

THE CONVERSE OF BADRIKIAN'S THEOREM

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1. Introduction. Let E be a locally convex Hausdorff space and E' be the topological dual of E . Denote by E'_S the dual with the weak * topology $\sigma(E', E)$. Let μ be a cylinder set measure on E' . Then μ is called a continuous cylinder set measure if the characteristic functional

$$\hat{\mu}(x) = \int_{E'} e^{i\langle x, a \rangle} d\mu(a), \quad x \in E$$

is continuous on E .

A family $\{\mu_\alpha\}$ of cylinder set measures on E' is called equicontinuous if the characteristic functionals $\{\hat{\mu}_\alpha(x)\}$ are equicontinuous on E . It is well-known that if E is a nuclear space, then each equicontinuous family $\{\mu_\alpha\}$ is tight on E' , that is, for every $\varepsilon > 0$ there exists $\sigma(E', E)$ -compact set K such that $\mu_\alpha(K) > 1 - \varepsilon$ for every α (Badrikian's theorem, see Badrikian [2], p. 177). We consider the converse problem.

Problem. Let E be a locally convex Hausdorff space. Suppose that each equicontinuous family $\{\mu_\alpha\}$ of cylinder set measures on E' is tight on E'_S . Then is E nuclear? We shall prove the following affirmative answer.

Theorem. Let p be $0 < p < 2$ and E be a barrelled space. If each equicontinuous family of discrete p -stable cylinder set measures on E' is tight on E'_S , then E is nuclear.

2. Preliminaries. The cylinder set measure μ on E' is called a continuous discrete p -stable cylinder set measure on E' if the characteristic functional $\hat{\mu}(x)$ is given by $\hat{\mu}(x) = \exp(-\|T(x)\|_{\ell_p}^p)$, $x \in E$, where $T : E \rightarrow \ell_p$ is a continuous linear operator, $\ell_p = \{(t_n) : \|(t_n)\|_{\ell_p} = (\sum_{n=1}^\infty |t_n|^p)^{1/p} < \infty\}$ and $0 < p \leq 2$. See Linde [6].

Let F, G be normed spaces and $0 < p < \infty$. A linear operator $S : F' \rightarrow G'$ is called p -summing if for every $\{a_n\} \subset F'$ with $\sum_{n=1}^\infty |\langle x, a_n \rangle|^p < \infty$ for every $x \in F$, it holds that $\sum_{n=1}^\infty \|S(a_n)\|_{G'}^p < \infty$. Each p -summing operator is q -summing for $p < q$. See Schwartz [11].

Let E be a locally convex Hausdorff space. For a closed absolutely convex neighborhood U of 0, we set $N(U) = \{x \in E : p_U(x) = 0\}$ where

$p_U(x) = \inf\{t > 0 : x \in tU\}$. Denote by $x(U)$ the equivalence class corresponding to $x \in E$ in the quotient space $E(U) = E/N(U)$. $E(U)$ is a normed space with norm $p[x(U)] = p_U(x)$ for $x \in E$.

For a closed absolutely convex bounded subset A of E , we set $E(A) = \{x \in E : x \in tA \text{ for some } t > 0\}$. $E(A)$ is a linear subspace of E . We put the norm on $E(A)$ by $p_A(x) = \inf\{t > 0 : x \in tA\}$ for $x \in E(A)$.

For a neighborhoods U of 0 in E , the polar $U^\circ = \{a \in E' : |\langle x, a \rangle| \leq 1 \text{ for every } x \in U\}$ is weakly compact absolutely convex subset of E'_S . The normed space $E'(U^\circ)$ is a Banach space and $E(U)' = E'(U^\circ)$ by the duality $\langle x(U), a \rangle = \langle x, a \rangle$.

For zero neighborhoods U, V with $V \subset U$ we define a canonical mapping $E(V, U) : E(V) \rightarrow E(U)$ by associating $x(U)$ with $x(V)$. For two closed absolutely convex bounded subsets A and B with $A \subset B$, it holds that $E(A) \subset E(B)$ and the canonical mapping $E(A, B) : E(A) \rightarrow E(B)$ is defined by $E(A, B)(x) = x$ for $x \in E(A)$.

A locally convex Hausdorff space E is called nuclear if it has a fundamental system $U_F(E)$ of zero neighborhoods which has the following property (see Pietsch [8], 4.1.2):

For each $U \in U_F(E)$ there exists $V \in U_F(E)$ with $V \subset U$ such that the canonical mapping $E'(U^\circ, V^\circ) : E'(U^\circ) \rightarrow E'(V^\circ)$ is 2-summing.

For the theory of locally convex space, we refer to Schaefer [10].

3. Converse of Badrikian's theorem. We consider the converse of Badrikian's theorem as formulated in Section 1.

Lemma 1. *Let E be a barrelled space, F be a Banach space and $0 < p \leq 2$. Let $\pi : E \rightarrow F$ be a continuous linear mapping and $\pi' : F' \rightarrow E'$ be the adjoint of π . Suppose that for each equicontinuous family of discrete p -stable cylinder set measures $\{\mu_\alpha\}$ on F' , the image $\{\pi'(\mu_\alpha)\}$ is $\sigma(E', E)$ -tight on E'_S . Then there is a zero neighborhood U of E such that for each continuous discrete p -stable cylinder set measure μ on F' , $\pi'(\mu)$ is $\sigma(E'(U^\circ), E(U))$ -Radon on $E'(U^\circ)$.*

Remark 1. If the assumption of Lemma 1 is satisfied, then for every continuous p -stable cylinder set measure on F' , by the 0-1 law of the p -stable measure (Dudley and Kanter [3]), there exists a zero neighborhood

$V = V(\mu)$ which depends on each μ such that $\pi'(\mu)(E'(V^\circ)) = 1$. This lemma asserts that V is independent of each continuous p -stable cylinder set measure μ .

Proof. Let $B = \{T : T \text{ is a continuous linear operator from } F \text{ to } \ell_p \text{ with norm } \|T\| \leq 1\}$. Consider the equicontinuous family $\{\mu_T : T \in B\}$, where $\hat{\mu}_T(y) = \exp(-\|T(y)\|_{\ell_p}^p)$, $y \in F$. Then the family $\{\pi'(\mu_T) : T \in B\}$ is $\sigma(E', E)$ -tight. There exists a $\sigma(E', E)$ -compact set K such that $\inf\{\pi'(\mu_T)(K) : T \in B\} > 0$. Since E is barrelled, there exists a zero neighborhood U such that $K \subset U^\circ$. By the 0-1 law (Dudley and Kanter [3]), it follows that $\pi'(\mu_T)(E'(U^\circ)) = 1$ for every $T \in B$. For every continuous p -stable cylinder set measure μ on F' with $\hat{\mu}(y) = \exp(-\|S(y)\|_{\ell_p}^p)$, where $S : F \rightarrow \ell_p$ is a continuous linear operator, we have $\pi'(\mu)(E'(U^\circ)) = 1$. In fact, for μ_1 with $\hat{\mu}_1(y) = \exp(-\|S_1(y)\|_{\ell_p}^p)$ where $S_1(y) = S(y)/\|S\|$, $\|S_1\| = 1$, it holds that $\pi'(\mu_1)(E'(U^\circ)) = 1$. Since $\pi'(\mu)$ is the image of $\pi'(\mu_1)$ by the constant multiple: $a \rightarrow \|S\|a$, we have $\pi'(\mu)(E'(U^\circ)) = 1$. This proves the lemma.

Lemma 2. Let F, G be Banach spaces, $\psi : G \rightarrow F$ be a continuous linear mapping and $\psi' : F' \rightarrow G'$ be the adjoint of ψ . Let $0 < p < 2$ and $(a_i) \subset F'$ be $\sum_{i=1}^\infty |\langle x, a_i \rangle|^p < \infty$ for every $x \in F$. Let μ be a continuous discrete p -stable cylinder set measure on F' with $\hat{\mu}(x) = \exp(-\sum_{i=1}^\infty |\langle x, a_i \rangle|^p)$. Suppose that the image $\psi'(\mu)$ is $\sigma(G', G)$ -Radon on G'_S . Then it holds that $\sum_{i=1}^\infty \|\psi'(a_i)\|_{G'}^p < \infty$.

Proof. We follow Linde [6], Cor. 6.5.2 and Maurey [7], Prop. 2b). For every N let λ_N, τ_N be the cylinder set measures on G' with

$$\begin{aligned} \hat{\lambda}_N(z) &= \exp(-\sum_{n=1}^N |\langle z, \psi'(a_n) \rangle|^p) \\ \hat{\tau}_N(z) &= \exp(-\sum_{n=N+1}^\infty |\langle z, \psi'(a_n) \rangle|^p), \quad z \in G. \end{aligned}$$

Then we have $\lambda_N * \tau_N = \psi'(\mu)$, where $*$ denotes the convolution. By the manner same to Hoffmann-Jørgensen [4], Th. 2.6, it holds that for every $q < p$

$$\int_{G'} \|a\|_{G'}^q d\lambda_N(a) \leq K \int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a),$$

where $K = 2^{1-q}$ for $0 < q < 1$ and $K = 1$ for $q \geq 1$. Let $\{f_n(\omega)\}$ be a sequence of independent identically distributed symmetric p -stable random variables on a probability space (Ω, P) with the characteristic functional

$e^{-|t|^p}$. Let q be fixed such that $0 < q < p$. For every N , we set

$$S_N(\omega) = \sum_{n=1}^N \psi'(a_n) f_n(\omega).$$

S_N is a random variable which values in a finite-dimensional subspace of G' and the distribution of S_N is λ_N . If we set

$$H_N(\omega) = \max_{1 \leq n \leq N} \|\psi'(a_n) f_n(\omega)\|_{G'}$$

then by Kwapien [5], Remark 1, it follows that

$$\begin{aligned} \int_{\Omega} H_N(\omega)^q dP(\omega) &\leq 8C^2 \int_{\Omega} \|S_N(\omega)\|_{G'}^q dP(\omega) \\ &= 8C^2 \int_{G'} \|a\|_{G'}^q d\lambda_N(a), \end{aligned}$$

where $C = 1$ for $0 < q < 1$ and $C = 2^{q-1}$ for $q \geq 1$. Consequently, we have

$$\int_{\Omega} H_N(\omega)^q dP(\omega) \leq 8C^2 K \int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a).$$

Since $\psi'(\mu)$ is a p -stable $\sigma(G', G)$ -Radon measure on G' and $0 < q < p$, we have

$$L = \int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a) < \infty$$

(see de Acosta [1], Linde [6], Cor. 6.7.5). Thus we have

$$\int_{\Omega} \max_{1 \leq n \leq N} \|\psi'(a_n) f_n(\omega)\|_{G'}^q dP(\omega) \leq 8C^2 K L < \infty$$

for every $N = 1, 2, \dots$. Letting $N \rightarrow \infty$, we have

$$\int_{\Omega} \sup_n \|\psi'(a_n f_n(\omega))\|_{G'}^q dP(\omega) < \infty.$$

Hence there exists $R > 0$ such that

$$\begin{aligned} &P(\omega : \sup_n \|\psi'(a_n f_n(\omega))\|_{G'} \leq R) \\ &= \prod_{n=1}^{\infty} \{1 - P(\omega : |f_n(\omega)| > R/\|\psi'(a_n)\|_{G'})\} > 0, \end{aligned}$$

where we have used the independence of $\{f_n(\omega)\}$. This implies that

$$\sum_{n=1}^{\infty} P(\omega : |f_n(\omega)| > R/\|\psi'(a_n)\|_{G'}) < \infty.$$

We remark that for every n ,

$$\int_{\Omega} \|\psi'(a_n)f_n(\omega)\|_{G'}^q dP(\omega) = \|\psi'(a_n)\|_{G'}^q \int_{\Omega} |f_n(\omega)|^q dP(\omega) \leq 8C^2KL,$$

that is, $\sup_n \|\psi'(a_n)\|_{G'} < \infty$. Furthermore, it is known that $P(\omega : |f_n(\omega)| > t) \sim t^{-p}$ as $t \rightarrow \infty$, so we obtain for sufficiently large R ,

$$P(\omega : |f_n(\omega)| > R/\|\psi'(a_n)\|_{G'}) \sim \|\psi'(a_n)\|_{G'}^p/R^p.$$

Hence it follows that $\sum_{n=1}^{\infty} \|\psi'(a_n)\|_{G'}^p < \infty$.

Lemma 3. *Let F, G be Banach spaces and $0 < p < 2$. Let $\psi : G \rightarrow F$ be a continuous linear mapping and $\psi' : F' \rightarrow G'$ be the adjoint of ψ . Suppose that for every continuous discrete p -stable cylinder set measure μ on F' , the image $\psi'(\mu)$ is $\sigma(G', G)$ -Radon. Then $\psi' : F' \rightarrow G'$ is p -summing.*

Proof. For each $(a_i) \subset F'$ with $\sum_{i=1}^{\infty} |\langle y, a_i \rangle|^p < \infty$ for every y in F , let μ be a continuous p -stable cylinder set measure on F' satisfying $\hat{\mu}(y) = \exp(-\sum_{i=1}^{\infty} |\langle y, a_i \rangle|^p)$. Then $\psi'(\mu)$ is $\sigma(G', G)$ -Radon. By Lemma 2, it follows that $\sum_{i=1}^{\infty} \|\psi'(a_i)\|_{G'}^p < \infty$, which proves the assertion.

Theorem 1. *Let E be a barrelled space and p be $0 < p < 2$. Suppose that each equicontinuous family of discrete p -stable cylinder set measures on E'_S is $\sigma(E', E)$ -tight. Then E is nuclear.*

Proof. Let $\{U_{\alpha}\}$ be a fundamental system of zero neighborhoods of E . Let α be arbitrarily fixed. For every continuous linear operator $T : E(U_{\alpha}) \rightarrow \ell_p$, let μ_T be the corresponding continuous discrete p -stable cylinder set measure on $E'(U_{\alpha}^{\circ})$ with $\hat{\mu}_T(x) = \exp(-\|T(x)\|_{\ell_p}^p)$, $x \in E(U_{\alpha})$. Let $\tau_{\alpha} : E'(U_{\alpha}^{\circ}) \rightarrow E'$ be the canonical injection. Then by the assumption, image $\tau_{\alpha}(\mu_T)$ is $\sigma(E', E)$ -Radon on E'_S . By Lemma 1, there exists $\beta = \beta(\alpha)$ with $U_{\beta} \subset U_{\alpha}$ (depends only on α , not on each T) such that for every continuous p -stable cylinder set measure μ on $E'(U_{\alpha}^{\circ})$, $\tau_{\alpha}(\mu)$ is Radon on $E'(U_{\beta}^{\circ})$. By Lemma 3, the canonical injection $E'(U_{\alpha}^{\circ}, U_{\beta}^{\circ})$ is p -summing and also 2-summing since $p < 2$. This completes the proof.

Remark 2. In Theorem 1, the barrelledness is essential. Let τ_S be the Sazonov topology on the infinite-dimensional Hilbert space H . Then

τ_S is not nuclear but each τ_S -equicontinuous family of cylinder set measures is tight, see Sazonov [9].

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