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IDEALS AND SYMMETRIC BI-DERIVATIONS OF PRIME AND SEMI-PRIME RINGS *

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Throughout this paper, R will represent an associative ring and $Z(R)$ will denote the center of R. We shall write $[x,y]$ for $xy-yx$. A mapping $B(.,.): R\times R\to R$ is called symmetric if $B(x,y)=B(y,x)$ holds for all pairs $x,y\in R$. A mapping $f: R\to R$ defined by $f(x)=B(x,x)$ is called the trace of B, where $B(.,.): R\times R\to R$ is a symmetric mapping. It is obvious that, if $B(,.,): R\times R\to R$ is a symmetric mapping which is also bi-additive (i.e. additive in both arguments), then the trace f of B satisfies the relation $f(x+y)=f(x)+f(y)+2B(x,y)$ for all $x,y\in R$. A symmetric bi-additive mapping $D(.,.): R\times R\to R$ is called a symmetric bi-derivation if $D(xy,z)=xD(y,z)+D(x,z)y$ is fulfilled for all $x,y,z\in R$. Then the relation $D(x,yz)=yD(x,z)+D(x,y)z$ is also fulfilled for all $x,y,z\in R$.

In [4], J. Vukman has proved some results concerning symmetric bi-derivation on prime and semi-prime rings. We shall substitute R for a non-zero ideal I of R and show that the results which are obtained in [4, Theorems 1, 2 and 3] are also valid in this situation.

We shall need the following well-known lemmas.

Lemma 1 ([3, Lemmas 2.1 and 2.2]). *Let $D: R\to R$ be a derivation of a prime ring R and I a non-zero ideal of R. Suppose that either* (i) *$aD(x)=0$ for all $x\in I$ or* (ii) *$D(x)a=0$ all $x\in I$ holds. Then $a=0$ or $D=0$.*

Lemma 2 ([1, Lemma 1]). *Let R be a prime ring and I a non-zero right ideal of R. If I is commutative, then R is commutative.*

The following lemma is a generalization of [2, Lemma 3.10].

Lemma 3. *Let R be a prime ring of* char$R\neq 2$ *and I a non-zero ideal of R. Let a, b be fixed elements of R. If $axb+bxa=0$ is fulfilled for all $x\in I$, then either $a=0$ or $b=0$.*

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Proof. We have $a(xbrax)b = -(b.xbr.a)xa = axbr(bxa) = -axbraxb$ for all $r \in R$, and so $2(axb)r(axb) = 0$ for all $r \in R$. Since R is a prime ring of $\mathrm{char}R \neq 2$, we get $axb = 0$ for all $x \in I$. Hence $a = 0$ or $b = 0$.

Lemma 4. *Let R be a prime ring of* $\mathrm{char}R \neq 2$ *and I a non-zero left (or right) ideal of R. Let $D(.,.) : R \times R \to R$ be a symmetric bi-derivation and d the trace of D. Suppose that $d(x) = 0$ for all $x \in I$. Then $d = 0$, that is, $D = 0$.*

Proof. We have

$$d(x) = 0 \text{ for all } x \in I. \tag{1}$$

The linearization of (1) gives us $d(x) + d(y) + 2D(x,y) = 0$ for all $x, y \in I$. Since $d(x) = d(y) = 0$ and $\mathrm{char}R \neq 2$, then

$$D(x,y) = 0 \text{ for all } x, y \in I. \tag{2}$$

Substituting y by ry ($r \in R$) in (2), we arrive at $D(x,r)y = 0$ for all $x, y \in I$ and $r \in R$. Since the left annihilator of a non-zero left ideal is zero, we have

$$D(x,r) = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{3}$$

Now, substituting x by rx in (3), we get $d(r)x = 0$ for all $x \in I$ and $r \in R$. Hence $d(r)$ is an element of the left annihilator of I. As the above, $d(r) = 0$ for all $r \in R$.

Theorem 1. *Let R be a non-commutative prime ring and I a non-zero ideal of R. Let $D(.,.) : R \times R \to R$ be a symmetric bi-derivation such that $D(I,I) \subset I$ and d the trace of D.*

(a) *If* $\mathrm{char}R \neq 2$ *and* $[x, d(x)] = 0$ *for all* $x \in I$, *then* $D = 0$.

(b) *If* $\mathrm{char}R \neq 2, 3$ *and* $[x, d(x)] \in Z(R)$ *for all* $x \in I$, *then* $D = 0$.

Proof. (a) I is not a commutative ideal of R by Lemma 2. Since I is a non-zero ideal of a prime ring R of $\mathrm{char}R \neq 2$, I itself is a non-commutative prime ring of $\mathrm{char}\, I \neq 2$. Therefore, $d(x) = 0$ for all $x \in I$ by [4, Theorem 1] and $d(r) = 0$ for all $r \in R$ by Lemma 4.

(b) Since $\mathrm{char}\, I \neq 2, 3$, we have $[x, d(x)] = 0$ for all $x \in I$ by the proof of [4, Theorem 2]. Hence $d(r) = 0$ for all $r \in R$ by (a).

Theorem 2. *Let R be a prime ring of* char$R \neq 2$ *and I a non-zero ideal of R. Suppose that there exist symmetric bi-derivations $D_1(.,.)$: $R \times R \to R$ and $D_2(.,.)$: $R \times R \to R$ such that $D_1(d_2(x), x) = 0$ for all $x \in I$, where d_2 denotes the trace of D_2. Then either $D_1 = 0$ or $D_2 = 0$.*

Proof. It is enough to show that $d_1(I) = 0$ or $d_2(I) = 0$ by Lemma 4. We get, by the proof of [4, Theorem 3],

$$d_2(x)D_1(x, y) + d_1(x)D_2(x, y) = 0 \text{ for all } x, y \in I, \tag{4}$$

and

$$d_1(x)yd_2(x) + d_2(x)yd_1(x) = 0 \text{ for all } x, y \in I. \tag{5}$$

Suppose that $d_1(I) \neq 0$ and $d_2(I) \neq 0$. Then there exist x_1, $x_2 \in I$ such that $d_1(x_1) \neq 0$ and $d_2(x_2) \neq 0$. In particular, $d_1(x_1)yd_2(x_1) + d_2(x_1)yd_1(x_1) = 0$ for all $y \in I$ by (5). Since $d_1(x_1) \neq 0$, we have $d_2(x_1) = 0$ by Lemma 3. Similarly, we get $d_1(x_2) = 0$. Then the equation (4) reduces to the equation $d_2(x_2)D_1(x_2, y) = 0$. Using this relation and Lemma 1, we obtain that $D_1(x_2, y) = 0$ holds for all $y \in I$ because of $d_2(x_2) \neq 0$ (recall that the mapping $y_1 \mapsto D_1(x_2, y)$ is a derivation). In particular, we have $D_1(x_2, x_1) = 0$. In the same way, we get $D_2(x_1, x_2) = 0$. Substituting $x_1 + x_2$ for y, we have $d_1(y) = d_1(x_1) + d_1(x_2) + 2D_1(x_1, x_2) = d_1(x_1) \neq 0$, and by the similar argument we have $d_2(y) \neq 0$. Hence we have $d_1(y) \neq 0$ and $d_2(y) \neq 0$; a contradiction by (5) and Lemma 3. So we get $d_1(I) = 0$ or $d_2(I) = 0$.

Difinition. Let R be a ring and I be a non-zero left (resp. right) ideal of R. We shall say that a mapping $D(.,.)$: $R \times R \to R$ acts as a left (resp. right) R-homomorphism on I if $D(rx, y) = rD(x, y)$ and $D(x, ry) = rD(x, y)$ (resp. $D(xr, y) = D(x, y)r$ and $D(x, yr) = D(x, y)r$) for all $x, y \in I$ and $r \in R$.

Let S be a set. $\ell_R(S)$ (resp. $r_R(S)$) will denote the left (resp. right) annihilator of S.

Theorem 3. *Let R be a ring and I a non-zero left (resp. right) ideal of R such that $\ell_R(I) = 0$ (resp. $r_R(I) = 0$). Let $D(.,.)$: $R \times R \to R$ be a symmetric bi-derivation. If D acts as a left (resp. right) R-homomorphism on I, then $D = 0$.*

Proof. Suppose that I is a left ideal such that $\ell_R(I) = 0$ and D acts as a left R-homomorphism on I. Then $rD(x,y) = D(x,ry) = D(x,r)y + rD(x,y)$ for all $x,y \in I$, $r \in R$ and $D(x,r)y = 0$ for all $x,y \in I$, $r \in R$. Hence $D(x,r) \in \ell_R(I) = 0$. Then we have $0 = D(sx,r) = sD(x,r) + D(s,r)x = D(s,r)x$ for all $x \in I$ and $r,s \in R$. As the above, $D(r,s) = 0$ for all $r,s \in R$.

Corollary 1. *Let R be a prime ring and I a non-zero left (resp. right) ideal of R. Let $D(.,.): R \times R \to R$ be a symmetric bi-derivation. If D acts as a left (resp. right) R-homomorphism on R, then $D = 0$.*

Corollary 2. *Let R be a semi-prime ring and $D(.,.): R \times R \to R$ a symmetric bi-derivation. If D acts as a left (or right) R-homomorphism on R, then $D = 0$.*

References

[1] Y. Hirano, A. Kaya and H. Tominaga: On a theorem of Mayne, Math. J. Okayama Univ. **25**(1983), 125–132.
[2] I. N. Herstein: Topics in Ring Theory, Univ. of Chicago Press, Chicago, 1969.
[3] K. Kaya: Prime rings with α-derivations, Hacettepe Bull. of Noth. Sci. and Eng. **16**(1987), 63–71.
[4] J. Vukman: Symmetric bi-derivations on prime and semi-prime rings, Aequationes Math. **38**(1989), 245–254.

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