Math. J. Okayama Univ. **35**(1993), 181-188

ON FINITENESS, COMMUTATIVITY, AND PERIODICITY IN RINGS

HOWARD E. BELL[1] and ABRAHAM A. KLEIN

Let R be an associative ring; let N be the set of nilpotent elements of R; and let P be the set of potent elements, i.e.

$P = \{x \in R|$ there exists $n = n(x) > 1$ such that $x^n = x\}$. Call R periodic if for each $x \in R$ there exist distinct positive integers m and n for which $x^m = x^n$; and as in [8], define R to be weakly periodic if $R = P + N$. It is known that all periodic rings are weakly periodic [1], but whether the converse holds is an open question.

In this paper we first present new proofs for two known finiteness theorems, and we then present some finiteness results for prime rings. In the final section, making use of our results for prime rings, we establish sufficient conditions for finiteness, commutativity, or periodicity of weakly periodic rings.

1. Preliminaries. Throughout the paper, $\mathbb{Z}$ will denote the ring of integers and $\mathbb{Z}^+$ the set of positive integers. For the ring R, Z will be the center and D the set of (not necessarily two-sided) zero divisors. If Y is either an element or a subset of R, then $A_\ell(Y)$, $A_r(Y)$, and $\langle Y \rangle$ will denote the left and right annihilators of Y and the subring generated by Y. The symbol $\mathcal{P}(R)$ will denote the maximal periodic ideal of R, the existence of which was established in [5].

The following results will be needed in our proofs.

Theorem 1.1 (Laffey, [10]). *If R is a ring in which every commutative subring is finite, then R is finite.*

Theorem 1.2 (Chacron [6]). *Let R have the property that for each $x \in R$, there exists $n \in \mathbb{Z}^+$ and $p(t) \in \mathbb{Z}[t]$ for which $x^n = x^{n+1}p(x)$. Then R is periodic.*

Theorem 1.3 (Herstein [9]). *If R is a periodic ring with $N \subseteq Z$,*

[1]Supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A3961.

then R is commutative.

Theorem 1.4 (Bell and Guerriero [4]). *If R is a periodic ring in which every proper noncentral subring of zero divisors is finite, then R is either finite or commutative.*

Theorem 1.5 (Bell and Tominaga [2]). *If R is a weakly periodic ring in which N is commutative, then R is periodic.*

2. Two finiteness proofs. We give first a new and simpler proof of a theorem of Putcha and Yaqub.

Theorem 2.1 ([13]). *If $R \neq N$ and $R \backslash N$ is finite, then R is finite.*

Proof. Since $R \backslash N$ is finite and power closed, it must contain an idempotent $e \neq 0$; and we can write

$$(1) \qquad R = eR + A_r(e) = eRe + eR \cap A_\ell(e) + A_r(e).$$

Now the mapping $x \to e + x$ is one-to-one on R; and it maps $A_r(e)$ into $R \backslash N$, for if $x \in A_r(e)$, $e(e+x) = e$ and hence $e + x \notin N$. Thus $A_r(e)$ is finite, and likewise $A_\ell(e)$. By (1), therefore, we need only show that eRe is finite. But e is an identity in eRe, so if $x \in eRe \cap N$, $e + x$ is invertible in eRe. It follows that $x \to e + x$ maps $eRe \cap N$ into $R \backslash N$, hence $eRe \cap N$ is finite. Since $eRe \backslash N$ is finite by hypothesis, eRe is finite.

We now give a new proof of a well-known theorem of Szele [14]. Our proof is not necessarily shorter or more elementary than Szele's; but it avoids the use of structure theory and fits better into the general context of finiteness theorems.

Theorem 2.2. *If the ring R has both ascending chain condition and descending chain condition on subrings, then R is finite.*

Proof. Since $\langle x \rangle \supseteq \langle x^2 \rangle \supseteq \langle x^4 \rangle \supseteq \cdots$ must become stationary, there exists n such that $x^{2^n} \in \langle x^{2^{n+1}} \rangle$; therefore R is periodic by Theorem 1.2, and every potent element has finite additive order. We now show by induction on index of nilpotence that nilpotent elements are also of finite additive order.

If $x^2 = 0$, then the subring $\langle x \rangle$ is a zero subring on a cyclic group. Since the infinite cyclic group does not have dcc on subgroups, $\langle x \rangle$ must be finite. Proceeding under the appropriate inductive hypothesis, note that if $x^n = 0$, $n \geq 3$, then x^2 has index $< n$, hence is of finite order k. Then $(kx)^2 = 0$, hence kx is of finite order and so is x.

As mentioned earlier, every element of a periodic ring is a sum of a potent element and a nilpotent element, hence $(R, +)$ is a torsion group. Consider now an arbitrary commutative subring S of R, which inherits the chain conditions on R. By acc, S is finitely generated as a ring; and since S is also periodic with $(S, +)$ a torsion group, S is finite. By Theorem 1.1 it follows that R is finite.

Our proof actually yields a bit more than Szele's theorem — specifically, if R has acc on commutative subrings and dcc on monogenic subrings, then R is finite. We mention that the symmetric condition — acc on monogenic subrings and dcc on commutative subrings — also implies finiteness; this can be seen from an argument in [11].

3. Finiteness results for prime rings. Our first lemma is an application of an interesting result of Lewin on subrings of finite index.

Lemma 3.1. *If R is an infinite prime ring, then for any $c \in R\backslash\{0\}$, the right ideal cR is infinite.*

Proof. If cR is finite, the kernel of the map $x \to cx$ has finite index in R. This kernel is clearly a subring of R, hence by a result of Lewin [12] contains an ideal I of finite index. Since R is infinite, $I \neq \{0\}$, contradicting the fact that in a prime ring the left annihilator of a nonzero ideal is trivial.

Corollary 3.2. *Any nonzero one-sided ideal of an infinite prime ring is infinite.*

Theorem 3.3. *An infinite prime ring which is not a domain contains an infinite zero subring.*

Proof. It is well known that a prime ring R which is not a domain contains nonzero nilpotent elements, hence a nonzero element a with $a^2 = 0$. Assume that every zero subring is finite. Then aRa is finite, and the kernel K of the map $x \to axa$ is an additive subgroup of finite index. Thus we have $R = \cup_{i=1}^{n}(K + b_i)$ for suitable elements $b_1, b_2, \cdots, b_n \in R$.

Now $aKa = \{0\}$, so the subring $aRaK$ is a zero subring, hence is finite. Moreover, $aRab_i$ is finite for each $i = 1, 2, \cdots, n$; and since $aRaR = aRa(\cup_{i=1}^{n}(K + b_i)) \subseteq aRaK + \Sigma_{i=1}^{n} aRab_i$, we conclude that $aRaR$ is finite. But $aRa \neq \{0\}$ because R is prime; hence, for $c \in aRa\backslash\{0\}$, we get cR finite, contrary to Lemma 3.1.

4. Weakly periodic rings.

We begin by extending Theorem 1.4 to weakly periodic rings.

Theorem 4.1. *If R is a weakly periodic ring in which every proper noncentral subring of zero divisors is finite, then R is either finite or commutative.*

The proof will require several lemmas.

Lemma 4.2. *If R is a weakly periodic ring with N=D, then R is periodic.*

Proof. We need only show that N is an ideal, for in that case for each $x \in R$ we have $n = n(x) > 1$ for which $x - x^n \in N$, so that R is periodic by Theorem 1.2.

We may assume that $R \neq N$. Thus, there exist nonzero potent elements, hence nonzero idempotents, all of which are regular. There is therefore a unique nonzero idempotent, which must be an identity element 1; and every nonzero potent element is invertible. Letting U be the set of units, we complete the proof by showing that $R = U \cup N$, a condition which is known to imply that N is an ideal.

Accordingly let $a \in R\backslash N$, and write $a = u + b$, where $u \neq 0$ is potent and $b \in N$. Now $u \in U$ and $u^{-1}b \in D = N$, hence $1 + u^{-1}b \in U$ and hence $a = u(1 + u^{-1}b) \in U$.

Lemma 4.3. *If R is a weakly periodic ring in which every proper noncentral subring is finite, then R is either finite or commutative.*

Proof. If R is not commutative and $x \in R\backslash Z$, then $\langle x, Z\rangle$ is proper and noncentral, hence finite. Therefore Z is finite, and R is finite by Theorem 1.1.

Lemma 4.4. *If R is a weakly periodic ring in which $D\backslash N \subseteq Z$, then* *every subring of zero divisors is periodic.*

Proof. Let $w \in D\backslash N$ and write $w = y + b$, with $y^k = y$, $k > 1$, and $b \in N$. Since $w \in Z$, $[y, b] = 0$; hence $w^k - w \in N$. The same condition is satisfied by all $w \in N$, hence Theorem 1.2 shows that every subring of zero divisors is periodic.

Proof of Theorem 4.1. By Lemma 4.2 and Theorem 1.4, we may assume that $N \neq D$. If $D\backslash N \not\subseteq Z$, then for $d \in (D\backslash N)\backslash Z$, $\langle d\rangle$ is a proper noncentral subring of zero divisors, hence is finite. On the other hand, if $D\backslash N \subseteq Z$, Lemma 4.4 shows that $\langle d\rangle$ is periodic for any choice of $d \in D\backslash N$. Thus, in any case we get a nonzero idempotent zero divisor e, which we may assume to be a right zero divisor.

Write $R = eR + A_r(e)$. Now $A_r(e) \neq R$; and if $eR = R$, our theorem follows by Lemma 4.3. Thus, we assume that eR and $A_r(e)$ are both proper subrings of zero divisors, so that each is finite or central. If both are finite, then R is finite; and if both are central, R is commutative. Therefore we assume that one of eR and $A_r(e)$ is finite and noncommutative and the other is central; and we denote these by F and C respectively. Since F is noncommutative, Theorem 1.3 gives $y \in F \cap N$ which is not in Z. Suppose $y^k = 0 \neq y^{k-1}$. Since $CF = \{0\}$, we have $vy^{k-1} = 0$ for all $v \in C$; hence $\langle y, C\rangle$ is a proper noncentral subring of zero divisors. Thus C is finite and R is finite.

Corollary 4.5. *If R satisfies the hypotheses of Theorem 4.1, then R is periodic.*

Proof. By Theorem 1.5, commutative weakly periodic rings are periodic.

Corollary 4.6. *If R is any weakly periodic ring with only finitely many noncentral zero divisors, then R is either finite or commutative.*

Proof. Suppose R has proper noncentral subrings of zero divisors. Let S be any such subring, and let $d \in S\backslash Z$. If $x \in S \cap Z$, then $d+x \in D\backslash Z$, hence there are only finitely many such x and S is therefore finite. By Theorem 4.1, R is either finite or commutative.

As noted in [3], the finiteness hypothesis in Corollary 4.6 cannot be weakened to the condition that R has only finitely many noncentral nilpotent elements. However, we do have the following result, which may be regarded as an extension of Theorem 1.5.

Theorem 4.7. *If R is a weakly periodic ring with only finitely many noncentral nilpotent elements, then R is periodic.*

A fundamental difficulty with weak periodicity is that it is not obviously inherited by subrings; indeed, if we knew that commutative subrings of weakly periodic subrings are weakly periodic, it would follow by Theorem 1.5 that all weakly periodic rings are periodic. However, the following lemma gets us around this difficulty in the proof of Theorem 4.7.

Lemma 4.8. *If R is a weakly periodic ring, every ideal I of R is weakly periodic.*

Proof. Let $b \in I$ and write $b = a + u$, $a \in P$, $u \in N$, $u^k = 0$. We prove by induction that for $i = 0, 1, \cdots, k$, any product of i a's and $k - i$ u's is in I. For $i = 0$, this is just the statement that $u^k = 0 \in I$. To proceed from i to $i + 1$, consider any product π of $i + 1$ a's and $k - (i + 1)$ u's, and write $\pi = vaw$ where vw is a product of i a's and $k - i - 1$ u's. By the inductive hypothesis, $vuw \in I$; hence $\pi = vbw - vuw \in I$, and the induction is complete.

The case $i = k$ gives $a^k \in I$; and since $a \in P$, $a \in I$ and $u = b - a \in I$. Thus, I is weakly periodic.

We also require

Lemma 4.9. *Let R be any ring and I a periodic ideal. Then any nilpotent element of R/I is of the form $u + I$ for some nilpotent element $u \in R$.*

Proof. Suppose $x + I$ is nilpotent in R/I. Then $x^k \in I$ for some $k \in \mathbb{Z}^+$, hence there exist distinct $m, n \in \mathbb{Z}^+$ with $n < m$ and $x^{km} = x^{kn}$. It is easily shown that $u = x - x^{1+k(m-n)} \in N$; and since $x^{1+k(m-n)} \in I$, we have $x + I = u + I$.

Proof of Theorem 4.7. Assume there is a counterexample R, which by Theorem 1.5 must have noncentral nilpotent elements. For $x \in N \backslash Z$ and $y \in N \cap Z$, $x + y \in N \backslash Z$; hence $N \cap Z$ is finite and N is finite. We may suppose, therefore, that R is a counterexample with a minimal number of nilpotent elements.

Consider $\bar{R} = R/\mathcal{P}(R)$, where $\mathcal{P}(R)$ is the maximal periodic ideal. Clearly $\bar{R}$ is weakly periodic, and by Lemma 4.9 it has no more noncentral nilpotent elements than R has. Moreover, $\bar{R}$ is not periodic, since otherwise we could use Theorem 1.2 to get R periodic. Replacing R by $\bar{R}$, we assume that R has no nonzero periodic ideals.

Let I be any nonzero ideal of R. Since I is weakly periodic, it must contain nonzero nilpotent elements; otherwise it would be a nonzero periodic ideal. If I contained some but not all of the nonzero nilpotent elements of R, our minimality assumption would force I to be periodic; therefore I contains all nonzero nilpotent elements of R. It follows that R is subdirectly irreducible.

Now the heart H of a subdirectly irreducible ring is either a simple ring or a zero ring [7, Lemma 75]; and under our assumption, H must be simple. Thus, H is a simple counterexample with only finitely many nilpotent elements, contrary to Theorem 3.3. This completes the proof.

REFERENCES

[1] H. E. BELL: A commutativity study for periodic rings, Pacific J. Math. **70**(1977), 29–36.

[2] H. E. BELL and H. TOMINAGA: On periodic rings and related rings, Math. J. Okayama Univ. **28**(1986), 101–103.

[3] H. E. BELL: Some conditions for finiteness of a ring, Int. J. Math. Math. Sci. **11**(1988), 239–242.

[4] H. E. BELL and F. GUERRIERO: Some conditions for finiteness and commutativity of rings, Int. J. Math. Math. Sci. **13**(1990), 535–544.

[5] H. E. BELL and A. A. KLEIN: On rings with Engel cycles, Canad. Math. Bull. **34**(1991), 295–300.

[6] M. CHACRON: On a theorem of Herstein, Canad. J. Math. **21**(1969), 1348–1353.

[7] N. J. DIVINSKY: Rings and Radicals, Univ. of Toronto Press, 1965.

[8] J. GROSEN, H. TOMINAGA and A. YAQUB: On weakly periodic rings, periodic rings and commutativity theorems, Math. J. Okayama Univ. **32**(1990), 77–81.

[9] I. N. HERSTEIN: A note on rings with central nilpotent elements, Proc. Amer. Math. Soc. **5**(1954), 620.

[10] T. J. LAFFEY: On commutative subrings of infinite rings, Bull. London Math. Soc. **4**(1972), 3–5.

[11] T. J. LAFFEY: Infinite rings with all proper subrings finite, Amer. Math. Monthly **81**(1974), 270–272.

[12] J. LEWIN: Subrings of finite index in finitely-generated rings, J. Algebra **5**(1967), 84–88.

[13] M. S. PUTCHA and A. YAQUB: Rings with a finite set of non-nilpotents, Int. J. Math. Math. Sci. **2**(1979), 121–126.

[14] T. SZELE: On a finiteness criterion for modules, Publ. Math. Debrecen **3**(1954), 253–256.

HOWARD E. BELL
DEPARTMENT OF MATHEMATICS
BROCK UNIVERSITY
ST. CATHARINES, ONTARIO
CANADA L2S 3A1

ABRAHAM A. KLEIN
SACKLER FACULTY OF EXACT SCIENCES
SCHOOL OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
RAMAT AVIV
69978 TEL-AVIV
ISRAEL

(*Received August 5, 1992*)