T(\nabla)_{X,Y}\\
&+ \mu(\xi)(T(\nabla P)_{X,Y} - PT(\nabla)_{X,Y}).
\end{aligned}$$

For the contracted curvature form, we have

Corollary 6.4. *If (M, ∇, W) is totally A-umbilical, then*

$$\begin{aligned}
k(A, B, \nabla)_Y Z &= (n-1)\mu(B_Y PZ),\\
k(A, D, B)_Y Z &= -(n-1)\mu(QB_Y Z),\\
k(\nabla, A, B)_Y Z &= \mu(B_Y Z)trP - \mu(B_{PY} Z),\\
k(A, D, D)_Y \xi &= (n-1)\mu(S(D)_Y \xi),\\
k(\nabla, A, D)_Y \xi &= (D^*_{PY}\mu)(\xi) - (D^*_Y \mu)(\xi)trP + \mu(Q\xi)t(\nabla)_Y,\\
k(\nabla, \nabla, A)_Y \xi &= -\mu(\xi)(t(\nabla P)_Y - t(P\nabla)_Y),\\
k(A, B, A) &= 0
\end{aligned}$$

and

$$\begin{aligned}
k(p\overline{\nabla}, \overline{\nabla}, \overline{\nabla}i)_Y Z = {} & k(\nabla)_Y Z - (n-1)(\mu(B_Y PZ) - \mu(QB_Y Z))\\
& - \mu(B_Y Z)trP + \mu(B_{PY} Z),\\
k(p\overline{\nabla}, \overline{\nabla}, \overline{\nabla}j)_Y \xi = {} & -(n-1)\mu(S(D)_Y \xi) - (D^*_{PY}\mu)(\xi) + (D^*_Y \mu)(\xi)trP\\
& - \mu(Q\xi)t(\nabla)_Y + \mu(\xi)(t(\nabla P)_Y - t(P\nabla)_Y).
\end{aligned}$$

Corollary 6.5. *If (M, ∇, W) is totally A-umbilical and $\overline{\nabla}$ is torsion-free, then*

$$\begin{aligned}
&K(B, A, D)_{X,Y}\xi = 0, \quad K(D, B, A)_{X,Y}\xi = 0,\\
&K(\nabla, A, D)_{X,Y}\xi = (D^*_X \mu)(\xi)PY - (D^*_Y \mu)(\xi)PX,\\
&K(\nabla, \nabla, A)_{X,Y}\xi = -\mu(\xi)T(\nabla P)_{X,Y}
\end{aligned}$$

and

$$\begin{aligned}
qK(\overline{\nabla})_{X,Y} j\xi = {} & K(D)_{X,Y}\xi + \mu(\xi)T(BP)_{X,Y},\\
pK(\overline{\nabla})_{X,Y} j\xi = {} & -\mu(S(D)_Y \xi)X + \mu(S(D)_X \xi)Y - (D^*_X \mu)(\xi)PY\\
& + (D^*_Y \mu)(\xi)PX + \mu(\xi)T(\nabla P)_{X,Y}.
\end{aligned}$$

Note that Corollary 3.10 implies $T(\nabla P) = T(BP) = 0$ if $\overline{\nabla P}$ is torsion-free. Let g be a metric on M. Corollary 6.4 implies

Corollary 6.6. *If (M, ∇, W) is totally A-umbilical, then*

$$\begin{aligned}
\kappa(p\overline{\nabla}, \overline{\nabla}, \overline{\nabla}i) = {} & \kappa(\nabla) - (n-1)(tr_g(\mu(BP)) - tr_g(\mu(QB)))\\
& - tr_g(\mu(B))trP + tr_g(\mu(B_P)).
\end{aligned}$$

Let $\overline{g}$ be a metric on $\overline{M}$ and (M, ∇, g) be an adapted submanifold of $(\overline{M}, \overline{\nabla}, \overline{g})$. We define $\mu^* \in \Gamma(W)$ by requiring $\overline{g}(\mu^*, \xi) = \mu(\xi)$. For the mean curvature vector field $H \in \Gamma(W)$, Corollary 4.2 implies

Corollary 6.7. *If (M, ∇, g) is totally A-umbilical and $\overline{\nabla}$ is metrical, then we have*

$$ {}^tQB_XY = g(PY, X)\mu^* \text{ and } {}^tQH = \frac{1}{n}\mu^* trP. $$

REFERENCES

[1] N. ABE: General connections on vector bundles, Kodai Math. J. **8**(1985), 322–329.

[2] N. ABE:Geometry of certain first order differential operators and its applications to general connections, Kodai Math. J. **11**(1988), 205–223.

[3] N. ABE: Metric general connections and torsion forms, SUT J. of Math. **26**(1990), 155–167.

[4] N. ABE, H. NEMOTO and S. YAMAGUCHI: On general connections satisfying $\nabla I = \omega \otimes I$, Acta Sci. Math. **53**(1989), 255–266.

[5] A. BEJANCU: Geometry of vertical vector bundle and its applications, Math. Rep. Toyama Univ. **10**(1987), 133–168.

[6] A. BEJANCU and T. OTSUKI: General Finsler connections on a Finsler vector bundle, Kodai Math. J. **10**(1987), 143–152.

[7] C. S. HOUH: Submanifolds in a Riemannian manifolds with general connections, Math. J. Okayama Univ. **12**(1963), 1–37.

[8] K. NOMIZU and U. PINKALL: Cubic form theorem for affine immersions, Results in Math. **13**(1988), 338–362.

[9] H. NEMOTO: On differential geometry of general connections, TRU Math. **21**(1985), 67–94.

[10] T. OTSUKI: Tangent bundles of order 2 and general connections, Math. J. Okayama Univ. **8**(1958), 143–180.

[11] T. OTSUKI: On General connections I, Math. J. Okayama Univ. **9**(1960), 99–164.

[12] T. OTSUKI: On General connections II, Math J. Okayama Univ. **10**(1961), 113–124.

[13] T. OTSUKI: On General connections, Kodai Math. Sem Rep. **13**(1961), 152–166.

[14] T. OTSUKI: On metric general connections, Proc. Japan Acad. **37**(1962), 183–188.

[15] T. OTSUKI: A note on metric general connections, Proc. Japan Acad. **38**(1962), 409–413.

[16] T. OTSUKI: General connections $A\Gamma A$ and the patallelism of Levi-Civita, Kodai Math. Sem. Rep. **14**(1962), 40–52.

[17] T. OTSUKI: On basic curves in spaces with normal general connections, Kodai Math. Sem. Rep. **14**(1962), 110–118.

[18] T. OTSUKI: On curvatures of spaces with normal general connecions I, Kodai Math. Sem. Rep. **15**(1963), 52–61.

[19] T. OTSUKI: On curvatures of spaces with normal general connections II, Kodai Math. Sem. Rep. **15**(1963), 184–194.

[20] T. OTSUKI: A note on general connections, Proc. Japan Acad. **41**(1965), 6–10.
[21] T. OTSUKI: General connections, Math J. Okayama Univ. **32**(1990), 227–242.
[22] T. OTSUKI and C. S. HOUH: Ricci's formula for normal general connections and its applications, Kodai Math. Sem. Rep. **17**(1965), 74–84.

NAOTO ABE
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SCIENCE UNIVERSITY OF TOKYO

HIROAKI NEMOTO
MATHEMATICAL LABORATRY
COLLEGE OF AGRICULTURE AND VETRINARY MEDICINE
NIHON UNIVERSITY

SEIICHI YAMAGUCHI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SCIENCE UNIVERSITY OF TOKYO

(Received September 23, 1992)