Math. J. Okayama Univ. 35(1993), 193-228

THE QUASI KO_*-TYPES OF THE STUNTED MOD 4 LENS SPACES

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1. Introduction Let KU and KO be the complex and the real K-spectrum respectively. For any CW-spectrum X its KU-homology KU_*X is regarded as a ($Z/2$-graded) abelian group with involution because the complex K-spectrum KU possesses the conjugation $\psi_C^{-1} : KU \to KU$. Given CW-spectra X and Y we say that X is *quasi KO_*-equivalent* to Y if there exists an equivalence $\tilde{f} : KO \wedge X \to KO \wedge Y$ of KO-module spectra (see [10]). If X is quasi KO_*-equivalent to Y, then KO_*X is isomorphic to KO_*Y as an KO_*-module, and in addition KU_*X is isomorphic to KU_*Y as an abelian group with involution. In [12] and [13] we have completely determined the quasi KO_*-types of the real projective space RP^n and its stunted projective space $RP^n_{m+1} = RP^n/RP^m$. In this paper we are interested in the standard mod 4 lens space $L^n(4)$ instead of the real projective space RP^n. Our purpose of this paper is to determine the quasi KO_*-types of the mod 4 lens space L^n and its stunted lens space $L^n_{m+1} = L^n/L^m$ along the line of [12], [13] or [15], where we simply denote by L^{2k+1} the $(2k+1)$-dimensional standard mod 4 lens space $L^k(4)$ and by L^{2k} its $2k$-skeleton $L^k_0(4)$.

Let $SZ/2^r (r \geq 1)$ be the Moore spectrum of type $Z/2^r$, and $i : \Sigma^0 \to SZ/2^r$ and $j : SZ/2^r \to \Sigma^1$ be the bottom cell inclusion and the top cell projection respectively. The stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$ of order 2 adomits an extension $\bar{\eta} : \Sigma^1 SZ/2^r \to \Sigma^0$ and a coextension $\tilde{\eta} : \Sigma^2 \to SZ/2^r$ satisfying $\bar{\eta} i = \eta$ and $j\tilde{\eta} = \eta$. In [10] and [11] we introduced some elementary spectra M_r, P_r, MP_r, $V_{r,s}$, $V'_{r,s}$, $W_{r,s}$ and so on $(r, s \geq 1)$ constructed as the cofibers of the following maps respectively: $i\eta : \Sigma^1 \to SZ/2^r$, $\tilde{\eta} : \Sigma^2 \to SZ/2^r$, $i\eta \vee \tilde{\eta} : \Sigma^1 \vee \Sigma^2 \to SZ/2^r$, $i\bar{\eta} : \Sigma^1 SZ/2^s \to SZ/2^r$, $\tilde{\eta} j : \Sigma^1 SZ/2^s \to SZ/2^r$, $i\bar{\eta} + \tilde{\eta} j : \Sigma^1 SZ/2^s \to SZ/2^r$ and so on, although these elementary spectra X_r and $X_{r,s}$ were written to be X_{2^r} and $X_{2^r,2^s}$. In particular we note that the elementary spectrum $W_{r,r}$ coincides with the smash product $P \wedge SZ/2^r$ where P denotes the cofiber of the stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$. In this paper we moreover introduce some new small spectra $U_{r,t,s}$, $V_{r,t,s}$, $MU_{r,t,s}$, $PU_{r,t,s}$ and so on $(r, t, s \geq 1)$ constructed as the cofibers of the following maps respectively: $(i\bar{\eta}, \tilde{\eta} j)$:

$\Sigma^1 SZ/2^s \to SZ/2^r \vee SZ/2^t$, $(i\bar{\eta}, i_V\bar{\eta}j) : \Sigma^1 SZ/2^s \to SZ/2^r \vee V_t$, $i\eta \vee \bar{\eta}jj_V$: $\Sigma^1 \vee \Sigma^{-1}V_{r,s} \to SZ/2^t$, $\bar{\eta} \vee i\bar{\eta}j'_V$: $\Sigma^2 \vee \Sigma^{-1}V'_{t,s} \to SZ/2^r$ and so on, where $V_t = V_{t-1,1}$ and $i_V : SZ/2^{t-1} \to V_t$ is the canonical inclusion, and $j_V : V_{r,s} \to \Sigma^2 SZ/2^s$ and $j'_V : V'_{t,s} \to \Sigma^2 SZ/2^s$ are the canonical projections.

Given CW-spectra X and Y we say that X has *the same $\mathscr{C}$-type* as Y if KU_*X is isomorphic to KU_*Y as an abelian group with involution (cf. [3, 4.1]). Dualizing the KU-cohomology KU^*L^n with the conjugation ψ_C^{-1} calculated in [5] (or [7]) we can observe the $\mathscr{C}$-type of the mod 4 lens space L^n (Proposition 5.1).

Proposition 1. *The suspended mod 4 lens space $\Sigma^1 L^n (n \geq 2)$ has the same $\mathscr{C}$-type as the following small spectrum : $U_{t-1,2t+1,t}$, $MU_{t-1,2t+1,t}$, $SZ/2^t \vee W_{2t+1,t+1}$, $\Sigma^0 \vee SZ/2^t \vee W_{2t+1,t+1}$ according as $n = 4t$, $4t+1$, $4t+2$, $4t+3$. Here $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$.*

More generally we can observe the $\mathscr{C}$-type of the stunted mod 4 lens spaces L^n_{m+1} (Corollary 5.3, Proposition 5.4 and (5.12)).

Proposition 2. i) *The stunted mod 4 lens spaces L^{4m+n}_{4m+1} and L^{4m+n}_{4n} have the same $\mathscr{C}$-types as L^n and $\Sigma^0 \vee L^n$ respectively.*

ii) *The suspended stunted mod 4 lens space $\Sigma^1 L^{4m+n+2}_{4m+3} (n \geq 2)$ has the same $\mathscr{C}$-type as the following small spectrum : $U_{2t,t,t}$, $\Sigma^0 \vee U_{2t,t,t}$, $SZ/2^{2t+2} \vee W_{t,t}$, $M_{2t+2} \vee W_{t,t}$ according as $n = 4t$, $4t+1$, $4t+2$, $4t+3$.*

iii) *The stunted mod 4 lens space $L^{4m+n+2}_{4m+2} (n \geq 1)$ has the same $\mathscr{C}$-type as $\Sigma^2 \vee L^{4m+n+2}_{4m+3}$.*

For a CW-spectrum X having the same $\mathscr{C}$-type as one of the small spectrum appearing in Propositions 1 and 2 ii) we can determine its quasi KO_*-type by developing the same method as adopted in [10] or [11] (Proposition 3.1 and Theorem 3.3). Applying this result to the mod 4 lens space L^n we can easily determine its quasi KO_*-type (cf. [4] and [9]).

Theorem 3. *The suspended mod 4 lens space $\Sigma^1 L^n (n \geq 2)$ is quasi KO_*-equivalent to the following small spectrum : $U_{2r-1,4r+1,2r}$, $MU_{2r-1,4r+1,2r}$, $V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^4 \vee V_{2r} \vee W_{4r+1,2r+1}$, $V_{2r,4r+3,2r+1}$, $MU_{2r,4r+3,2r+1}$, $SZ/2^{2r+1} \vee W_{4r+3,2r+2}$, $\Sigma^0 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ according as $n = 8r$, $8r+1$, $\cdots$, $8r+7$. Here $V_0 \vee W_{1,1}$ should be replaced* *by* $\Sigma^2 SZ/4$.

In order to investigate the quasi KO_*-types of the stunted mod 4 lens spaces L^n_{m+1} in general, we discuss separately in the following three cases (cf. [15]): i) $L^{2k+n}_{2k+1} (n \geq 2)$, ii) $L^{2k+2\ell}_{2k} (\ell \geq 1)$ and iii) $L^{2k+2\ell+1}_{2k} (\ell \geq 0)$. By a quite similar argument to the non-stunted case we can also determine the quasi KO_*-types of L^{2k+n}_{2k+1} and $DL^{2k+2\ell}_{2k}$ where DX denotes the S-dual of X (Theorem 5.8). Dualizing our result for $DL^{2k+2\ell}_{2k}$ we can immediately determine the quasi KO_*-type of $L^{2k+2\ell}_{2k}$ (Theorem 5.9). In order to establish the rest case we construct certain maps $f_{k,\ell} : Y_{k,\ell} \to X_{k,\ell}$ modelled on the bottom cell inclusions $i : \Sigma^{2k-4m+2} \to \Sigma^{-4m+1} L^{2k+2\ell+1}_{2k+1}$ with $k = 2m$ or $2m-1$, and then prove that the cofiber of each map $f_{k,\ell}$ has the same quasi KO_*-type as $\Sigma^{-4m+1} L^{2k+2\ell+1}_{2k+2}$. Using this fact we can determine the quasi KO_*-type of $L^{2k+2\ell+1}_{2k}$, too (Theorem 5.11). Consequently we can obtain the following main result (cf. [6, Theorem 2] and [8, Theorem 2]).

Theorem 4. i) $\Sigma^{-4m} L^{4m+n}_{4m+1}$ *is quasi* KO_**-equivalent to* L^n.

ii) $\Sigma^{-4m} L^{4m+n}_{4m}$ *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^0 \vee L^n$.

iii) $\Sigma^{-4m+1} L^{4m+n-2}_{4m-1} (n \geq 2)$ *is quasi* KO_**-equivalent to the following small spectrum* : $U_{4r,2r,2r}$, $\Sigma^0 \vee U_{4r,2r,2r}$, $SZ/2^{4r+2} \vee W_{2r,2r}$, $M_{4r+2} \vee W_{2r,2r}$, $V_{4r+2,2r+1,2r+1}$, $\Sigma^4 \vee V_{4r+2,2r+1,2r+1}$, $V_{4r+4} \vee W_{2r+1,2r+1}$, $M_{4r+4} \vee W_{2r+1,2r+1}$ *according as* $n = 8r, 8r+1, \cdots, 8r+7$.

iv) $\Sigma^{-4m+1} L^{4m+n-2}_{4m-2} (n \geq 1)$ *is quasi* KO_**-equivalent to the following small spectrum* : $PU_{4r+1,2r,2r}$, $\Sigma^0 \vee PU_{4r+1,2r,2r}$, $P_{4r+3} \vee W_{2r,2r}$, $\Sigma^4 MP_{4r+3,2r+1,2r+1}$, $\Sigma^4 PU_{4r+3,2r+1,2r+1}$, $\Sigma^4 \vee \Sigma^4 PU_{4r+3,2r+1,2r+1}$, $\Sigma^4 P_{4r+5} \vee W_{2r+1,2r+1}$, $\Sigma^4 MP_{4r+5} \vee W_{2r+1,2r+1}$ *according as* $n = 8r, 8r+1, \cdots, 8r+7$ *where* $PU_{1,0,0} = \Sigma^{-1}$.

Let γ be the canonical complex line bundle over $L^{2k+1} = L^k(4)$ or its restriction to $L^{2k} = L^k_0(4)$, and $r\gamma$ denote its realification. The $4m$-dimensional real vector bundle $2mr\gamma$ over L^n is KO-orientable and its Thom complex $T(2mr\gamma)$ is homeomorphic to the stunted mod 4 lens space L^{4m+n}_{4m}. So we remark that Theorem 4 ii) may be proved by means of [13, (3.8)] in a different way from ours.

This paper is organized as follows. In §2 we introduce some new small spectra $U_{r,t,s}$, $V_{r,t,s}$, $MU_{r,t,s}$, $PU_{r,t,s}$, $R'U_{r,t,s}$ and so on, and compute their KU-homologies with the conjugation ψ_C^{-1} and their KO-homologies. In

§3 we determine its quasi KO_*-type for a CW-spectrum X having the same $\mathscr{C}$-type as $U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee U_{r,t,s}$ (Theorem 3.3). In §4 we consider several maps $f : \Sigma^{2i} \to X$ to construct desired maps $f_{k,\ell} : Y_{k,\ell} \to X_{k,\ell}$, and study their cofibers $C(f)$ and their induced homomorphisms $f_* : KU_0\Sigma^{2i} \to KU_0X$. In §5 we first investigate the behavior of the conjugations ψ_C^{-1} on $KU^*L^n_{m+1}$ and $KU_*L^n_{m+1}$, and then prove our main result separetely in the three case as stated above (Theorems 5.8, 5.9 and 5.11).

2. Small spectra $U_{r,t,s}$, $V_{r,t,s}$, $MU_{r,t,s}$ and $PU_{r,t,s}$

2.1. We first construct small spectra $U_{r,s+1}$ and $U'_{r+1,s} (r, s \geq 1)$ as the cofibers of following maps respectively:

$$(i\tilde{\eta}, \tilde{\eta}j) : \Sigma^1 SZ/2^s \to SZ/2^r \vee SZ/2 \quad \text{and}$$
$$i\tilde{\eta} \vee \tilde{\eta}j : \Sigma^1 SZ/2 \vee \Sigma^1 SZ/2^s \to SZ/2^r.$$

According to [11, Lemma 1.1] these new spectra $U_{r,s+1}$ and $U'_{r+1,s}$ may be given as the cofibers of the following compsite maps

$$i\tilde{\eta}j'_V : \Sigma^{-1}V'_{s+1} \to SZ/2^r \quad \text{and} \quad i_V\tilde{\eta}j : \Sigma^1 SZ/2^s \to V_{r+1}$$

respectively where $V'_{s+1} = V'_{1,s}$, $V_{r+1} = V_{r,1}$ and $j'_V : V'_{s+1} \to \Sigma^2 SZ/2^s$ and $i_V : SZ/2^r \to V_{r+1}$ are the canonical projection and inclusion. For the convenience' sake we set $U_{r,1} = SZ/2^{r+1}$ and $U'_{1,s} = \Sigma^2 SZ/2^{s+1}$. Evidently there holds the S-duality $U'_{r,s} = \Sigma^3 DU_{s,r}$. It is easily seen that these new spectra $U_{r,s}$ and $U'_{r,s}$ have the same $\mathscr{C}$-type as $V_{r,s}$ and $V'_{r,s}$ respectively. As is implicitly established in [10, Theorem 5.2] or [11, Thorem 4.2], we can observe more precisely that

(2.1) *$U_{r,s}$ and $U'_{r,s}$ have the same quasi KO_*-types as $\Sigma^6 V_{s-1,r+1}$ and $\Sigma^2 V'_{s+1,r-1}$ respectively, where $V_{0,t} = \Sigma^2 SZ/2^t$ and $V'_{t,0} = SZ/2^t$.*

We here introduce new small spectra $U_{r,t,s}$, $U'_{r,t,s}$, $V_{r,t+1,s}$, $V'_{r,t+1,s} (r, t, s \geq 1)$ constructed as the cofibers of the following maps respectively:

$$(2.2) \quad \begin{aligned} (i\tilde{\eta}, \tilde{\eta}j) &: \Sigma^1 SZ/2^s \to SZ/2^r \vee SZ/2^t, \\ i\tilde{\eta} \vee \tilde{\eta}j &: \Sigma^1 SZ/2^t \vee \Sigma^1 SZ/2^s \to SZ/2^r, \\ (i\tilde{\eta}, i_V\tilde{\eta}j) &: \Sigma^1 SZ/2^s \to SZ/2^r \vee V_{t+1}, \\ i\tilde{\eta}j'_V \vee \tilde{\eta}j &: \Sigma^{-1}V'_{t+1} \vee \Sigma^1 SZ/2^s \to SZ/2^r. \end{aligned}$$

Of course the new spectra $U_{r,1,s}$ and $U'_{r,1,s}$ coincide with the previous elementary spectra $U_{r,s+1}$ and $U'_{r+1,s}$ respectively. For the convenience' sake we set $V_{r,1,s} = V_{r,s+1}$ and $V'_{r,1,s} = V'_{r+1,s}$, and in addition $U_{r,0,s} = U'_{r,s,0} = V_{r,0,s} = V_{r,s}$, $U_{0,r,s} = U'_{r,0,s} = V'_{r,0,s} = V'_{r,s}$, $V_{0,r,s} = U'_{r,s}$ and $V'_{r,s,0} = U_{r,s}$. Evidently there hold the S-dualities $U_{r,t,s} = \Sigma^3 DU'_{s,t,r}$ and $V_{r,t,s} = \Sigma^3 DV'_{s,t,r}$. By a routine argument we can easily compute the KU-homologies with the conjugation ψ_C^{-1} and the KO-homologies of these new spectra.

Proposition 2.1. *When $X = U_{r,t,s}$, $V_{r,t,s}$, $U'_{r,t,s}$ and $V'_{r,t,s} (r,t,s \geq 1)$, $KU_1X = 0$ and KU_0X with the conjugation ψ_C^{-1} are given as follows:*

i) *"The $X = U_{r,t,s}$ or $V_{r,t,s}$ case"*

		$r<s<t$	$r<s\geq t$
KU_0X	$\cong$	$Z/2^t \oplus Z/2^s \oplus Z/2^r$	$Z/2^{s+1} \oplus Z/2^{t-1} \oplus Z/2^r$
ψ_C^{-1}	$=$	$\begin{pmatrix} 1 & 2^{t-s} & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -2^{s-t+2} & 0 \\ 0 & 1 & 0 \\ -1 & -2^{s-t+1} & 1 \end{pmatrix}$
		$r\geq s<t$	$r\geq s\geq t$
KU_0X	$\cong$	$Z/2^t \oplus Z/2^{r+1} \oplus Z/2^{s-1}$	$Z/2^{r+1} \oplus Z/2^s \oplus Z/2^{t-1}$
ψ_C^{-1}	$=$	$\begin{pmatrix} 1 & -2^{t-s} & 2^{t-s+1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -2^{s-t+1} \\ 0 & 0 & 1 \end{pmatrix}$

ii) *"The $X = U'_{r,t,s}$ or $V'_{r,t,s}$ case"*

		$s<r<t$	$s<r\geq t$
KU_0X	$\cong$	$Z/2^t \oplus Z/2^r \oplus Z/2^s$	$Z/2^{r+1} \oplus Z/2^{t-1} \oplus Z/2^s$
ψ_C^{-1}	$=$	$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 2^{r-s} \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 2^{r-s+1} \\ 1 & -1 & 2^{r-s} \\ 0 & 0 & -1 \end{pmatrix}$
		$s\geq r<t$	$s\geq r\geq t$
KU_0X	$\cong$	$Z/2^t \oplus Z/2^{s+1} \oplus Z/2^{r-1}$	$Z/2^{s+1} \oplus Z/2^r \oplus Z/2^{t-1}$
ψ_C^{-1}	$=$	$\begin{pmatrix} -1 & 0 & 0 \\ 2^{s-r+1} & -1 & -2^{s-r+2} \\ -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -2^{s-r+1} & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

Proposition 2.2. *For the small spectra $X = U_{r,t,s}$, $V_{r,t,s}$, $U'_{r,t,s}$ and $V'_{r,t,s}(r,t,s \geq 1)$ their KO-homologies $KO_iX (0 \leq i \leq 7)$ are tabled as follows :*

$X\backslash i$ =	0	1	2	3
$U_{r,t,s}$	$Z/2^r \oplus Z/2^t$	$Z/2$	$(*)_{s-1,t} \oplus Z/2$	$Z/2$
$V_{r,t,s}$	$Z/2^r \oplus Z/2^{t-1}$	0	$Z/2^s \oplus Z/2$	$Z/2$
$U'_{r,t,s}$	$Z/2^r$	0	$Z/2^{t-1} \oplus Z/2^{s+1}$	$Z/2$
$V'_{r,t,s}$	$(*)_{r-1,t}$	$Z/2$	$Z/2^t \oplus Z/2^{s+1}$	$Z/2$
	4	5	6	7
$U_{r,t,s}$	$Z/2^{r+1} \oplus Z/2^{t-1}$	0	$Z/2^s$	0
$V_{r,t,s}$	$Z/2^{r+1} \oplus Z/2^t$	$Z/2$	$(*)_{s-1,t}$	0
$U'_{r,t,s}$	$(*)_{r-1,t} \oplus Z/2$	$Z/2$	$Z/2^t \oplus Z/2^s$	0
$V'_{r,t,s}$	$Z/2^r \oplus Z/2$	0	$Z/2^{t-1} \oplus Z/2^s$	0

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,\ell} \cong Z/2^{k+1} \oplus Z/2$ *if* $\ell \geq 2$.

2.2. Choose a map $k_M : M_s \to \Sigma^1$ satisfying $k_M i_M = j : SZ/2^s \to \Sigma^1$ and $2^s k_M = \eta j_M : M_s \to \Sigma^1$ such that the sequence

$$\Sigma^0 \overset{2^s i_P}{\to} P \overset{i_{P,M}}{\to} M_s \overset{k_M}{\to} \Sigma^1 \tag{2.3}$$

is a cofiber sequence where $i_P : \Sigma^0 \to P$ is the bottom cell inclusion. For the convenience' sake we set $M_0 = \Sigma^2$ and $k_M = \eta : M_0 \to \Sigma^1$. It is easily seen that $[M_s, \Sigma^1] \cong Z/2^{s+1}$ which is generated by the map k_M for any $s \geq 0$. We here introduce new small spectra $MV'_{t,s}$, $QV_{r,t}$, $QU_{r,t+1}$, $V'M_{r,s}$, $MU_{r,t,s}$, $U'M_{r,t,s}$ and $V'M_{r,t+1,s} (r, s, t \geq 1)$ constructed as the cofibers of the following maps respectively:

$$\begin{aligned} i\eta \vee \tilde{\eta} j &: \Sigma^1 \vee \Sigma^1 SZ/2^s \to SZ/2^t, \\ \tilde{\eta}\eta \vee i\bar{\eta} &: \Sigma^3 \vee \Sigma^1 SZ/2^t \to SZ/2^r, \\ \tilde{\eta}\eta \vee i\bar{\eta} j'_V &: \Sigma^3 \vee \Sigma^{-1} V'_{t+1} \to SZ/2^r, \\ \tilde{\eta} k_M &: \Sigma^1 M_s \to SZ/2^r, \\ i\eta \vee \tilde{\eta} j j_V &: \Sigma^1 \vee \Sigma^{-1} V_{r,s} \to SZ/2^t, \\ i_V \tilde{\eta} k_M &: \Sigma^1 M_s \to V_{r,t}, \\ i_U \tilde{\eta} k_M &: \Sigma^1 M_s \to U_{r,t+1}. \end{aligned} \tag{2.4}$$

We moreover choose another map $h'_V : \Sigma^2 \to V'_{r,s}$ satisfying $j'_V h'_V = i : \Sigma^0 \to SZ/2^s$ and $2^s h'_V = i'_V \bar{\eta} : \Sigma^2 \to V'_{r,s}$ such that the sequence

$$\Sigma^2 \overset{h'_V}{\to} V'_{r,s} \overset{j'_{V,P}}{\to} P_r \overset{2^s j_P}{\to} \Sigma^3 \tag{2.5}$$

is a cofiber sequence where $j_P : P_r \to \Sigma^3$ is the top cell projection. Notice that $[\Sigma^2, V'_{1,s}] \cong Z/2^{s+2}$ which is generated by the map h'_V, and $[\Sigma^2, V'_{r,s}] \cong$

$Z/2^{s+1} \oplus Z/2$ whose direct summands are generated by the maps h'_V and $i'_V i\eta^2$ whenever $r \geq 2$ (cf. (1.4)). As is easily checked, the cofibers of the maps $i_V\tilde{\eta} : \Sigma^2 \to V_{r,t}$ and $i_U\tilde{\eta} : \Sigma^2 \to U_{r,t}$ coincide with those of the maps $i'_{V,U}h'_V : \Sigma^2 \to U'_{r,t,s}$ and $i'_{V,V}h'_V : \Sigma^2 \to V'_{r,t,s}$. Therefore the above new spectra $V'M_{r,s}$, $U'M_{r,t,s}$ and $V'M_{r,t+1,s}$ may be given as the cofibers of the following maps respectively:
$h'_V\eta : \Sigma^3 \to V'_{r,s}$, $i'_{V,U}h'_V\eta : \Sigma^3 \to U'_{r,t,s}$ and $i'_{V,V}h'_V\eta : \Sigma^3 \to V'_{r,t+1,s}$.

For the convenience' sake we set $QU_{r,1} = Q_{r+1}$, $V'M_{r,1,s} = V'M_{r+1,s}$, $V'M_{r,0} = Q_r$, $MU_{0,t,s} = MV'_{t,s}$, $U'M_{r,t,0} = QV_{r,t}$ and $V'M_{r,t,0} = QU_{r,t}$ where Q_r donotes the cofiber of the map $\tilde{\eta}\eta : \Sigma^3 \to SZ/2^r$.

Similarly to Propositions 2.1 and 2.2 we can easily compute the KU-homologies with the conjugation ψ_C^{-1} and the KO-homologies of these new spectra.

Proposition 2.3. *When $X = V'M_{r,s}$, $MU_{r,t,s}$, $U'M_{r,t,s}$ and $V'M_{r,t,s}$, $KU_1X = 0$ and KU_0X with the conjugation ψ_C^{-1} are given as follows :*

i) *"The $X = V'M_{r,s}(r \geq 1$ and $s \geq 0)$ case"*

$$\begin{array}{lccc} & & s < r & s \geq r \\ KU_0X & \cong & Z \oplus Z/2^r \oplus Z/2^s & Z \oplus Z/2^{s+1} \oplus Z/2^{r-1} \\ \psi_C^{-1} & = & \begin{pmatrix} 1 & 0 & 0 \\ -2^{r-s-1} & 1 & 2^{r-s} \\ 1 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -2^{s-r+2} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

ii) *"The $X = MU_{r,t,s}(r \geq 0$ and $s,t \geq 1)$ case"*

$$\begin{array}{lccc} & & r < s < t & r < s \geq t \\ KU_0X & \cong & Z \oplus Z/2^t \oplus Z/2^s \oplus Z/2^r & Z \oplus Z/2^{s+1} \oplus Z/2^{t-1} \oplus Z/2^r \\ \psi_C^{-1} & = & \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 2^{t-s} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2^{s-t+1} & -1 & -2^{s-t+2} & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2^{s-t+1} & 1 \end{pmatrix} \\ & & r \geq s < t & r \geq s \geq t \\ KU_0X & \cong & Z \oplus Z/2^t \oplus Z/2^{r+1} \oplus Z/2^{s-1} & Z \oplus Z/2^{r+1} \oplus Z/2^s \oplus Z/2^{t-1} \\ \psi_C^{-1} & = & \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & -2^{t-s} & 2^{t-s+1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2^{s-t} & 1 & -1 & -2^{s-t+1} \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

iii) *"The $X = U'M_{r,t,s}$ or $V'M_{r,t,s}(r,t \geq 1$ and $s \geq 0)$ case"*

	$s<r<t$	$s<r\geq t$
$KU_0X \cong$	$Z\oplus Z/2^t\oplus Z/2^r\oplus Z/2^s$	$Z\oplus Z/2^{r+1}\oplus Z/2^{t-1}\oplus Z/2^s$
$\psi_C^{-1} =$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2^{r-s-1} & -1 & 1 & 2^{r-s} \\ 1 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2^{r-s} & 1 & 0 & 2^{r-s+1} \\ 0 & 1 & -1 & 2^{r-s} \\ 1 & 0 & 0 & -1 \end{pmatrix}$
	$s\geq r<t$	$s\geq r\geq t$
$KU_0X \cong$	$Z\oplus Z/2^t\oplus Z/2^{s+1}\oplus Z/2^{r-1}$	$Z\oplus Z/2^{s+1}\oplus Z/2^r\oplus Z/2^{t-1}$
$\psi_C^{-1} =$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 2^{s-r+1} & -1 & -2^{s-r+2} \\ 0 & -1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -2^{s-r+1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

Proposition 2.4. *For the small spectra $X = MV'_{t,s}$, $QV_{r,t}$, $QU_{r,t}$, $V'M_{r,s}$, $MU_{r,t,s}$, $U'M_{r,t,s}$ and $V'M_{r,t,s}(r,t,s \geq 1)$ their KO-homologies $KO_iX(0 \leq i \leq 7)$ are tabled as follows :*

$X\backslash i =$	0	1	2	3
$MV'_{t,s}$	$Z/2^t$	0	$Z\oplus Z/2^{s+1}$	$Z/2$
$QV_{r,t}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2$	0
$QU_{r,t}$	$Z\oplus (*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2$	0
$V'M_{r,s}$	$Z\oplus Z/2^r$	$Z/2$	$(*)_{s,r}$	0
$MU_{r,t,s}$	$Z/2^r\oplus Z/2^t$	0	$Z\oplus Z/2^s\oplus Z/2$	$Z/2$
$U'M_{r,t,s}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2^{s+1}$	0
$V'M_{r,t,s}$	$Z\oplus (*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2^{s+1}$	0

$X\backslash i =$	4	5	6	7
$MV'_{t,s}$	$Z/2^t\oplus Z/2$	0	$Z\oplus Z/2^s$	0
$QV_{r,t}$	$Z\oplus (*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2$	0
$QU_{r,t}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2$	0
$V'M_{r,s}$	$Z\oplus Z/2^{r-1}$	0	$Z/2^{s+1}$	0
$MU_{r,t,s}$	$Z/2^{r+1}\oplus Z/2^t$	0	$Z\oplus Z/2^s$	0
$U'M_{r,t,s}$	$Z\oplus (*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2^{s+1}$	0
$V'M_{r,t,s}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2^{s+1}$	0

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,\ell} \cong Z/2^{k+1}\oplus Z/2$ *if* $\ell \geq 2$.

2.3. We next introduce new small spectra $V'P'_{r,s}$, $U'P'_{r,t,s}$ and $V'P'_{r,t+1,s}(r,t,s \geq 1)$ constructed as the cofibers of the following maps respectively:

(2.6) $(\tilde{\eta}j,\bar{\eta}):\Sigma^1SZ/2^s\to SZ/2^r\vee\Sigma^0$, $(i_V\tilde{\eta}j,\bar{\eta}):\Sigma^1SZ/2^s\to V_{r,t}\vee\Sigma^0$
and $(i_U\tilde{\eta}j,\bar{\eta}):\Sigma^1SZ/2^s\to U_{r,t+1}\vee\Sigma^0$.

For the convenience' sake we set $V'P'_{r,1,s}=V'P'_{r+1,s}$. Similarly to Propositions 2.3 and 2.4 we can easily compute the KU-homologies with the conjugation ψ_C^{-1} and the KO-homologies of these small spectra.

Proposition 2.5. i) *The small spectra $V'P'_{r,s}$, $U'P'_{r,t,s}$ and $V'P'_{r,t,s}(r,t,s\geq 1)$ have the same $\mathscr{C}$-types as $U'M_{r,t,s-1}$ and $V'M_{r,t,s-1}$ respectively, where $V'M_{r,0}=Q_r$, $U'M_{r,t,0}=QV_{r,t}$ and $V'M_{r,t,0}=QU_{r,t}$.*

ii) *Their KO-homologies $KO_iX(0\leq i\leq 7)$ are tabled as follows :*

$X\backslash i$ =	0	1	2	3
$V'P'_{r,s}$	$Z\oplus Z/2^r$	$Z/2$	$(*)_{s-1,r}$	0
$U'P'_{r,t,s}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2^s$	0
$V'P'_{r,t,s}$	$Z\oplus(*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2^s$	0
	4	5	6	7
$V'P'_{r,s}$	$Z\oplus Z/2^{r-1}$	0	$Z/2^s$	0
$U'P'_{r,t,s}$	$Z\oplus(*)_{r-1,t}$	$Z/2$	$Z/2^t\oplus Z/2^s$	0
$V'P'_{r,t,s}$	$Z\oplus Z/2^r$	0	$Z/2^{t-1}\oplus Z/2^s$	0

where $(*)_{k,1}\cong Z/2^{k+2}$ *and* $(*)_{k,\ell}\cong Z/2^{k+1}\oplus Z/2$ *if* $\ell\geq 2$.

We here consider the S-duals $PV_{r,s}$, $PU_{r,t,s}$ and $PV_{r,t+1,s}$ of $V'P'_{s,r}$, $U'P'_{s,t,r}$ and $V'P'_{s,t+1,r}(r,t,s\geq 1)$ obtained as the cofibers of the following maps respectively:

(2.7) $\bar{\eta}\vee i\bar{\eta}:\Sigma^2\vee\Sigma^1SZ/2^s\to SZ/2^r$, $\bar{\eta}\vee i\bar{\eta}j'_V:\Sigma^2\vee\Sigma^{-1}V'_{t,s}\to SZ/2^r$
and $\bar{\eta}\vee i\bar{\eta}j'_U:\Sigma^2\vee\Sigma^{-1}U'_{t+1,s}\to SZ/2^r$.

For the convenience' sake we set $PV_{r,1,s}=PV_{r,s+1}$. Evidently the S-dualities are given as $PV_{r,s}=\Sigma^3DV'P'_{s,r}$, $PU_{r,t,s}=\Sigma^3DU'P'_{s,t,r}$ and $PV_{r,t,s}=\Sigma^3DV'P'_{s,t,r}$. As a dual of Proposition 2.5 we can obtain the KU-homologies with the conjugation ψ_C^{-1} and the KO-homologies for these S-dual spectra.

Corollary 2.6. i) *The small spectra $PV_{r,s}$, $PU_{r,t,s}$ and $PV_{r,t,s}$ $(r,t,s\geq 1)$ have the same $\mathscr{C}$-types as the wedge sums $\Sigma^3\vee V_{r-1,s}$, $\Sigma^3\vee U_{r-1,t,s}$ and $\Sigma^3\vee V_{r-1,t,s}$ respectively, where $V_{0,s}=\Sigma^2SZ/2^s$, $U_{0,t,s}=V'_{t,s}$ and $V_{0,t,s}=U'_{t,s}$.*

ii) *Their KO-homologies $KO_iX(0\leq i\leq 7)$ are tabled as follows :*

$X\backslash i$ =	0	1	2	3
$PV_{r,s}$	$Z/2^r$	0	$Z/2^{s-1}$	Z
$PU_{r,t,s}$	$Z/2^r \oplus Z/2^t$	$Z/2$	$(*)_{s-1,t}$	Z
$PV_{r,t,s}$	$Z/2^r \oplus Z/2^{t-1}$	0	$Z/2^s$	Z
	4	5	6	7
$PV_{r,s}$	$(*)_{r-1,s}$	$Z/2$	$Z/2^s$	Z
$PU_{r,t,s}$	$Z/2^r \oplus Z/2^{t-1}$	0	$Z/2^s$	Z
$PV_{r,t,s}$	$Z/2^r \oplus Z/2^t$	$Z/2$	$(*)_{s-1,t}$	Z

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,\ell} \cong Z/2^{k+1} \oplus Z/2$ *if* $\ell \geq 2$.

2.4. Denote by Q'_t and R'_t the elementary spectra constructed as the cofibers of the maps $\eta\bar{\eta} : \Sigma^2 SZ/2^t \to \Sigma^0$ and $\eta^2\bar{\eta} : \Sigma^3 SZ/2^t \to \Sigma^0$ respectively. There elementary spectra Q'_t and R'_t are related via the obvious map $\lambda'_{Q,R} : \Sigma^1 Q'_t \to R'_t$ satisfying $\lambda'_{Q,R} i'_Q = i'_R \eta : \Sigma^1 \to R'_t$ and $j'_Q = j'_R \lambda'_{Q,R} : Q'_t \to \Sigma^3 SZ/2^t$. Choose a unique map $\tilde{h}_Q : \Sigma^5 \to Q'_t$ satisfying $j'_Q \tilde{h}_Q = \bar{\eta} : \Sigma^2 \to SZ/2^t$, and then set $\tilde{h}_R = \lambda'_{Q,R}\tilde{h}_Q : \Sigma^6 \to R'_t$ (see [11, (2.2)]). We here introduce new small spectra $R'V'_{t,s}$, $V'R'_{r,s}$, $R'U_{r,t,s}$, $U'R'_{r,t,s}$ and $V'R'_{r,t+1,s} (r,t,s \geq 1)$ constructed as the cofibers of the following maps respectively:

$$
\begin{aligned}
&\tilde{h}_R j : \Sigma^5 SZ/2^s \to R'_t, \\
&(\bar{\eta} j, \eta^2\bar{\eta}) : \Sigma^3 SZ/2^s \to \Sigma^2 SZ/2^r \vee \Sigma^0, \\
(2.8) \qquad &\tilde{h}_R j j_V : \Sigma^3 V_{r,s} \to R'_t, \\
&(i_V \bar{\eta} j, \eta^2\bar{\eta}) : \Sigma^3 SZ/2^s \to \Sigma^2 V_{r,t} \vee \Sigma^0, \\
&(i_U \bar{\eta} j, \eta^2\bar{\eta}) : \Sigma^3 SZ/2^s \to \Sigma^2 U_{r,t+1} \vee \Sigma^0.
\end{aligned}
$$

For the convenience' sake we set $V'R'_{r,1,s} = V'R'_{r+1,s}$ and $R'U_{0,t,s} = R'V'_{t,s}$. Choose a map $k'_V : \Sigma^2 V'_{t,s} \to \Sigma^0$ satisfying $k'_V i'_V = \eta\bar{\eta} : \Sigma^2 SZ/2^t \to \Sigma^0$, whose cofiber coincides with the cofiber of the map $\tilde{h}_Q j : \Sigma^4 SZ/2^s \to Q'_t$. Since the cofiber of the obvious map $\lambda'_{Q,R} : \Sigma^1 Q'_t \to R'_t$ is just the cofiber P of the map $\eta : \Sigma^1 \to \Sigma^0$, the above new small spectra $R'V'_{t,s}$ and $R'U_{r,t,s}$ may be given as the cofibers of the following maps respectively:

$$\eta k'_V : \Sigma^3 V'_{t,s} \to \Sigma^0 \quad \text{and} \quad \eta k'_V j_{U,V'} : \Sigma^3 U_{r,t,s} \to \Sigma^0.$$

For these new spectra we can easily compute their KU-homologies with the conjugation ψ_C^{-1} and their KO-homologies.

Proposition 2.7. i) *The small spectra* $R'V'_{t,s}$, $V'R'_{r,s}$, $R'U_{r,t,s}$, $U'R'_{r,t,s}$ *and* $V'R'_{r,t,s}(r,t,s \geq 1)$ *have the same* $\mathscr{C}$*-type as the wedge sums* $\Sigma^0 \vee V'_{t,s}$, $\Sigma^0 \vee \Sigma^2 V'_{r,s}$, $\Sigma^0 \vee U_{r,t,s}$, $\Sigma^0 \vee \Sigma^2 U'_{r,t,s}$ *and* $\Sigma^0 \vee \Sigma^2 V'_{r,t,s}$ *respectively.*

ii) *Their KO-homologies* $KO_iX(0 \leq i \leq 7)$ *are tabled as follows* :

$i \backslash X=$	$R'V'_{t,s}$	$V'R'_{r,s}$	$R'U_{r,t,s}$	$U'R'_{r,t,s}$	$V'R'_{r,t,s}$
0	$Z \oplus Z/2^{t-1} \oplus Z/2$	$Z \oplus Z/2^s$	$Z \oplus Z/2^{r+1} \oplus Z/2^{t-1}$	$Z \oplus Z/2^t \oplus Z/2^s$	$Z \oplus Z/2^{t-1} \oplus Z/2^s$
1	$Z/2$	$Z/2$	$Z/2$	$Z/2$	$Z/2$
2	$(*)_{s-1,t}$	$Z/2^r \oplus Z/2$	$(*)_{s-1,t}$	$Z/2^r \oplus Z/2$	$(*)_{r-1,t} \oplus Z/2$
3	0	$Z/2$	0	0	$Z/2$
4	$Z \oplus Z/2^{t-1}$	$Z \oplus (*)_{s-1,r}$	$Z \oplus Z/2^r \oplus Z/2^{t-1}$	$Z \oplus Z/2^{t-1} \oplus Z/2^s$	$Z \oplus Z/2^t \oplus Z/2^s$
5	$Z/2$	$Z/2$	$Z/2$	$Z/2$	$Z/2$
6	$(*)_{s,t}$	$Z/2^{r-1} \oplus Z/2$	$(*)_{s-1,t} \oplus Z/2$	$(*)_{r-1,t} \oplus Z/2$	$Z/2^r \oplus Z/2$
7	$Z/2$	0	$Z/2$	$Z/2$	0

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,\ell} \cong Z/2^{k+1} \oplus Z/2$ *if* $\ell \geq 2$.

3. The same quasi KO_*-type as $U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee U_{r,t,s}$

3.1. Let X be a CW-spectrum having the same $\mathscr{C}$-type as the wedge sum $Y \vee W_{r,s}$ where $Y = SZ/2^t$, $\Sigma^0 \vee SZ/2^t$ or M_t. In this case we note that there exists an isomorphism $KO_{2i+1}X \oplus KO_{2i+5}X \cong KO_{2i+1}Y \oplus KO_{2i+5}Y$ for any i. Using the same method as adopted in [10, Theorem 5.2] or [11, Theorem 4.2] we can easily determine the quasi KO_*-type of such a CW-spectrum X.

Proposition 3.1. *Let* Y *be the small spectrum* $SZ/2^t$, $\Sigma^0 \vee SZ/2^t$ *or* $M_t(t \geq 1)$. *If a CW-spectrum* X *has the same* $\mathscr{C}$*-type as the wedge sum* $Y \vee W_{r,s}$, *then it is quasi* KO_**-equivalent to one of the following wedge sum:* i) $\Sigma^{4i} SZ/2^t \vee W_{r,s}$ *and* $\Sigma^{4i} V_t \vee W_{r,s}$; ii) $\Sigma^{4i} \vee \Sigma^{4j} SZ/2^t \vee W_{r,s}$,

$\Sigma^{4i}\vee\Sigma^{4j}V_t\vee W_{r,s}$ *and* $\Sigma^{4i}R'_t\vee W_{r,s}$; iii) $\Sigma^{4i}M_t\vee W_{r,s}(i,j=0$ *or* $1)$, *according as* $Y=SZ/2^t$, $\Sigma^0\vee SZ/2^t$ *or* M_t.

Let KT be the self conjugate K-spectrum (which is sometimes denoted by KC). For the small spectra $Y=U_{r,t,s}$, $MU_{r,t,s}$ and $R'U_{r,t,s}(r,t,s\geq 1)$ their KT-homologies $KT_iY(0\leq i\leq 3)$ are easily calculated as follows:

(3.1)

$Y\backslash i$ =	0	1	2	3
$U_{r,t,s}$	$Z/2^r\oplus Z/2^t$	$(*)_{s-1,t}$	$Z/2^s\oplus Z/2$	$Z/2^{r+1}\oplus Z/2^{t-1}$
$MU_{r,t,s}$	$Z/2^r\oplus Z/2^t$	$Z\oplus Z/2^s$	$Z\oplus Z/2^s\oplus Z/2$	$Z/2^{r+1}\oplus Z/2^t$
$R'U_{r,t,s}$	$Z\oplus Z/2^r\oplus Z/2^t$	$(*)_{s-1,t}\oplus Z/2$	$Z/2^s\oplus Z/2$	$Z\oplus Z/2^{r+1}\oplus Z/2^{t-1}$

Let X be a CW-spectrum having the same $\mathscr{C}$-type as $\Sigma^0\vee U_{r,t,s}$. Then there exist two isomorphisms $\theta_1: KO_1X\oplus KO_5X\rightarrow KO_1\Sigma^0\oplus KO_1U_{r,t,s}\cong Z/2\oplus Z/2$ and $\theta_3: KO_3\oplus KO_7X\rightarrow KO_3U_{r,t,s}\cong Z/2$. Identify KT_0X and KT_2X with $KT_0\Sigma^0\oplus KT_0U_{r,t,s}\cong Z\oplus Z/2^r\oplus Z/2^t$ and $KT_2U_{r,t,s}\stackrel{\simeq}{\rightarrow} Z/2^s\oplus Z/2$ respectively. Then the composite homomorphisms $\theta_1(-\tau,\tau B_T^{-1})_*: KT_0X\rightarrow KO_1X\oplus KO_5X\stackrel{\simeq}{\rightarrow} Z/2\oplus Z/2$ is represented by the matrix $\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}: Z\oplus Z/2^r\oplus Z/2^t\rightarrow Z/2\oplus Z/2$, and $\theta_3(-\tau,\tau B_T^{-1})_*: KT_2X\rightarrow KO_3X\oplus KO_7X\stackrel{\simeq}{\rightarrow} Z/2$ is given by the second projection $(0\ \ 1): Z/2^s\oplus Z/2\rightarrow Z/2$. Consider the automorphism $\alpha_T: KT_0X\rightarrow KT_0X$ represented by the matrix $\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}$ on $Z\oplus Z/2^r\oplus Z/2^t$. By a routine computation we can easily get an automorphism $\alpha_C: KU_0X\rightarrow KU_0X$ such that $\psi_{C*}^{-1}\alpha_C=\alpha_C\psi_{C*}^{-1}: KU_0X\rightarrow KU_0X$ and $\alpha_C\zeta_*=\zeta_*\alpha_T: KT_0X\rightarrow KU_0X$. Therefore we may regard that the induced homomorphism $\theta_1(-\tau,\tau B_T^{-1})_*$ is represented by the matrix $\begin{pmatrix}1&0&0\\1&0&1\end{pmatrix}$ in place of $\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}$. In other words, it may be regarded that the isomorphism θ_1 is expressed by one of three kinds of matrices $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ on $Z/2\oplus Z/2$. Hence the induced homomorphism $(-\tau,\tau B_T^{-1})_*: KT_0X\rightarrow KO_1X\oplus KO_5X$ is given as one of the homomorphism represented by the following three kinds of matrices:

(3.2) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$: Z \oplus Z/2^r \oplus Z/2^t \to Z/2 \oplus Z/2.$$

Lemma 3.2. *For the small spectra $Y = U_{r,t,s}$, $MU_{r,t,s}$ and $R'U_{r,t,s}$ $(r,t,s \geq 1)$ the induced homomorphism $\tau_* : KT_{2i}Y \to KO_{2i+1}Y$ are represented by the following rows $M_{2i+1}(Y)$:*

i) $M_1(U_{r,t,s}) = (0\ 1) : Z/2^r \oplus Z/2^t \to Z/2$,
$M_3(U_{r,t,s}) = (0\ 1) : Z/2^s \oplus Z/2 \to Z/2$;

ii) $M_3(MU_{r,t,s}) = (0\ 0\ 1) : Z \oplus Z/2^s \oplus Z/2 \to Z/2$;

iii) $M_1(R'U_{r,t,s}) = (1\ 0\ 1) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2$,
$M_5(R'U_{r,t,s}) = (0\ 0\ 1) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2$,
$M_7(R'U_{r,t,s}) = (0\ \ 1) : Z/2^s \oplus Z/2 \to Z/2$.

Proof. We only show our result in the $Y = R'U_{r,t,s}$ case. The other cases are very easy. Since the small spectrum $R'U_{r,t,s}$ has the same $\mathscr{C}$-type as the wedge sum $\Sigma^0 \vee U_{r,t,s}$, the induced homomorphism $(-\tau, \tau B_T^{-1})_* : KT_0R'U_{r,t,s} \to KO_1R'U_{r,t,s} \oplus KO_5R'U_{r,t,s}$ restricted to $Z/2^r \subset Z \oplus Z/2^r \oplus Z/2^t$ becomes trivial by (3.2), and $\tau_* : KT_6R'U_{r,t,s} \to KO_7R'U_{r,t,s}$ is given by the second projection $(0\ 1) : Z/2^s \oplus Z/2 \to Z/2$. Consider the commutative diagram

$$\begin{array}{ccccc} KO_1R'_t & \xleftarrow{\tau_*} & KT_0R'_t & \xrightarrow{\tau B_{T*}^{-1}} & KO_5R'_t \\ \downarrow\cong & & \downarrow & & \downarrow\cong \\ KO_1R'U_{r,t,s} & \xleftarrow{\tau_*} & KT_0R'U_{r,t,s} & \xrightarrow{\tau B_{T*}^{-1}} & KO_5R'U_{r,t,s} \end{array}$$

where the vertical arrows are induced by the canonical inclusion $i_{R'U} : R'_t \to R'U_{r,t,s}$. The both sides arrows are isomorphisms and the central one is the obvious monomorphism $i : Z \oplus Z/2^t \to Z \oplus Z/2^r \oplus Z/2^t$. Our result is now immediate from [11, Lemma 3.2 iii)].

3.2. For a CW-spectrum X having the same $\mathscr{C}$-type as the small spectrum $Y = U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee R'U_{r,t,s}(r,t,s \geq 1)$ we determine its quasi KO_*-type by using the same method as adopted in [10, Theorem 5.2] or [11, Theorem 4.2].

Theorem 3.3. *Let Y be the small spectrum $U_{r,t,s}$, $MU_{r,t,s}$ or $\Sigma^0 \vee U_{r,t,s}(r,t,s \geq 1)$. If a CW-spectrum X has the same $\mathscr{C}$-type as the small spectrum Y, then it is quasi KO_*-equivalent to one of the following small* *spectra* : i) $\Sigma^{4i}U_{r,t,s}$ *and* $\Sigma^{4i}V_{r,t,s}$; ii) $\Sigma^{4i}MU_{r,t,s}$; iii) $\Sigma^{4i}\vee\Sigma^{4j}U_{r,t,s}$, $\Sigma^{4i}\vee\Sigma^{4j}V_{r,t,s}$ *and* $\Sigma^{4i}R'U_{r,t,s}(i,j=0$ *or* $1)$, *according as* $Y=U_{r,t,s}$, $MU_{r,t,s}$ *or* $\Sigma^0\vee U_{r,t,s}$.

Proof. We may assume that $KO_3X\cong Z/2$ and $KO_7X=0$ since $KO_3X\oplus KO_7X\cong Z/2$ in any case.

i) Note that $KO_1X\oplus KO_5X\cong Z/2$. Under the assumption that $KO_1X=KO_7X=0$ we first show that X is quasi KO_*-equivalent to the small spectrum $V_{r,t,s}$. Choose a map $g_V: V_{r,t,s}\to KT\wedge X$ such that $(\zeta\wedge 1)g_V: V_{r,t,s}\to KU\wedge X$ is a quasi KU_*-equivalence. For our purpose it is sufficient to find a map $h_V: V_{r,t,s}\to KO\wedge X$ satisfying $(c\wedge 1)h_V=(\zeta\wedge 1)g_V$ because such a map h_V is fortunately a quasi KO_*-equivalence by virtue of [10, Proposition 1.1]. When $t=1$ our assertion is immediately established as (2.1) is done. When $t\geq 2$ we consider the following cofiber sequence

$$\Sigma^1SZ/2^s\overset{(i\bar\eta,i_V\bar\eta j)}{\to}SZ/2^r\vee V_t\overset{i_V\vee i_{V,V}}{\to}V_{r,t,s}\overset{j_V}{\to}\Sigma^2SZ/2^s$$

obtained from (2.2). Recall that the elementary spectrum V_t is obtained as the cofiber of the map $2^{t-1}\bar i:\Sigma^0\to P_1'$ where P_1' denotes the cofiber of the map $\bar\eta:\Sigma^1SZ/2\to\Sigma^0$ and $\bar i:\Sigma^0\to P_1'$ is the bottom cell inclusion. Since the elementary spectrum P_1' has the same quasi KO_*-type as Σ^4, the composite map $(\eta\wedge 1)(\tau B_T^{-1}\wedge 1)g_Vi_{V,V}: V_t\to\Sigma^2KO\wedge X$ becomes trivial. On the other hand, it follows from Lemma 3.2 i) that the composite map $(\eta\wedge 1)(\tau B_T^{-1}\wedge 1)g_Vi_V: SZ/2^r\to\Sigma^2KO\wedge X$ is trivial, too. Now we can apply [10, Lemma 1.3] to get a map $h: SZ/2^s\to\Sigma^1KO\wedge X$ satisfying $hj_V=(\tau B_T^{-1}\wedge 1)g_V$ where the map g_V might be replaced by a new one suitably if necessary. Consider the map $k_V: V_t\to\Sigma^1$ of order 2^{t+1} satisfying $k_Vi_V=j: SZ/2^{t-1}\to\Sigma^1$ and $2^{t-1}k_V=\bar\eta j_V: V_t\to\Sigma^1$, whose fiber is the elementary spectrum P_1'. Since the map h admits a coextension $\tilde h:\Sigma^0\to KO\wedge X$ with $\tilde hj=h$, it is immediately seen that $(\eta\wedge 1)h=\tilde hk_Vi_V\bar\eta j=(0\vee\tilde hk_V)(i\bar\eta,i_V\bar\eta j):\Sigma^1SZ/2^s\to SZ/2^r\vee V_t\to\Sigma^1KO\wedge X$, and hence $(\eta\wedge 1)hj_V: V_{r,t,s}\to\Sigma^2KO\wedge X$ is trivial as desired.

By a similar argument to the above we can show that X is quasi KO_*-equivalent to the small spectrum $U_{r,t,s}$ under the assumption that $KO_5X=KO_7X=0$, whose proof is simpler than the above case.

ii) Under the assumption that $KO_1X=KO_5X=KO_7X=0$ it is sufficient to show that X is quasi KO_*-equivalent to the small spectrum $MU_{r,t,s}$. Choose a map $g_{MU}: MU_{r,t,s}\to KT\wedge X$ such that $(\zeta\wedge 1)g_{MU}$:

$MU_{r,t,s} \to KU \wedge X$ is a quasi KU_*-equivalence, and then consider the following cofiber sequence

$$\Sigma^1 \wedge \Sigma^{-1} V_{r,s} \overset{i\eta\wedge\bar{\eta}jj_V}{\longrightarrow} SZ/2^t \overset{i_{MU}}{\longrightarrow} MU_{r,t,s} \overset{j_{MU}}{\longrightarrow} \Sigma^2 \wedge V_{r,s}$$

obtained from (2.4). Since the composite map $(\eta \wedge 1)(\tau B_T^{-1} \wedge 1) g_{MU} i_{MU} : SZ/2^t \to \Sigma^2 KO \wedge X$ is trivial, we obtain a map $h_1 \vee h_2 : \Sigma^2 \vee V_{r,s} \to \Sigma^3 KO \wedge X$ satisfying $(h_1 \vee h_2) j_{MU} = (\tau B_T^{-1} \wedge 1) g_{MU}$ where the map g_{MU} might be replaced by a new one as in the above i) and the map h_1 is in fact trivial. Evidently there exists a map $h_3 : SZ/2^s \to KO \wedge X$ satisfying $h_3 j_V = (\eta \wedge 1) h_2$, and hence a map $h_4 : SZ/2^r \to \Sigma^1 KU \wedge X$ satisfying $h_4 i\bar{\eta} = (c \wedge 1) h_3$ and $h_2 i_V = (rB_C^{-1} \wedge 1) h_4$. Since the composite map $h_4 i : \Sigma^0 \to \Sigma^1 KU \wedge X$ is trivial, we get a map $h_5 : SZ/2^s \to \Sigma^1 KO \wedge X$ satisfying $(\eta \wedge 1) h_5 = h_3$, which admits a coextension $\tilde{h}_5 : \Sigma^0 \to KO \wedge X$ with $\tilde{h}_5 j = h_5$. Consequently we can observe that $(\eta \wedge 1) h_2 = \tilde{h}_5 \eta j j_V = 0$ as desired.

iii) Note that $KO_1X \oplus KO_5X \cong Z/2 \oplus Z/2$. By a similar argument to the above i) we can easily show that X is quasi KO_*-equivalent to the wedge sum $\Sigma^0 \vee U_{r,t,w}$ when $KO_5X = KO_7X = 0$, and to $\Sigma^4 \vee V_{r,t,s}$ when $KO_1X = KO_7X = 0$. So we assume that $KO_1X \cong KO_3X \cong KO_5X \cong Z/2$ and $KO_7X = 0$. According to (3.2) the induced homomorhisms $\tau_* : KT_{2i}X \to KO_{2i+1}X$, whose matrix representations are denoted by $M_{2i+1}(X)$, are divided into the following three types:

(1) $M_1(X) = (1\ 0\ 0)$, $M_5(X) = (0\ 0\ 1)$, $M_3(X) = (0\ 1)$;

(2) $M_1(X) = (0\ 0\ 1)$, $M_5(X) = (1\ 0\ 0)$, $M_3(X) = (0\ 1)$;

(3) $M_1(X) = (0\ 0\ 1)$, $M_5(X) = (1\ 0\ 1)$, $M_3(X) = (0\ 1)$

where $M_1(X), M_5(X) : Z \oplus Z/2^r \oplus Z/2^t \to Z/2$ and $M_3(X) : Z/2^s \oplus Z/2 \to Z/2$. By a similar argument to the above cases we can show that X is quasi KO_*-equivalent to the wedge sum $\Sigma^0 \vee V_{r,t,s}$ or $\Sigma^4 \vee U_{r,t,s}$ according as the case (1) or (2). In the case (3) we show that X is quasi KO_*-equivalent to the small spectrum $\Sigma^4 R'U_{r,t,s}$. Choose a map $g_{R'U} : \Sigma^4 R'U_{r,t,s} \to KT \wedge X$ such that $(\zeta \wedge 1) g_{R'U} : \Sigma^4 R'U_{r,t,s} \to KU \wedge X$ is a quasi KU_*-equivalence, and then consider the following cofiber sequence

$$\Sigma^3 U_{r,s} \overset{\tilde{h}_R jj_V}{\longrightarrow} R'_t \overset{i_{R'U}}{\longrightarrow} R'U_{r,t,s} \overset{j_{R'U}}{\longrightarrow} \Sigma^4 V_{r,s}$$

obtained from (2.8). Since the induced homomorphism $\tau_* : KT_0X \to KO_1X$ restricted to $Z \subset Z \oplus Z/2^r \oplus Z/2^t$ is trivial, the composite map $(\tau B_T^{-1} \wedge 1) g_{R'U} i_{R'U} i_{R'} : \Sigma^1 \to KO \wedge X$ becomes trivial. Hence we obtain a map $h_0 : \Sigma^5 SZ/2^t \to KO \wedge X$ satisfying $h_0 j'_R = (\tau B_T^{-1} \wedge 1) g_{R'U} i_{R'U}$. By virtue of Lemma 3.2 iii) we see that the composite homomorphism $(\tau B_T^{-1} \wedge 1)_* g_{R'U_*} : KO_4 R'U_{r,t,s} \to KO_5 X$ is trivial, and hence $h_{0*} : KO_0 SZ/2^t \to KO_5 X$ is trivial, too. This implies that the composite map $(\eta \wedge 1)(\tau B_T^{-1} \wedge 1) g_{R'U} i_{R'U} : \Sigma^2 R'_t \to KO \wedge X$ is trivial. So we obtain a map $h_1 : \Sigma^5 V_{r,s} \to KO \wedge X$ satisfying $h_1 j_{R'U} = (\tau B_T^{-1} \wedge 1) g_{R'U}$ where the map $g_{R'U}$ might be replaced by a new one as in the above i) or ii). Evidently the composite homomorphism $h_{1*} i_{V*} : KO_0 SZ/2^r \to KO_5 X$ is trivial. Therefore there exists a map $h_2 : \Sigma^8 SZ/2^s \to KO \wedge X$ satisfying $h_2 j_V = (\eta \wedge 1) h_1$, and hence a map $h_3 : \Sigma^7 SZ/2^r \to KU \wedge X$ such that $(c \wedge 1) h_2 = h_3 i \tilde{\eta}$ and $(r B_C^{-1} \wedge 1) h_3 = h_1 i_V$. Since the composite map $h_3 i : \Sigma^7 \to KU \wedge X$ is trivial, we get a map $h_4 : \Sigma^7 SZ/2^s \to KO \wedge X$ satisfying $(\eta \wedge 1) h_4 = h_2$, which admits a coextension $\tilde{h}_4 : \Sigma^8 \to KO \wedge X$ with $\tilde{h}_4 j = h_4$. Consequently we can observe that $(\eta \wedge 1) h_1 = \tilde{h}_4 \eta j j_V = \tilde{h}_4 j j'_R \tilde{h}_R j j_V$ because $j'_R \tilde{h}_R = \tilde{\eta} : \Sigma^2 \to SZ/2^t$, and hence $(\eta \wedge 1) h_1 j_{R'U} = 0$ as desired.

Combining Theorem 3.3 with Propositions 2.2, 2.4, 2.5 and 2.7 we obtain

Corollary 3.4. i) *$U'_{r,t,s}$ and $V'_{r,t,s} (r,t,s \geq 1)$ are quasi KO_*-equivalent to $\Sigma^2 U_{t-1,s+1,r}$ and $\Sigma^6 V_{t-1,s+1,r}$ respectively, where $U_{0,s+1,r} = V'_{s+1,r}$ and $V_{0,s+1,r} = U'_{s+1,r}$.*

ii) *$U'M_{r,t,s}$ and $\Sigma^4 V'M_{r,t,s} (r,t,s \geq 1)$ are quasi KO_*-equivalent to $\Sigma^2 MU_{t-1,s+1,r}$ where $MU_{0,s+1,r} = MV'_{s+1,r}$.*

iii) *$U'P'_{r,t,s}$ and $V'P'_{r,t,s} (r,t,s \geq 1)$ are quasi KO_*-equivalent to $U'M_{r,t,s-1}$ and $V'M_{r,t,s-1}$ respectively, where $U'M_{r,t,0} = QV_{r,t}$ and $V'M_{r,t,0} = QU_{r,t}$.*

iv) *$U'R'_{r,t,s}$ and $\Sigma^4 V'R'_{r,t,s} (r,t,s \geq 1)$ are quasi KO_*-equivalent to $R'U_{t-1,s+1,r}$ where $R'U_{0,s+1,r} = R'V'_{s+1,r}$.*

4. The cofibers of certain maps $f : \Sigma^{2i} \to X$

4.1. For any $k (0 \leq k \leq s)$ the cofiber of the map $2^k i : \Sigma^0 \to SZ/2^s$ is just the wedge sum $\Sigma^1 \wedge SZ/2^k$. Thus we have the following cofiber sequence

$$\Sigma^0 \xrightarrow{2^k i} SZ/2^s \xrightarrow{(j,\rho)} \Sigma^1 \vee SZ/2^k \xrightarrow{2^{s-k} \wedge (-j)} \Sigma^1 \tag{4.1}$$

where $\rho = \rho_{s,k} : SZ/2^s \to SZ/2^k$ is the obvious map. For any $s \geq 2$ we choose a map $h_V : \Sigma^2 \to V_{r,s}$ of order 2^{s-1} satisfying the equalities

$j_V h_V = 2i : \Sigma^0 \to SZ/2^s$ and $h_V j = -i_V i\bar{\eta} : \Sigma^1 SZ/2 \to V_{r,s}$, whose cofiber is the wedge sum $\Sigma^3 \wedge V_{r+1}$. Note that such a map h_V is uniquely chosen. As is easily calculated, $[\Sigma^2, V_{r,s}] \cong Z/2^{s-1} \oplus Z/2$ whose direct summands are generated by the maps h_V and $i_V\tilde{\eta}$ (cf. (1.4)). For any $k(1 \le k < s)$ we set

$$f_{V,k} = 2^{k-1}h_V + i_V\tilde{\eta} : \Sigma^2 \to V_{r,s}. \tag{4.2}$$

Lemma 4.1. *Assume that $1 \le k < s$. Then the cofiber of the map $f_{V,k} : \Sigma^2 \to V_{r,s}$ is the wedge sum $\Sigma^3 \vee W_{r,k}$, and the induced homomorphism $f_{V,k*} : KU_0\Sigma^2 \to KU_0V_{r,s}$ is given as follows:*

$$f_{V,k*}(1) = (2^k, 2^{k-1} + 2^{r-1}) \in Z/2^s \oplus Z/2^r \text{ when } r < s;$$
$$f_{V,k*}(1) = (2^r, 2^{k-1}) \in Z/2^{r+1} \oplus Z/2^{s-1} \quad \text{when } r \ge s.$$

Proof. Choose a map $f : \Sigma^2 \to V_{r,s}$ of order 2^{s-k} satisfying the equalities $j_V f = 2^k i : \Sigma^0 \to SZ/2^s$ and $-fj = i_V(i\bar{\eta} + \tilde{\eta}j) : \Sigma^1 SZ/2^k \to V_{r,s}$. Since such a map f is uniquely determined, it is exactly the map $f_{V,k}$ given in (4.2). When $r < s$ the induced homomorphism $h_{V*} : KU_0\Sigma^2 \to KU_0V_{r,s}$ is expressed as $h_{V*}(1) = (2, c) \in Z/2^s \oplus Z/2^r$ for some c. Using the behavior of the conjugation ψ_C^{-1} on $KU_0V_{r,s}$ we can immediately show that $c \equiv 1 \bmod 2^{r-1}$, thus $c = 1$ or $1 + 2^{r-1}$. Note that the cokernel of h_{V*} is isomorphic to KU_0V_{r+1} but not to KU_0W_{r+1} as an abelian group with involution. From this fact it follows that $c = 1$. On the other hand, when $r \ge s$ we may express as $h_{V*}(1) = (a, b) \in Z/2^{r+1} \oplus Z/2^{s-1}$ for some a, b with the relation $-a + 2b \equiv 2 \bmod 2^s$. Then it is immediately shown that $a \equiv 0 \bmod 2^r$, thus $(a, b) = (0, 1)$ or $(2^r, 1)$. We can also verify that $(a, b) = (0, 1)$ by a similar observation to the above case. Evidently the induced homomorphism $i_V\tilde{\eta}_* : KU_0\Sigma^2 \to KU_0V_{r,s}$ is given by $i_V\tilde{\eta}_*(1) = (0, 2^{r-1}) \in Z/2^s \oplus Z/2^r$ when $r < s$, and $i_V\tilde{\eta}_*(1) = (2^r, 0) \in Z/2^{r+1} \oplus Z/2^{s-1}$ when $r \ge s$. Therefore our result is now immediate.

Lemma 4.2. *Assume that $1 \le k < s$. Then there exists a map $f_{U,k} : \Sigma^2 \to U_{r,t,s}$ whose cofiber is the wedge sum $P_t \vee W_{r,k}$ and whose induced homomorphism $f_{U,k*} : KU_0\Sigma^2 \to KU_0U_{r,t,s}$ is given as follows :*

i) $f_{U,k*}(1) = (-2^{t-s+k-1}, 2^k, 2^{k-1} + 2^{r-1}) \in Z/2^t \oplus Z/2^s \oplus Z/2^r$ *when $r < s < t$;*

ii) $f_{U,k*}(1) = (2^k, 0, 2^{k-1} + 2^{r-1}) \in Z/2^{s+1} \oplus Z/2^{t-1} \oplus Z/2^r$ *when $r < s \ge t$;*

iii) $f_{U,k*}(1)=(-2^{t-s+k-1},2^r,2^{k-1})\in Z/2^t\oplus Z/2^{r+1}\oplus Z/2^{s-1}$ *when* $r\geq s<t$;

iv) $f_{U,k*}(1)=(2^r,2^{k-1},0)\in Z/2^{r+1}\oplus Z/2^s\oplus Z/2^{t-1}$ *when* $r\geq s\geq t$.

Proof. Consider the cofiber sequence

$$SZ/2^t\xrightarrow{i_U}U_{r,t,s}\xrightarrow{j_{U,V}}V_{r,s}\xrightarrow{\tilde{\eta}jj_V}\Sigma^1SZ/2^t.$$

Using the map $f_{V,k}:\Sigma^2\to V_{r,s}$ we choose a map $f_{U,k}:\Sigma^2\to U_{r,t,s}$ satisfying the equalities $j_{U,V}f_{U,k}=f_{V,k}:\Sigma^2\to V_{r,s}$, $2^{s-k}f_{U,k}=i_U\tilde{\eta}:\Sigma^2\to U_{r,t,s}$ and $f_{U,k}jj_W=0:W_{r,k}\to\Sigma^1U_{r,t,s}$ so that its cofiber is the wedge sum $P_t\vee W_{r,k}$. Then we can easily check that the induced homomorphism $f_{U,k*}:KU_0\Sigma^2\to KU_0U_{r,t,s}$ is expressed as desired, after replacing the map $f_{U,k}$ by $(1+2^{s-k})f_{U,k}=f_{U,k}+i_U\tilde{\eta}$ if necessary.

Recall that the small spectrum $MU_{r,t,s}$ is given as the cofiber of the composite map $i_Ui\eta:\Sigma^1\to U_{r,t,s}$ where $i_U:SZ/2^t\to U_{r,t,s}$ is the canonical inclusion. Using the map $f_{U,k}:\Sigma^2\to U_{r,t,s}$ obtained in Lemma 4.2 we consider the composite map

$$f_{MU,k}=i_{U,MU}f_{U,k}:\Sigma^2\to MU_{r,t,s}\tag{4.3}$$

for any $k(1\leq k<s)$, where $i_{U,MU}:U_{r,t,s}\to MU_{r,t,s}$ is the canonical inclusion.

Lemma 4.3. *Assume that* $1\leq k<s$. *Then the cofiber of the map* $f_{MU,k}:\Sigma^2\to MU_{r,t,s}$ *is the wedge sum* $MP_t\vee W_{r,k}$ *and the induced homomorphism* $f_{MU,k*}:KU_0\Sigma^2\to KU_0MU_{r,t,s}$ *is given by* $f_{MU,k*}(1)=(0,f_{U,k*}(1))\in Z\oplus KU_0U_{r,t,s}$ *where* $f_{U,k*}(1)$ *is precisely expressed in Lemma 4.2.*

Proof. The cofiber of the map $f_{MU,k}$ coincides with that of the map $(i_Pi\eta,0):\Sigma^1\to P_t\vee W_{r,k}$ where $i_P:SZ/2^t\to P_t$ is the canonical inclusion. Thus it is exactly the wedge sum $MP_t\vee W_{r,k}$ as desired. The latter part of our result is obvious.

4.2. Recall that the small spectrum $U'_{r,t,s}$ is obtained as the cofiber of the map $i_V\tilde{\eta}j:\Sigma^1SZ/2^s\to V_{r,t}$. Using the map $f_{V,k}:\Sigma^2\to V_{r,t}$ given in (4.2) we consider the composite map

$$f'_{U,k}=i_{V,U'}f_{V,k}:\Sigma^2\to U'_{r,t,s}\tag{4.4}$$

for any $k(1 \leq k < t)$, where $i_{V,U'} : V_{r,t} \to U'_{r,t,s}$ is the canonical inclusion.

Lemma 4.4. *Assume that $s < k < t$. Then the cofiber of the map $f'_{U,k} : \Sigma^2 \to U'_{r,t,s}$ is the wedge sum $\Sigma^3 \vee W_{r,k} \vee \Sigma^2 SZ/2^s$, and the induced homomorphism $f'_{U,k*} : KU_0\Sigma^2 \to KU_0U'_{r,t,s}$ is given as follows :*

i) $f'_{U,k*}(1) = (2^k, 2^{k-1} + 2^{r-1}, 0) \in Z/2^t \oplus Z/2^r \oplus Z/2^s$ *when* $s < r < t$;

ii) $f'_{U,k*}(1) = (2^r, 2^{k-1}, 0) \in Z/2^{r+1} \oplus Z/2^{t-1} \oplus Z/2^s$ *when* $s < r \geq t$;

iii) $f'_{U,k*}(1) = (2^k, 2^s - 2^{s-r+k}, 2^{k-1}) \in Z/2^t \oplus Z/2^{s+1} \oplus Z/2^{r-1}$ *when* $s \geq r < t$.

Proof. Under the assumption that $k > s$ we observe that $f_{V,k}j = f_{V,k}j\rho_{s,k} = i_V(i\bar{\eta} + \tilde{\eta}j)\rho_{s,k} = -i_V\tilde{\eta}j : \Sigma^1 SZ/2^s \to V_{r,t}$. Therefore the cofiber of the composite map $f'_{U,k} = i_{V,U'}f_{V,k}$ coincides with the wedge sum $\Sigma^3 \vee W_{r,k} \vee \Sigma^2 SZ/2^s$ because Lemma 4.1 says that the cofiber of the map $f_{V,k}$ is just the wedge sum $\Sigma^3 \vee W_{r,k}$ when $1 \leq k < t$. By a routine computation we can easily show the latter part of our result.

For the map $h'_V : \Sigma^2 \to V'_{r,s}$ chosen in (2.5) its induced homomorphism $h'_{V*} : KU_0\Sigma^2 \to KU_0V'_{r,s}$ is expressed as follows:

$$(4.5) \quad \begin{array}{ll} \text{i)} & h'_{V*}(1) = (-2^{r-s-1}, 1) \in Z/2^r \oplus Z/2^s \ \textit{when } r > s; \\ \text{ii)} & h'_{V*}(1) = (1, 0) \in Z/2^{s+1} \oplus Z/2^{r-1} \quad \textit{when } r \leq s. \end{array}$$

Here the map h'_V might be replaced by $(1 + 2^s)h'_V = h'_V + i'_V\tilde{\eta}$ if necessary. Recall that the small spectrum $MV'_{r,s}$ is given as the cofiber of the composite map $i'_V i\eta : \Sigma^1 \to V'_{r,s}$. For any $k(0 \leq k \leq s)$ we set

$$(4.6) \quad h'_{V,k} = 2^k h'_V : \Sigma^2 \to V'_{r,s} \text{ and } h'_{MV,k} = i'_{V,MV}h'_{V,k} : \Sigma^2 \to MV'_{r,s}$$

where $i'_{V,MV} : V'_{r,s} \to MV'_{r,s}$ is the cononical inclusion.

Lemma 4.5. *Assume that $0 \leq k \leq s$. Then the cofibers of the maps $h'_{V,k} : \Sigma^2 \to V'_{r,s}$ and $h'_{MV,k} : \Sigma^2 \to MV'_{r,s}$ are the wedge sums $P_r \vee \Sigma^2 SZ/2^k$ and $MP_r \vee \Sigma^2 SZ/2^k$ respectively, and the induced homomorphisms $h'_{V,k*} : KU_0\Sigma^2 \to KU_0V'_{r,s}$ and $h'_{MV,k*} : KU_0\Sigma^2 \to KU_0MV'_{r,s}$ are given by $h'_{V,k*}(1) = 2^k h'_{V*}(1)$ and $h'_{MV,k*}(1) = (0, 2^k h'_{V*}(1)) \in Z \oplus KU_0V'_{r,s}$ where $h'_{V*}(1)$ is precisely expressed in (4.5).*

Proof. Choose a map $f_k : \Sigma^2 \to V'_{r,s}$ satisfying the equalities $j'_V f_k = 2^k i : \Sigma^0 \to SZ/2^s$, $2^{s-k}f_k = i'_V\tilde{\eta} : \Sigma^2 \to V'_{r,s}$ and $f_k j = 0 : \Sigma^1 SZ/2^k \to V'_{r,s}$ so that its cofiber is the wedge sum $P_r \vee \Sigma^2 SZ/2^k$. Evidently $f_s =$

$i'_V\tilde{\eta} = 2^s h'_V$, and in general $f_k = 2^k h'_V + a i'_V \tilde{\eta}$ for some a because $i'_V i\eta^2 j$: $\Sigma^1 SZ/2^k \to V'_{r,s}$ is never trivial when $r \geq 2$. Note that the cofiber of the maps $h'_{V,k}$ and $f_k = (1+2^{s-k}a)h'_{V,k}$ coincide in the $k < s$ case. Now the first of our result is easily shown. The latter part is obvious.

4.3. The cofiber of the map $2^\ell ij : \Sigma^{-1}SZ/2^k \to SZ/2^s$ is just the wedge sum $SZ/2^{s+k-\ell} \vee SZ/2^\ell$ for any $\ell \leq \mathrm{Min}(s,k)$. Thus there exists a cofiber sequence

$$\Sigma^{-1}SZ/2^k \overset{2^\ell ij}{\to} SZ/2^s \overset{(\rho,\rho)}{\to} SZ/2^{s+k-\ell} \vee SZ/2^\ell \overset{\rho\vee(-\rho)}{\to} SZ/2^k$$

in which each of ρ is the obvious map. As is easily seen, under the assumption that $\ell \leq \mathrm{Min}(s, k-1)$ we get a map $g'_{V,\ell}; \Sigma^1 SZ/2^k \to V'_{r,s}$ satisfying the equalities $j'_V g'_{V,\ell} = 2^\ell ij : SZ/2^k \to \Sigma^1 SZ/2^s$, $g'_V\rho = i'_V(i\bar{\eta} + \tilde{\eta}j)$: $\Sigma^1 SZ/2^{s+k-\ell} \to V'_{r,s}$ and $g'_{V,\ell}\rho = 0 : \Sigma^1 SZ/2^\ell \to V'_{r,s}$ so that its cofiber is the wedge sum $W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell$. Thus we have the following cofiber sequence

$$\Sigma^1 SZ/2^k \overset{g'_{V,\ell}}{\to} V'_{r,s} \overset{(\rho_{V'},w,\rho j'_V)}{\to} W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell \overset{\rho jw\vee(-\rho)}{\to} \Sigma^2 SZ/2^k.$$

Assume that $0 \leq \ell < k \leq s+k-\ell < t$. Using the above two maps $h'_{V,k} : \Sigma^2 \to V'_t$ and $g'_{V,\ell} : \Sigma^1 SZ/2^k \to V'_{r,s}$, we then obtain a map $f'_{V,k,\ell} : \Sigma^2 \to V'_{r,t,s}$ making the following diagram with four cofiber sequences commutative

$$(4.7)\quad \begin{array}{ccccccc}
 & & \Sigma^2 & = & \Sigma^2 & & \\
 & & \downarrow f'_{V,k,\ell} & & \downarrow h'_{V,k} & & \\
V'_{r,s} & \to & V'_{r,t,s} & \to & V'_t & \overset{i'_V i\bar{\eta} j'_V}{\to} & \Sigma^1 V'_{r,s} \\
\| & & \downarrow \varphi & & \downarrow & & \| \\
V'_{r,s} & \to & P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell & \to & P_1 \vee \Sigma^2 SZ/2^k & \overset{0\vee g'_{V,\ell}}{\to} & \Sigma^1 V'_{r,s} \\
 & & \downarrow \psi & & \downarrow & & \\
 & & \Sigma^3 & = & \Sigma^3 & &
\end{array}$$

because $g'_{V,\ell}\rho_{t-1,k} j'_V = i'_V(i\bar{\eta} + \tilde{\eta}j)\rho_{t-1,s+k-\ell} j'_V = i'_V i\bar{\eta} j'_V$.

Lemma 4.6. *Assume that* $0 \leq \ell \leq s < r < s+k-\ell < t$. *Then there exists a map* $f'_{V,k,\ell} : \Sigma^2 \to V'_{r,t,s}$ *whose cofiber is the wedge sum* $P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell$ *and whose induced homomorphism* $f'_{V,k,\ell *} : KU_0\Sigma^2 \to$

$KU_0V'_{r,t,s}$ is given by $f'_{V,k,\ell *}(1) = (2^k, 2^{r-s+\ell-1}, -2^\ell) \in Z/2^t \oplus Z/2^r \oplus Z/2^s$.

Proof. Use the commutative diagram (4.7). The induced homomorphism $f'_{V,k,\ell *} : KU_0\Sigma^2 \to KU_0V'_{r,t,s}$ is expressed as $f'_{V,k,\ell *}(1) = (2^k, b, c) \in Z/2^t \oplus Z/2^r \oplus Z/2^s$ with the relation $b + 2^{r-s-1}c \equiv 0 \bmod 2^{r-1}$. On the other hand, the induced homomorphism $\varphi_* : KU_0V'_{r,t,s} \to KU_0W_{r,s+k-\ell} \oplus KU_0\Sigma^2 SZ/2^\ell$ is represented by a certain matrix $\begin{pmatrix} x & 2^{m+1} & u \\ y & -1 & v \\ z & 0 & 1 \end{pmatrix} : Z/2^t \oplus Z/2^r \oplus Z/2^s \to Z/2^{s+k-\ell+1} \oplus Z/2^{r-1} \oplus Z/2^\ell$ with the relations $x + 2^{m+1}y - 2^{k-\ell}z \equiv 1 \bmod 2^k$ and $u + 2^{m+1}v \equiv 2^{k-\ell} \bmod 2^{s+k-\ell}$ where $m = s - r + k - \ell \geq 1$. Using the behavior of the conjugation ψ_C^{-1} on $KU_0W_{r,s+k-\ell} \oplus KU_0\Sigma^2 SZ/2^\ell$ we see that $u + 2(2^m - 1)v \equiv 2^{r-s} \bmod 2^r$. Hence it follows that $u \equiv 2^{k-\ell+1}(1 - 2^{m-1}) \bmod 2^{s+k-\ell}$ and $v \equiv 2^{r-s-1}(2^m - 1) \bmod 2^{r-1}$, thus $u = 2^{k-\ell+1}u'$ for some u' and $v = 2^{r-s-1}v'$ for some odd v' whenever $s \geq 1$. Since $\varphi f'_{V,k,\ell} : \Sigma^2 \to P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell$ is trivial, it is shown that $2^k x + 2^{m+1}b + cu \equiv 0 \bmod 2^{s+k-\ell+1}$, $-b + cv \equiv 0 \bmod 2^{r-1}$ and $c \equiv 0 \bmod 2^\ell$, thus $b = 2^{r-s+\ell-1}b'$ and $c = 2^\ell c'$ for some b', c', and in addition $x + b' + 2c'u' \equiv 0 \bmod 2^{s-\ell+1}$. Moreover we notice that $b' + c' \equiv 0 \bmod 2^{s-\ell}$ because $b + 2^{r-s-1}c \equiv 0 \bmod 2^{r-1}$. Consequently $f'_{V,k,\ell *}(1) = (2^k, 2^{r-s+\ell-1}b', -2^\ell b') \in Z/2^t \oplus Z/2^r \oplus Z/2^s$ for some odd b'. In this case we may take $b' = 1$ by replacing suitably the direct sum decomposition of $KU_0V'_{r,t,s} \cong KU_0V'_t \oplus KU_0V'_{r,s}$ if necessary.

Since the map $\bar\eta : \Sigma^1 SZ/2 \to \Sigma^0$ has order 4 we can choose a map $k'_P : P'_1 \to \Sigma^0$ satisfying $k'_P i'_P = 4 : \Sigma^0 \to \Sigma^0$ and $ik'_P = \bar\eta_{1,2} j'_P : P'_1 \to SZ/4$, where cofiber is the small spectrum U_1 constructed as the cofiber of the map $\bar\eta_{1,2} : \Sigma^2 SZ/2 \to SZ/4$ with $j\bar\eta_{1,2} = \bar\eta$ (see [14, 1.1]). Composing this map $k'_P : P'_1 \to \Sigma^0$ before the map $f'_{V,k,\ell} : \Sigma^2 \to V'_{r,t,s}$ obtained in (4.7) we get a map

$$g'_{V,k,\ell} = f'_{V,k,\ell} k'_P : \Sigma^2 P'_1 \to V'_{r,t,s} \tag{4.8}$$

when $0 \leq \ell < k \leq s + k - \ell < t$.

Lemma 4.7. *Assume that $0 \leq \ell < s < r < s + k - \ell < t$. Then the cofiber of the map $g'_{V,k,\ell} : \Sigma^2 P'_1 \to V'_{r,t,s}$ is quasi KO_*-equivalent to the wedge sum $\Sigma^7 \vee W_{r,s+k-\ell} \vee \Sigma^6 V_{\ell+1}$, and the induced homomorphism $g'_{V,k,\ell *} : KU_0\Sigma^2 P'_1 \to KU_0V_{r,t,s}$ is given by $g'_{V,k,\ell *}(1) = (2^{k+1}, 2^{r-s+\ell}, -2^{\ell+1}) \in$*

$Z/2^t \oplus Z/2^r \oplus Z/2^s$.

Proof. The cofiber of the composite map $g'_{V,k,\ell} = f'_{V,k,\ell}k'_P$ coincides with the fiber of the map $i_U\psi = \psi_1 \vee \psi_2 \vee \psi_3 : P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell \to \Sigma^3 U_1$ where $\psi = 2^{t-k-1}j_P \vee (-2^{s-\ell}jj_w) \vee j : P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 SZ/2^\ell \to \Sigma^3$. Since P_1 and U_1 have the same quasi KO_*-types as Σ^7 and $\Sigma^6 SZ/2$ respectively, it follows that $[P_1, \Sigma^3 KO \wedge U_1] \cong KO_6 SZ/2 = 0$ and $[\Sigma^0, KO \wedge U_1] \cong [\Sigma^2, KO \wedge SZ/2] \cong Z/4$. Obviously the composite map $(\iota_R \wedge 1)\psi_1 : P_1 \to \Sigma^3 KO \wedge U_1$ is trivial where $\iota_R : S \to KO$ denotes the unit of KO. Since $2\tilde{\eta}jj_W = i\eta^2 jj_W : W_{r,s+k-\ell} \to \Sigma^1 SZ/2$ is trivial, the composite map $(\iota_R \wedge 1)\psi_2 : W_{r,s+k-\ell} \to \Sigma^3 KO \wedge U_1$ becomes trivial under the assumption that $\ell < s$. On the other hand, the cofiber of the map $\psi_3 = i_U j : \Sigma^{-1} SZ/2^\ell \to U_1$ coincides with the small spectrum $U_{\ell+1}$ obtained as the cofiber of the map $2^\ell k'_P : P'_1 \to \Sigma^0$, which has the same quasi KO_*-type as $\Sigma^4 V_{\ell+1}$ (see [14, (1.4)]). Using these facts we observe that the cofiber of the map $g'_{V,k,\ell}$ is quasi KO_*-equivalent to the wedge sum $P_1 \vee W_{r,s+k-\ell} \vee \Sigma^2 U_{\ell+1}$ and hence it is quasi KO_*-equivalent to the wedge sum $\Sigma^7 \vee W_{r,s+k-\ell} \vee \Sigma^6 V_{\ell+1}$ as desired. Since the induced homomorphism $k'_{P*} : KU_0 P'_1 \to KU_0 \Sigma^0$ is the multiplication by 2 on Z, the latter part of our result is immediate from Lemma 4.6.

4.4. For any $s \geq 1$ we choose a map $h_W : \Sigma^2 \to W_{r,s}$ of order 2^s satisfying the equalities $j_W h_W = 2i : \Sigma^0 \to SZ/2^s$, $2^{s-1}h_W = i_W\tilde{\eta} : \Sigma^2 \to W_{r,s}$ and $h_W j = -i_W i\bar{\eta} : \Sigma^1 SZ/2 \to W_{r,s}$ so that its cofiber is the small spectrum $PV_{r,1}$ constructed in (2.7). Evidently $[\Sigma^2, W_{r,s}] \cong Z/2^s$ which is generated by the map h_W. After the map h_W is replaced by $(1+2^{s-1})h_W = h_W + i_W\tilde{\eta}$ if necessary, the induced homomorphism $h_{W*} : KU_0\Sigma^2 \to KU_0 W_{r,s}$ is expressed as follows:

(4.9) i) $h_{W*}(1) = (-2^{r-s+1}, 1) \in Z/2^{r+1} \oplus Z/2^{s-1}$ when $r > s$;
ii) $h_{W*}(1) = (2, 1) \in Z/2^s \oplus Z/2^r$ when $r = s$;
iii) $h_{W*}(1) = (2 - 2^{s-r+1}, 1) \in Z/2^{s+1} \oplus Z/2^{r-1}$ when $r < s$.

For any $k (1 \leq k \leq s)$ we set

(4.10) $$h_{W,k} = 2^{k-1}h_W : \Sigma^2 \to W_{r,s}\ .$$

Lemma 4.8. *Assume that $1 \leq k \leq s$. Then the cofiber of the map $h_{W,k} : \Sigma^2 \to W_{r,s}$ is the small spectrum $PV_{r,k}$ and the induced homomorphism $h_{W,k*} : KU_0\Sigma^2 \to KU_0 W_{r,s}$ is given by $h_{W,k*}(1) = 2^{k-1}h_{W*}(1)$ where $h_{W*}(1)$ is precisely expressed in* (4.9).

Proof. Similarly to the proof of Lemma 4.5 we choose a map $f_k : \Sigma^2 \to W_{r,s}$ satisfing the equalities $j_W f_k = 2^k i : \Sigma^0 \to SZ/2^s$, $2^{s-k} f_k = i_W \tilde{\eta} : \Sigma^2 \to W_{r,s}$ and $f_k j = i_W i \tilde{\eta} : \Sigma^1 SZ/2^k \to W_{r,s}$ so that its cofiber is the small spectrum $PV_{r,k}$. Evidently $f_s = i_W \tilde{\eta} = h_{W,s}$ and $f_k = h_{W,k}$ or $h_{W,k} + i_W \tilde{\eta}$ when $k < s$. Since the cofibers of the maps $h_{W,k}$ and $h_{W,k} + i_W \tilde{\eta} = (1 + 2^{s-k}) h_{W,k}$ coincide under the assumption that $k < s$, our result is immediate.

Denote by $\tilde{\eta}_V : \Sigma^2 \to V_t$ the composite map $i_V \tilde{\eta} : \Sigma^2 \to SZ/2^{t-1} \to V_t$ when $t \geq 2$, and the bottom cell inclusion $i : \Sigma^2 \to \Sigma^2 SZ/2$ when $t = 1$. Using the maps $h_{W,k} : \Sigma^2 \to W_{r,s}$, $\tilde{\eta} : \Sigma^2 \to SZ/2^t$ and $\tilde{\eta}_V : \Sigma^2 \to V_t$ we consider the two maps

$$(4.11) \qquad \begin{aligned} f_{W,k} &= (h_{W,k}, \tilde{\eta}) : \Sigma^2 \to W_{r,s} \vee SZ/2^t \quad \text{and} \\ f_{WV,k} &= (h_{W,k}, \tilde{\eta}_V) : \Sigma^2 \to W_{r,s} \vee V_t \end{aligned}$$

for any $k (1 \leq k \leq s)$.

Lemma 4.9. *Assume that* $1 \leq k \leq s$. *The cofibers of the maps* $f_{W,k} : \Sigma^2 \to W_{r,s} \vee SZ/2^t$ *and* $f_{WV,k} : \Sigma^2 \to W_{r,s} \vee V_t$ *are the small spectra* $PU_{r,t,k}$ *and* $PV_{r,t,k}$ *respectively, and the induced homomorphisms* $f_{W,k*} : KU_0 \Sigma^2 \to KU_0 W_{r,s} \oplus KU_0 SZ/2^t$ *and* $f_{WV,k*} : KU_0 \Sigma^2 \to KU_0 W_{r,s} \oplus KU_0 V_t$ *are given by* $f_{W,k*}(1) = f_{WV,k*}(1) = (h_{W,k*}(1), 2^{t-1}) \in KU_0 W_{r,s} \oplus Z/2^t$ *where* $h_{W,k*}(1)$ *is expressed in Lemma* 4.8.

Proof. The cofiber of the map $f_{W,k}$ coincides with that of the composite map $\tilde{\eta}(2^{s-k} \vee (-j)) j_{PV} : PV_{r,k} \to \Sigma^3 \vee \Sigma^2 SZ/2^k \to \Sigma^3 \to \Sigma^1 SZ/2^t$. Note that $(i\eta^2 \vee 0) j_{PV} : PV_{r,k} \to \Sigma^1 SZ/2^t$ is trivial because $(\tilde{\eta} \vee i \tilde{\eta}) j_{PV} : PV_{r,k} \to \Sigma^1 SZ/2^r$ is trivial. Hence the above composite map $\tilde{\eta}(2^{s-k} \vee (-j)) j_{PV}$ is rewritten to be $(0 \vee (-\tilde{\eta} j)) j_{PV}$. Therefore the cofiber of the map $f_{W,k}$ coincides with of the map $\tilde{\eta} \vee i \tilde{\eta} j'_V : \Sigma^2 \vee \Sigma^{-1} V'_{t,k} \to SZ/2^r$. Thus it is the small spectrum $PU_{r,t,s}$ given in (2.7). By a similar argument we can also observe that the cofiber of the map $f_{WV,k}$ is the small spectrum $PV_{r,t,s}$. The latter part of our result is obvious.

4.5. For any $k (0 \leq k \leq r)$ and $\ell (0 \leq \ell \leq t)$ we set

$$(4.12) \quad g_{M,k} = 2^k i_M i : \Sigma^0 \to M_r \quad \text{and} \quad g_{W,\ell} = 2^\ell i_W i : \Sigma^0 \to W_{t,s}.$$

The cofiber of the map $g_{M,k}$ is the wedge sum $\Sigma^1 \vee M_k$ where $M_0 = \Sigma^2$.

Thus we have the following cofiber sequence

$$\Sigma^0 \overset{g_{M,k}}{\to} M_r \overset{(k_M,\rho_M)}{\to} \Sigma^1 \vee M_k \overset{2^{r-k}\vee(-k_M)}{\to} \Sigma^1$$

in which the map k_M is appearing in (2.3). On the other hand, the cofiber of the map $g_{W,t}$ is the wedge sum $\Sigma^1 \vee W_{t,s}$, and when $\ell < t$ the cofiber of the map $g_{W,\ell}$ coincides with the small spectrum $VM'_{\ell,s}$ constructed as the cofiber of the map $(\eta j, i\bar{\eta}) : \Sigma^1 SZ/2^s \to \Sigma^1 \vee SZ/2^\ell$. Note that $VM'_{0,s} = \Sigma^1 M'_s$ and $VM'_{\ell,s}$ is the S-dual of $MV'_{s,\ell}$ given in (2.4), thus $VM'_{\ell,s} = \Sigma^3 DMV'_{s,\ell}$. We see immediately that the induced homomorphism $g_{M,k*} : KU_0\Sigma^0 \to KU_0M_r$ and $g_{W,\ell*} : KU_0\Sigma^0 \to KU_0W_{t,s}$ are expressed as follows:

(4.13) i) $g_{M,k*}(1) = (0, 2^k) \in Z \oplus Z/2^r$;
ii) $g_{W,\ell*}(1) = (2^{\ell+s-t+1}, -2^\ell) \in Z/2^{s+1} \oplus Z/2^{t-1}$ *when* $t < s$;
iii) $g_{W,\ell*}(1) = (0, 2^\ell) \in Z/2^s \oplus Z/2^t$ *when* $t = s$;
iv) $g_{W,\ell*}(1) = (2^{\ell+1} - 2^{t-s+\ell+1}, 2^\ell) \in Z/2^{t+1} \oplus Z/2^{s-1}$ *when* $t > s$.

Using these two maps $g_{M,k} : \Sigma^0 \to M_r$ and $g_{W,\ell} : \Sigma^0 \to W_{t,s}$ we consider the map

(4.14) $$g_{MW,k,\ell} = (g_{M,k}, g_{W,\ell}) : \Sigma^0 \to M_r \vee W_{t,s}$$

for any $k(0 \le k \le r)$ and $\ell(0 \le \ell \le t)$.

Lemma 4.10. *Assume that* $0 \le k \le r$ *and* $0 \le \ell \le t \le r - k + \ell$. *Then the cofiber of the map* $g_{MW,k,\ell} : \Sigma^0 \to M_r \vee W_{t,s}$ *is the wedge sum* $\Sigma^1 \vee MU_{\ell,t+k-\ell,s}$ *or* $\Sigma^1 \vee M_k \vee W_{t,s}$ *according as* $k > \ell < t$ *or otherwise, and the induced homomorphism* $g_{MW,k,\ell*} : KU_0\Sigma^0 \to KU_0M_r \oplus KU_0W_{t,s}$ *is given by* $g_{MW,k,\ell*}(1) = (0, 2^k, g_{W,\ell*}(1)) \in Z \oplus Z/2^r \oplus KU_0W_{t,s}$ *where* $g_{W,\ell*}(1)$ *is precisely expressed in* (4.13).

Proof. The cofiber of the map $g_{MW,k,\ell}$ is obtained as the cofiber of the composite map $g_{W,\ell}(2^{r-k} \vee (-k_M)) : \Sigma^0 \vee \Sigma^{-1}M_k \to W_{t,s}$. Evidently the latter map is rewritten to be $0 \vee (-2^\ell i_W i k_M)$ under the assumption that $r - k + \ell \ge t$. Set $g_{MW,\ell} = 2^\ell i_W i k_M : \Sigma^{-1}M_k \to W_{t,s}$. Then the cofiber of the map $g_{MW,k,\ell}$ is just the wedge sum of Σ^1 and the cofiber of the map $g_{MW,\ell}$. Since $2^k i k_M = i\eta j_M : \Sigma^{-1}M_k \to SZ/2^r$, it is easily seen that the map $g_{MW,\ell} : \Sigma^{-1}M_k \to W_{t,s}$ is trivial if $k \le \ell$ or $t = \ell$. Therefore the cofiber of the map $g_{MW,\ell}$ is exactly the wedge sum $M_k \vee W_{t,s}$ in the $k \leq \ell$ or $t = \ell$ case. Assume that $k > \ell < t$. Then the cofiber of the map $2^\ell k_M : \Sigma^{-1}M_k \to \Sigma^0$ is just the wedge sum $P \vee SZ/2^\ell$ because the cofiber of the map $k_M : \Sigma^{-1}M_k \to \Sigma^0$ is the elementary spectrum P. Evidently the cofiber of the composite map $2^\ell ik_M : \Sigma^{-1}M_k \to SZ/2^t$ is the wedge sum $M_{t+k-\ell} \vee SZ/2^\ell$. Thus there exists a cofiber sequence

$$\Sigma^{-1}M_k \overset{2^\ell ik_M}{\to} SZ/2^t \overset{(\varphi_1,\varphi_2)}{\to} M_{t+k-\ell} \vee SZ/2^\ell \to M_k.$$

Here $\varphi_1 i = 2^{k-\ell} i_{P,M} i_P : \Sigma^0 \to M_{t+k-\ell}$, $k_M\varphi_1 = j : SZ/2^t \to \Sigma^1$ and $\varphi_2 i = i : \Sigma^0 \to SZ/2^\ell$ in which the map $i_{P,M} : P \to M_{t+k-\ell}$ is appearing in (2.3). As is easily seen, $\varphi_1\tilde{\eta} = i_M\tilde{\eta} : \Sigma^2 \to M_{t+k-\ell}$ and $\varphi_2 = \rho_{t,\ell} + ai\eta j : SZ/2^t \to SZ/2^\ell$ for some a where $\rho_{t,\ell}$ is the obvious map. Since $2i_P\tilde{\eta} : \Sigma^1 SZ/2^s \to P$ is trivial, it follows that $\varphi_1(i\tilde{\eta} + \tilde{\eta}j) = i_M\tilde{\eta}j : \Sigma^1 SZ/2^s \to M_{t+k-\ell}$ and $\varphi_2(i\tilde{\eta} + \tilde{\eta}j) = i\tilde{\eta}(1 + ai\eta j) : \Sigma^1 SZ/2^s \to SZ/2^\ell$. Hence we can observe that when $k > \ell < t$ the cofiber of the map $g_{MW,\ell} = 2^\ell i_W ik_M$ coincides with that of the map $(i_M\tilde{\eta}j, i\tilde{\eta}(1 + ai\eta j)) : \Sigma^1 SZ/2^s \to M_{t+k-\ell} \vee SZ/2^\ell$, which is exactly the desired spectrum $MU_{\ell,t+k-\ell,s}$.

5. The stunted mod 4 lens spaces

5.1. Let $L^k(4)$ be the $(2k+1)$-dimensional standard mod 4 lens space and $L_0^k(4)$ its $2k$-skeleton. For simplicity we set $L^{2k+1} = L^k(4)$ and $L^{2k} = L_0^k(4)$. Recall the structure of KU-cohomology KU^*L^n (see [5] or [7]). The inclusion $i : L^{2k} \to L^{2k+1}$ induces an isomorphism $i^* : KU^0L^{2k+1} \overset{\cong}{\to} KU^0L^{2k}$, and $KU^1L^{2k+1} \cong Z$ and $KU^1L^{2k} = 0$. The ring $KU^0L^{2k+1} \cong KU^0L^{2k}$ is generated by $\sigma = \gamma - 1$, whose multiplicative structure is given by the two relations $(\sigma+1)^4 = 1$ and $\sigma^{k+1} = 0$. Here γ denotes the canonical complex line bundle over $L^{2k+1} = L^k(4)$ or its restriction to $L^{2k} = L_0^k(4)$. According to [5, Theorem 4.6] the KU-cohomology $KU^0L^{2k+1} \cong KU^0L^{2k}(k = 2m$ or $2m+1)$ is explicitly given as follows:

$$KU^0L^{4m+1} \cong KU^0L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$$
$$KU^0L^{4m+3} \cong KU^0L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$$

whose direct summands are generated by the elements σ, $\sigma(1)$ and $\sigma(1)\sigma + 2^{m+1}\sigma$ in the former case, and σ, $\sigma(1) + 2^{m+1}\sigma$ and $\sigma(1)\sigma$ in the latter case, where $\sigma = \gamma - 1$ and $\sigma(1) = \gamma^2 - 1 = \sigma^2 + 2\sigma$.

We next study the behavior of the complex Adams operation ψ_C^r on $KU^0L^{2k+1} \cong KU^0L^{2k}$ after changing the above direct summands slightly as follows:

(5.1) i) $KU^0L^{4m+1} \cong KU^0L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$
with generators σ, $\sigma(1)\sigma + \sigma(1)$ *and* $\sigma(1)\sigma + 2^{m+1}\sigma$, *and*
ii) $KU^0L^{4m+3} \cong KU^0L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$
with generators σ, $\sigma(1)\sigma + \sigma(1) + 2^{m+1}\sigma$ *and* $\sigma(1) + 2^{m+1}\sigma$.

Since $\psi_C^{r+4}\sigma = \psi_C^r\sigma$ and $\psi_C^{r+2}\sigma(1) = \psi_C^r\sigma(1)$, it is evident that $\psi_C^{r+4} = \psi_C^r$ on $KU^0L^{2k+1} \cong KU^0L^{2k}$. As is easily calculated, the complex Adams operation ψ_C^r on $KU^0L^{2k+1} \cong KU^0L^{2k}(k = 2m$ or $2m+1)$ is given as follows:

(5.2) i) $\psi_C^{4s} = 0$ and $\psi_C^{4s+1} = 1$;

ii) $\psi_C^{4s+2} = \begin{pmatrix} 2^{m+1} & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} -2^{m+1} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$;

iii) $\psi_C^{4s+3} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1-2^{m+1} & 2^{m+2} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

which operate respectively on $KU^0L^{4m+1} \cong KU^0L^{4m} \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$ and $KU^0L^{4m+3} \cong KU^0L^{4m+2} \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ whose direct summands are given as in (5.1) i) and ii). Here the matrices behave always as left action.

Dualizing (5.1) and (5.2) we can study the behavior of the complex Adams operation ψ_C^r on $KU_*L^n \otimes Z[1/r]$, and in particular the conjugation ψ_C^{-1} on KU_*L^n. Note that $KU_{-1}L^{2k} \cong KU^0L^{2k}$, $KU_{-1}L^{2k+1} \cong KU_{-1}\Sigma^{2k+1} \oplus KU_{-1}L^{2k}$ and $KU_0L^{2k} = KU_0L^{2k+1} = 0$. By virtue of (5.1) the induced homomorphism $i^* : KU^0L^{2k+2} \to KU^0L^{2k+1}$ is actually represented by the following matrix $A_k(k = 2m$ or $2m+1)$:

(5.3) $A_{2m} = \begin{pmatrix} 1 & 2^{m+1} & 2^{m+2} \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $A_{2m+1} = \begin{pmatrix} 1 & -2^{m+1} & 2^{m+2} \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

where $A_{2m} : Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \to Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$ and $A_{2m+1} : Z/2^{2m+3} \oplus Z/2^{m+1} \oplus Z/2^m \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$. Therefore the induced homomorphism $i_* : KU_{-1}L^{2k+1} \to KU_{-1}L^{2k+2}$ is given by the following matrix $B_k(k = 2m$ or $2m+1)$:

$$(5.4)\quad B_{2m} = \begin{pmatrix} x & 2 & 0 & 0 \\ y & 1 & 1 & 0 \\ z & 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B_{2m+1} = \begin{pmatrix} u & 2 & 0 & 0 \\ v & -1 & 2 & 0 \\ w & 1 & 1 & -1 \end{pmatrix}$$

where $B_{2m} : Z \oplus Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1} \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ and $B_{2m+1} : Z \oplus Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m \to Z/2^{2m+3} \oplus Z/2^{m+1} \oplus Z/2^m$. Since the above induced homomorphism i_* is an epimorphism in any case, it follows that x and u must be odd. Using this fact we show

Proposition 5.1. *The suspended* mod 4 *lens space* $\Sigma^1 L^n (n \geq 2)$ *has the same* $\mathscr{C}$*-type as the small spectrum* $U_{m-1,2m+1,m}$, $MU_{m-1,2m+1,m}$, $SZ/2^m \vee W_{2m+1,m+1}$ *or* $\Sigma^0 \vee SZ/2^m \vee W_{2m+1,m+1}$ *according as* $n = 4m$, $4m+1$, $4m+2$ *or* $4m+3$, *where* $W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $n = 2$ *and* 3 *cases.*

Proof. The $n = 2k$ case is just shown as the dual of (5.2). On the other hand, the conjugation ψ_C^{-1} on $KU_{-1}L^{2k+1}$ is represented by the following matrix:

$$\psi_C^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ a & 1 & 2^{m+1} & 0 \\ b & 0 & -1 & 0 \\ c & 0 & -1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ d & 1-2^{m+1} & 2^{m+2} & 0 \\ e & 1 & -1 & 0 \\ f & 0 & 0 & 1 \end{pmatrix}$$

according as $k = 2m$ or $2m+1$. Using the equality $\psi_C^{-1} i_* = i_* \psi_C^{-1}$: $KU_{-1}L^{2k+1} \to KU_{-1}L^{2k+2}$ we get immediately that $a = x + 2^m a'$, $b = 0$, $c = x - z$, $d = 2^{m+1}d'$, $e = -d'$ and $f = 0$. As is easily verified, we may take $x = -1$, $a' = c = 0$ and $d' = 0$ after changing the direct sum decomposition of $KU_{-1}L^{2k+1} \cong Z \oplus KU_{-1}L^{2k}$ suitably if necessary. Now our result is immediate from Propositions 2.1 and 2.3.

5.2. The stunted mod 4 lens space $L^n/L^m (n > m \geq 0)$ is simply written to be L^n_{m+1} as usual. We here study the behavior of the conjugations ψ_C^{-1} on $KU^*L^n_{m+1}$ and $KU_*L^n_{m+1}$. Similarly to (5.3) the induced homomorphism i^* : $KU^0L^{2\ell} \to KU^0L^{2k} (\ell > k)$ is represented by the following matrix $A_{\ell,k}$:

$$(5.5)\quad \begin{aligned}
A_{2n,2m} &= \begin{pmatrix} 1 & 0 & 2^{n+1}-2^{m+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
A_{2n+1,2m} &= \begin{pmatrix} 1 & 2^{n+1} & 2^{n+1}+2^{m+1} \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \\
A_{2n+1,2m+1} &= \begin{pmatrix} 1 & 2^{n+1}-2^{m+1} & 2^{n+1}-2^{m+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
A_{2n+2,2m+1} &= \begin{pmatrix} 1 & -2^{m+1} & 2^{n+2} \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

where $A_{2n,2m} : Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1} \to Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$, $A_{2n+1,2m} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$, $A_{2n+1,2m+1} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ and $A_{2n+2,2m+1} : Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n \to Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$.

The projection $p : L^{2\ell} \to L^{2\ell}_{2k+1}$ induces a monomorphism $p^* : KU^0L^{2\ell}_{2k+1} \to KU^0L^{2\ell}$, which is represented by the following matrix $C_{k,\ell}$:

$$(5.6)\quad \begin{aligned}
C_{2m,2n} &= \begin{pmatrix} 2^{2m} & 0 & 0 \\ 0 & 2^m & 0 \\ 2^{2m-1}-2^{m-1} & 0 & 2^m \end{pmatrix}, \\
C_{2m,2n+1} &= \begin{pmatrix} 2^{2m} & 0 & 0 \\ 2^{2m-1}-2^{m-1} & 2^m & 0 \\ 2^{m-1}-2^{2m-1} & 0 & 2^m \end{pmatrix}, \\
C_{2m+1,2n+1} &= \begin{pmatrix} 2^{2m+1} & 0 & 0 \\ 2^{2m} & 2^m & 0 \\ 2^m-2^{2m} & -2^m & 2^{m+1} \end{pmatrix}, \\
C_{2m+1,2n+2} &= \begin{pmatrix} 2^{2m+1} & 0 & 0 \\ 2^m & 2^{m+1} & 0 \\ 2^{2m}-2^m & 0 & 2^m \end{pmatrix}
\end{aligned}$$

where $C_{2m,2n} : Z/2^{2n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m-1} \to Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1}$, $C_{2m,2n+1} : Z/2^{2n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m} \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$, $C_{2m+1,2n+1} : Z/2^{2n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m-1} \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$ and $C_{2m+1,2n+2} : Z/2^{2n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m} \to Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n$.

Using the equality $p^*\psi_C^{-1} = \psi_C^{-1}p^* : KU^0L^{2\ell}_{2k+1} \to KU^0L^{2\ell}$ we can easily show the following result by virtue of Proposition 2.1.

Proposition 5.2. *The S-dual $DL_{2k+1}^{2\ell+2k}$ of the stunted* mod 4 *lens space $L_{2k+1}^{2\ell+2k}(\ell \geq 1)$ has the same $\mathcal{C}$-type as the small spectrum $U_{2n,n,n}$, $SZ/2^n \vee W_{2n+1,n+1}$, $U_{n-1,2n+1,n}$ or $SZ/2^{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$ in the $(k,\ell) = (2m,1)$ case.*

Dualizing Proposition 5.2 we can immediately obtain

Corollary 5.3. *The suspended stunted* mod 4 *lens space $\Sigma^1 L_{2k+1}^{2\ell+2k}$ $(\ell \geq 1)$ has the same $\mathcal{C}$-type as the small spectrum $U_{n-1,2n+1,n}$, $SZ/2^n \vee W_{2n+1,n+1}$, $U_{2n,n,n}$ or $SZ/2^{2n+2} \vee W_{n,n}$ according as $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ or $(2m+1,2n+1)$, where $W_{1,1}$ should be replaced by $\Sigma^2 SZ/4$ in the $(k,\ell) = (2m,1)$ case.*

The induced homomorphism $p_* : KU_{-1}L^{2\ell} \to KU_{-1}L_{2k+1}^{2\ell}$ is represented by the following matrix $C'_{\ell,k}$ dual to (5.6):

$$(5.7)\quad \begin{aligned}
C'_{2n,2m} &= \begin{pmatrix} 1 & 0 & 2^{n+1}-2^{n-m+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
C'_{2n+1,2m} &= \begin{pmatrix} 1 & 2^{n+1}-2^{n-m+1} & 2^{n-m+1}-2^{n+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
C'_{2n+1,2m+1} &= \begin{pmatrix} 1 & 2^{n+1} & 2^{n-m+1}-2^{n+1} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\
C'_{2n+2,2m+1} &= \begin{pmatrix} 1 & 2^{n-m+1} & 2^{n+2}-2^{n-m+2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

where $C'_{2n,2m} : Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1} \to Z/2^{2n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m-1}$, $C'_{2n+1,2m} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$, $C'_{2n+1,2m+1} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n-2m+1} \oplus Z/2^{n-m} \oplus Z/2^{n-m-1}$ and $C'_{2n+2,2m+1} : Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n \to Z/2^{2n-2m+2} \oplus Z/2^{n-m} \oplus Z/2^{n-m}$.

Notice that $KU_{-1}L_{2k+1}^{2\ell+1} \cong KU_{-1}\Sigma^{2\ell+1} \oplus KU_{-1}L_{2k+1}^{2\ell}$ and $KU_0 L_{2k+1}^{2\ell+1} = 0$. The induced homomorphism $p_* : KU_{-1}L^{2\ell+1} \to KU_{-1}L_{2k+1}^{2\ell+1}$ is represented by the matrix

$$(5.8)\quad \begin{pmatrix} 1 & 0 \\ 0 & C'_{\ell,k} \end{pmatrix} : Z \oplus KU_{-1}L^{2\ell} \to Z \oplus KU_{-1}L_{2k+1}^{2\ell}$$

in which the matrix $C'_{\ell,k}$ is explicitly expressed in (5.7). Since the induced homomorphism $p_*: KU_{-1}L^{2\ell+1} \to KU_{-1}L^{2\ell+1}_{2k+1}$ is an epimorphism, we can easily show the following result by means of Proposition 5.1.

Proposition 5.4. *The suspended stunted* mod 4 *lens space* $\Sigma^1 L^{2\ell+2k+1}_{2k+1}(\ell \geq 1)$ *has the same* $\mathscr{C}$*-type as the small spectrum* $MU_{n-1,2n+1,n}$, $\Sigma^0 \vee SZ/2^n \vee W_{2n+1,n+1}$, $\Sigma^0 \vee U_{2n,n,n}$ *or* $M_{2n+2} \vee W_{n,n}$ *according as* $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ *or* $(2m+1,2n+1)$, *where* $W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $(k,\ell) = (2m,1)$ *case.*

5.3. By means of (5.6) we can easily give the matrix representation of the induced homomorphism $j^*: KU^0L^{2\ell+2k}_{2k+1} \to KU^0L^{2\ell+2k}_{2k-1}$ where $j: L^{2\ell+2k}_{2k-1} \to L^{2\ell+2k}_{2k+1}$ denotes the canonical projection. Note that $KU^0L^{2\ell+2k}_{2k} \cong KU^0\Sigma^{2k} \oplus KU^0L^{2\ell+2k}_{2k+1}$ and $KU^1L^{2\ell+2k}_{2k} = 0$. Then the bottom cell collapsing $j: L^{2\ell+2k}_{2k-1} \to L^{2\ell+2k}_{2k}$ induces an epimorphism $j^*: KU^0L^{2\ell+2k}_{2k} \to KU^0L^{2\ell+2k}_{2k-1}$, which is represented by the following matrix $B_{k,\ell}$ similarly to (5.4):

$$
(5.9)\qquad
\begin{aligned}
B_{2m,2n} &= \begin{pmatrix} x_1 & 2 & 0 & 0 \\ y_1 & -1 & 1 & 0 \\ z_1 & 1 & 0 & 2 \end{pmatrix}, \quad B_{2m,2n+1} = \begin{pmatrix} x_2 & 2 & 0 & 0 \\ y_2 & -1 & 2 & 0 \\ z_2 & -1 & 1 & 1 \end{pmatrix} \\
B_{2m+1,2n} &= \begin{pmatrix} x_3 & 2 & 0 & 0 \\ y_3 & 1 & 1 & 0 \\ z_3 & 0 & -1 & 2 \end{pmatrix}, \quad B_{2m+1,2n+1} = \begin{pmatrix} x_4 & 2 & 0 & 0 \\ y_4 & 1 & 2 & 0 \\ z_4 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

where $B_{2m,2n}: Z \oplus Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1} \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$, $B_{2m,2n+1}: Z \oplus Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n$, $B_{2m+1,2n}: Z \oplus Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1} \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$ and $B_{2m+1,2n+1}: Z \oplus Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n$. Notice that all of $x_i (1 \leq i \leq 4)$ must be odd. By a quite similar argument to Proposition 5.1 we show

Proposition 5.5. *The S-dual* $DL^{2\ell+2k}_{2k}$ *of the stunted* mod 4 *lens space* $L^{2\ell+2}_{2k}(\ell \geq 1)$ *has the same* $\mathscr{C}$*-type as the small spectrum* $\Sigma^0 \vee U_{2n,n,n}$, $\Sigma^0 \vee SZ/2^n \vee W_{2n+1,n+1}$, $MU_{n-1,2n+1,n}$ *or* $M_{2n+2} \vee W_{n,n}$ *according as* $(k,\ell) = (2m,2n)$, $(2m,2n+1)$, $(2m+1,2n)$ *or* $(2m+1,2n+1)$, *where* $W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $(k,\ell) = (2m,1)$ *case.*

Proof. By virtue of Proposition 5.2 the conjugation ψ_C^{-1} on $KU^0L_{2k}^{2\ell+2k}$ is expressed by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ b_1 & 1 & -1 & -2 \\ c_1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_2 & 1-2^{n+1} & 2^{n+2} & 0 \\ b_2 & 1 & -1 & 0 \\ c_2 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ a_3 & 1 & 2^{n+1} & 0 \\ b_3 & 0 & -1 & 0 \\ c_3 & 0 & -1 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 \\ a_4 & 1 & 0 & 0 \\ b_4 & 0 & -1 & -1 \\ c_4 & 0 & 0 & 1 \end{pmatrix}$$

according as $(k,\ell) = (2m,2n), (2m,2n+1), (2m+1,2n)$ or $(2m+1,2n+1)$. Using the equality $j^*\psi_C^{-1} = \psi_C^{-1}j^* : KU^0L_{2k}^{2\ell+2k} \to KU^0L_{2k-1}^{2\ell+2k}$ we see that i) $a_1 = c_1 = 0$, $b_1 = -y_1 - z_1$; ii) $a_2 = 2^{n+1}a_2'$, $b_2 = -a_2'$, $c_2 = 0$; iii) $a_3 = x_3 + 2^n a_3'$, $b_3 = 0$, $c_3 = z_3$; and iv) $a_4 = x_4$, $b_4 = -z_4$, $c_4 = -2z_4$. As in the proof of Proposition 5.2 we may take $a_2' = a_3' = 0$, $x_3 = x_4 = -1$ and $b_1 = c_3 = b_4 = c_4 = 0$. Thus a_i, b_i and $c_i (1 \le i \le 4)$ are taken to be zero except a_3 and a_4, while $a_3 = a_4 = 1$ as desired. Now our result is immediate from Propositions 2.1 and 2.3.

By means of (5.7) we can represent the induced homomorphism $j_* : KU_{-1}L_{2k-1}^{2\ell+2k} \to KU_{-1}L_{2k+1}^{2\ell+2k}$ by the following matrix $D_{\ell,k}$:

(5.10)
$$D_{2n,2m} = \begin{pmatrix} 1 & -2^{n+1} & 2^{n+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_{2n+1,2m} = \begin{pmatrix} 1 & -2^{n+1} & -2^{n+2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$D_{2n,2m+1} = \begin{pmatrix} 1 & 2^{n+1} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, D_{2n+1,2m+1} = \begin{pmatrix} 1 & 2^{n+1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $D_{2n,2m} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1}$, $D_{2n+1,2m} : Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$, $D_{2n,2m+1} : Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n \to Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1}$ and $D_{2n+1,2m+1} : Z/2^{2n+3} \oplus Z/2^{n+1} \oplus Z/2^n \to Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n$.

Evidently the induced homomorphism $j_* : KU_{-1}L_{2k-1}^{2\ell+2k+1} \to KU_{-1}L_{2k+1}^{2\ell+2k+1}$ is represented by the matrix

$$(5.11) \qquad \begin{pmatrix} 1 & 0 \\ 0 & D_{\ell,k} \end{pmatrix} : Z \oplus KU_{-1}L_{2k-1}^{2\ell+2k} \to Z \oplus KU_{-1}L_{2k+1}^{2\ell+2k}$$

in which the matrix $D_{\ell,k}$ is explicitly expressed in (5.10).

Consider the exact sequence

$$0 \to KU_1L^n_{2k} \to KU_1L^n_{2k+1} \to KU_0\Sigma^{2k} \to KU_0L^n_{2k} \to KU_0L^n_{2k+1} \to 0$$

induced by the cofiber sequence $\Sigma^{2k} \to L^n_{2k} \to L^n_{2k+1} \to \Sigma^{2k+1} (n > 2k)$ where $KU_0L^n_{2k+1} = 0$. Assume that $KU_0L^n_{2k} = 0$, and then $KU_1L^n_{2k+1} \cong Z \oplus KU_1L^n_{2k}$. When $n = 2\ell$ this is evidently a contradiction because $KU_1L^{2\ell}_{2k+1} \otimes Q = 0$. In the $n = 2\ell + 1$ case our assumption implies that $KU_1L^{2\ell+1}_{2k} \cong KU_1L^{2\ell}_{2k+1}$ because $KU_1L^{2\ell+1}_{2k+1} \cong Z \oplus KU_1L^{2\ell}_{2k+1}$. In $KU_1L^{2\ell+1}_{2k-1}$ there exists an element of order $2^{\ell-k+2}$, but in $KU_1L^{2\ell+1}_{2k+1}$ there exist no elements of order $2^{\ell-k+2}$ under our assumption. As is easily checked, this is a contradiction, too. Therefore it is verified that $KU_0L^n_{2k} \cong Z$, and hence there exist isomorphisms

(5.12) $i_* : KU_0\Sigma^{2k} \overset{\cong}{\to} KU_0L^n_{2k}$ and $j_* : KU_{-1}L^n_{2k} \overset{\cong}{\to} KU_{-1}L^n_{2k+1}$

for any $n > 2k$ where $i : \Sigma^{2k} \to L^n_{2k}$ and $j : L^n_{2k} \to L^n_{2k+1}$ denote the bottom cell inclusion and collapsing respectively.

Using (5.11) and (5.12) we can immediately show

Lemma 5.6. *The induced homomorphism* $i_* : KU_{-1}\Sigma^{2k+1} \to KU_{-1}L^{2\ell+2k+1}_{2k+1} (\ell \geq 1)$ *is identified with the homomorphism* $\varphi_{k,\ell}$ *defined as follows*:

$$\varphi_{2m,2n}(1) = (0, 2^{2n-1}, 2^{n-1}, 0) \in Z \oplus Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1};$$
$$\varphi_{2m,2n+1}(1) = (0, 2^{2n}, 2^{n-1}, 2^{n-1}) \in Z \oplus Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n;$$
$$\varphi_{2m+1,2n}(1) = (0, 2^{2n-1}, 2^{n-1}, 0) \in Z \oplus Z/2^{2n+1} \oplus Z/2^n \oplus Z/2^{n-1};$$
$$\varphi_{2m+1,2n+1}(1) = (0, 2^{2n}, 0, 2^{n-1}) \in Z \oplus Z/2^{2n+2} \oplus Z/2^n \oplus Z/2^n.$$

5.4. In order to determine the quasi KO_*-types of the stunted mod 4 lens spaces $L^n_{m+1} = L^n/L^m$ we only need the following part (cf. [15, Lemma 2.2]), although $KO^*L^n_{m+1}$ and hence $KO_*L^n_{m+1}$ are completely calculated in [6, Theorem 2] and [8, Theorem 2].

Lemma 5.7. i) $KO_{4m}L^{4m+n}_{4m+1} = 0 = KO_{4m}L^{4m+n}_{4m-1}$ *if* $n \equiv 1, 2, 3, 4, 5 \bmod 8$.

ii) $KO_{4m+4}L^{4m+n}_{4m+1} = 0 = KO_{4m+4}L^{4m+n}_{4m-1}$ *if* $n \equiv 0, 1, 5, 6, 7 \bmod 8$.

iii) $KO_{4m+6}L^{4m+n}_{4m+1} = 0 = KO_{4m+6}L^{4m+n}_{4m-1}$ *for all* n.

iv) $KO^{4m-3}L^{4m+2\ell}_{4m} = 0 = KO^{4m-3}L^{4m+2\ell}_{4m-2}$ *if* $\ell \equiv 1, 2 \bmod 4$.

v) $KO^{4m-7}L_{4m}^{4m+2\ell} = 0 = KO^{4m-7}L_{4m-2}^{4m+2\ell}$ *if* $\ell \equiv 0, 3 \bmod 4$.
vi) $KO^{4m-5}L_{4m}^{4m+2\ell} = 0 = KO^{4m-5}L_{4m-2}^{4m+2\ell}$ *for all* ℓ.

Using Corollary 5.3, Propositions 5.4 and 5.5 and Lemma 5.7 and then applying Proposition 3.1 and Theorem 3.3, we can first determine the quasi KO_*-types of L_{2k+1}^{2k+n} and $DL_{2k}^{2k+2\ell}$.

Theorem 5.8. i) $\Sigma^{-4m+1}L_{4m+1}^{4m+n}(n \geq 2)$ *is quasi* KO_**-equivalent to the following small spectrum:* $U_{2r-1,4r+1,2r}$, $MU_{2r-1,4r+1,2r}$, $V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^4 \vee V_{2r} \vee W_{4r+1,2r+1}$, $V_{2r,4r+3,2r+1}$, $MU_{2r,4r+3,2r+1}$, $SZ/2^{2r+1} \vee W_{4r+3,2r+2}$, $\Sigma^0 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ *according as* $n = 8r$, $8r+1, \cdots, 8r+7$. *Here* $V_0 \vee W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $n = 2$ *and* 3 *cases.*

ii) $\Sigma^{-4m+1}L_{4m-1}^{4m+n-2}(n \geq 2)$ *is quasi* KO_**-equivalent to the following small spectrum:* $U_{4r,2r,2r}$, $\Sigma^0 \vee U_{4r,2r,2r}$, $SZ/2^{4r+2} \vee W_{2r,2r}$, $M_{4r+2} \vee W_{2r,2r}$, $V_{4r+2,2r+1,2r+1}$, $\Sigma^4 \vee V_{4r+2,2r+1,2r+1}$, $V_{4r+4} \vee W_{2r+1,2r+1}$, $M_{4r+4} \vee W_{2r+1,2r+1}$ *according as* $n = 8r$, $8r+1, \cdots, 8r+7$.

iii) $\Sigma^{4m}DL_{4m}^{4m+2\ell}(\ell \geq 1)$ *is quasi* KO_**-equivalent to the following small spectrum:* $\Sigma^0 \vee U_{4r,2r,2r}$, $\Sigma^0 \vee \Sigma^4 V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^0 \vee \Sigma^4 V_{4r+2,2r+1,2r+1}$, $\Sigma^0 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ *according as* $\ell = 4r$, $4r+1$, $4r+2$, $4r+3$. *Here* $\Sigma^4 V_0 \vee W_{1,1}$ *should be replaced by* $\Sigma^{-2}SZ/4$ *in the* $\ell = 1$ *case.*

iv) $\Sigma^{4m}DL_{4m-2}^{4m+2\ell-2}(\ell \geq 1)$ *is quasi* KO_**-equivalent to the following small spectrum:* $MU_{2r-1,4r+1,2r}$, $M_{4r+2} \vee W_{2r,2r}$, $\Sigma^4 MU_{2r,4r+3,2r+1}$, $\Sigma^4 M_{4r+4} \vee W_{2r+1,2r+1}$ *according as* $\ell = 4r$, $4r+1$, $4r+2$, $4r+3$.

From [10, Corollary 1.8] we recall that

(5.13) *two finite spectra* X *and* Y *have the same quasi* KO_**-type if and only if their* S*-duals* DX *and* DY *have the same quasi* KO_**-type.*

By virtue of Theorem 5.8 iii) and iv) and (5.13) we can next determine the quasi KO_*-types of $L_{2k}^{2k+2\ell}$ with the aid of Corollary 3.4.

Theorem 5.9. i) $\Sigma^{-4m+1}L_{4m}^{4m+2\ell}(\ell \geq 1)$ *is quasi* KO_**-equivalent to the following small spectrum :* $\Sigma^1 \vee U_{2r-1,4r+1,2r}$, $\Sigma^1 \vee V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^1 \vee V_{2r,4r+3,2r+1}$, $\Sigma^1 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ *according as* $\ell = 4r$, $4r+1$, $4r+2$, $4r+3$. *Here* $V_0 \vee W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $\ell = 1$ *case.*

ii) $\Sigma^{-4m+1}L_{4m-2}^{4m+2\ell-2}(\ell \geq 1)$ *is quasi* KO_**-equivalent to the following small spectrum :* $PU_{4r+1,2r,2r}$, $P_{4r+3} \vee W_{2r,2r}$, $\Sigma^4 PU_{4r+3,2r+1,2r+1}$, $\Sigma^4 P_{4r+5} \vee W_{2r+1,2r+1}$ *according as* $\ell = 4r,\ 4r+1,\ 4r+2,\ 4r+3$.

5.5. Using the maps appearing in Lemmas 4.3, 4.4, 4.5, 4.7, 4.9 and 4.10 we here consider the following maps $f_{k,\ell} : Y_{k,\ell} \to X_{k,\ell}$ modelled on the bottom cell inclusions $i : \Sigma^{2k-4m+2} \to \Sigma^{-4m+1}L_{2k+1}^{2k+2\ell+1}$ with $k = 2m$ or $2m-1$:

(5.14)
(1) $f_{2m,1} = (0, i) : \Sigma^2 \to \Sigma^4 \vee \Sigma^2 SZ/4$;
(2) $f_{2m,2} = h'_{MV,0} : \Sigma^2 \to MV'_{3,1}$;
(3) $f_{2m,2n+2} = f_{MU,n} : \Sigma^2 \to MU_{n,2n+3,n+1}$;
(4) $f_{2m,4r-1} = (0, f_{W,2r-1}) : \Sigma^2 \to \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$;
(5) $f_{2m,4r+1} = (0, f_{WV,2r}) : \Sigma^2 \to \Sigma^4 \vee W_{4r+1,2r+1} \vee V_{2r}$;
(6) $f_{2m-1,1} = i_M i : \Sigma^0 \to M_2$;
(7) $f_{2m-1,2} = (0, i_V i) : \Sigma^0 \to \Sigma^4 \vee V_{2,2}$;
(8) $f_{2m-1,4r} = (0, f'_{U,2r}) : \Sigma^0 \to \Sigma^0 \vee \Sigma^{-2}U'_{2r,4r+1,2r-1}$;
(9) $f_{2m-1,4r+2} = (0, g'_{V,4r,2r}) : \Sigma^4 P'_1 \to \Sigma^4 \vee \Sigma^2 V'_{2r+1,4r+3,2r}$;
(10) $f_{2m-1,2n+1} = g_{MW,2n,n-1} : \Sigma^0 \to M_{2n+2} \vee W_{n,n}$

$(n, r \geq 1)$ where the small spectrum P'_1 has the same quasi KO_*-type as Σ^4. According to Theorem 5.8 i) and ii) combined with Corollary 3.4, the small spectrum $X_{k,\ell}$ has the same quasi KO_*-type as $\Sigma^{-4m+1}L_{2k+1}^{2k+2\ell+1}$ where $k = 2m$ or $2m-1$. Using Lemmas 4.3, 4.4, 4.5, 4.7, 4.9 and 4.10 with the aid of (5.13) and Corollary 3.4, we can observe that

(5.15) i) *the cofiber of the map* $f_{k,\ell}$ *has the same quasi* KO_**-type as the following small spectrum* $Z_{k,\ell}$: $\Sigma^4 \vee \Sigma^3$, MP_3, $MP_{2n+3} \vee W_{n,n}$, $\Sigma^0 \vee PU_{4r-1,2r-1,2r-1}$, $\Sigma^4 \vee \Sigma^4 PU_{4r+1,2r,2r}$, $\Sigma^1 \vee \Sigma^2$, $\Sigma^1 \vee \Sigma^4 \vee \Sigma^2 SZ/4$, $\Sigma^1 \vee \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$, $\Sigma^1 \vee \Sigma^4 \vee W_{4r+1,2r+1} \vee V_{2r}$, $\Sigma^1 \vee MU_{n-1,2n-1,n}$ *corresponding to the case* (1), (2), $\cdots$, (10) *of* (5.14), *and moreover*

ii) *the induced homomorphism* $f_{k,\ell *} : KU_0 Y_{k,\ell} \to KU_0 X_{k,\ell}$ *is identified (up to signs) with the homomorphism* $\varphi_{k,\ell}$ *defined in Lemma* 5.6.

Proposition 5.10. *Let* X *and* Y *be CW-spectra having the same quasi* KO_**-types as* $X_{k,\ell}$ *and* $Y_{k,\ell}$ *given in* (5.14) *respectively. Let* $f : Y \to X$ *be a map whose induced homomorphism* $f_* : KU_0 Y \to KU_0 X$ *is* *identified with the homomorphism* $\varphi_{k,\ell}$ *defined in Lemma* 5.6. *Then the cofiber of the map* f *is quasi* KO_**-equivalent to the small spectrum* $Z_{k,\ell}$ *appearing in* (5.15) i).

Proof. Choose quasi KO_*-equivalences $h_0 : Y \to KO \wedge Y_{k,\ell}$ and $h_1 : X \to KO \wedge X_{k,\ell}$ satisfying $(c \wedge f_{k,\ell})h_0 = (c \wedge 1)h_1 f : Y \to KU \wedge X_{k,\ell}$ where $c : KO \to KU$ denotes the complexification. It is sufficient to show that the equaility $(1 \wedge f_{k,\ell})h_0 = h_1 f : Y \to KO \wedge X_{k,\ell}$ holds in any case. By means of [10, Propositions 4.2 and 4.5] and Propositions 2.2 and 2.4 it is immediate that $[Y, \Sigma^1 KO \wedge X_{k,\ell}] \cong [Y_{k,\ell}, \Sigma^1 KO \wedge X_{k,\ell}] = 0$ except $(k,\ell) = (2m, 4r-1)$. Therefore our assertion that the equaility $(1 \wedge f_{k,\ell})h_0 = h_1 f$ holds is valid unless $(k,\ell) = (2m, 4r-1)$. In the $(k,\ell) = (2m, 4r-1)$ case we next show that our assertion is also valid after changing the quasi KO_*-equivalence $h_1 : X \to KO \wedge X_{2m,4r-1}$ suitably if necessary. As is easily seen, we can choose a certain map $g = (a\eta^2, 2^{2r-2}h_W, \tilde{\eta} + bi\eta^2) : \Sigma^2 \to \Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$ with $a, b \in Z/2$ satisfying $(1 \wedge g)h_0 = h_1 f : Y \to KO \wedge (\Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1})$. Consider the involution α on $\Sigma^0 \vee W_{4r-1,2r} \vee SZ/2^{2r-1}$ represented by the matrix $\begin{pmatrix} 1 & 0 & a\eta j \\ 0 & 1 & 0 \\ 0 & 0 & 1 + bi\eta j \end{pmatrix}$, and replace the quasi KO_*-equivalence h_1 by the composite map $h_1' = (1 \wedge \alpha)h_1$. Then we get the equalities $(c \wedge 1)h_1' = (c \wedge 1)h_1$ and $(1 \wedge f_{k,\ell})h_0 = h_1' f$ for the new quasi KO_*-equivalence h_1'. Hence our assertion is valid even if $(k,\ell) = (2m, 4r-1)$.

Combining Proposition 5.10 with Lemma 5.6 we can finally determine the quasi KO_*-types of $L_{2k}^{2k+2\ell+1}$.

Theorem 5.11. i) $\Sigma^{-4m+1} L_{4m}^{4m+2\ell+1} (\ell \geq 0)$ *is quasi* KO_**-equivalent to the following small spectrum:* $\Sigma^1 \vee MU_{2r-1,4r+1,2r}$, $\Sigma^1 \vee \Sigma^4 \vee V_{2r} \vee W_{4r+1,2r+1}$, $\Sigma^1 \vee MU_{2r,4r+3,2r+1}$, $\Sigma^1 \vee \Sigma^0 \vee SZ/2^{2r+1} \vee W_{4r+3,2r+2}$ *according as* $\ell = 4r$, $4r+1$, $4r+2$, $4r+3$. *Here* $MU_{-1,1,0} = \Sigma^2$ *and* $V_0 \vee W_{1,1}$ *should be replaced by* $\Sigma^2 SZ/4$ *in the* $\ell = 1$ *case.*

ii) $\Sigma^{-4m+1} L_{4m-2}^{4m+2\ell-1} (\ell \geq 0)$ *is quasi* KO_**-equivalent to the following small spectrum:* $\Sigma^0 \vee PU_{4r+1,2r,2r}$, $\Sigma^4 MP_{4r+3} \vee W_{2r,2r}$, $\Sigma^4 \vee \Sigma^4 PU_{4r+3,2r+1,2r+1}$, $\Sigma^4 MP_{4r+5} \vee W_{2r+1,2r+1}$ *according as* $\ell = 4r$, $4r+1$, $4r+2$, $4r+3$ *where* $PU_{1,0,0} = \Sigma^{-1}$.

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(Received September 9, 1992)

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