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THE PROPERTY (DF) FOR REGULAR RINGS WHOSE PRIMITIVE FACTOR RINGS ARE ARTINIAN

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In [4] and [5], we have studied the property that the direct sum of two directly finite projective modules is directly finite. We call this property (DF). In this paper, we shall investigate the property (DF) for regular rings R whose primitive factor rings are artinian. In §1, We give a criterion of the directly finiteness of projective modules over these rings (Theorem 1.3), and using this criterion, we can characterize the property (DF) for regular rings R whose primitive factor rings are artinian (Theorem 1.4). In §2, for those regular rings R we shall investigate the transivity of the property (DF) from the maximal right quotient ring $Q(R)$ of R to R. We can obtain the main result that for a regular ring R whose primitive factor rings are artinian, if $Q(R)$ has the property (DF), then so does R (Theorem 2.5).

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules.

§1. Preliminaries and directly finite projective modules

Notation. For two R-modules P and Q, we use $P \lesssim Q$ (resp. $P \lesssim \oplus Q$) to mean that P is isomorphic to a submodule of Q (resp. a direct summand of Q). For a submodule P of an R-module Q, $P < \oplus Q$ and $P \leqq_e Q$ mean that P is a direct summand of Q and an essential submodule of Q respectively. For a cardinal number k and an R-module P, kP denotes a direct sum of k-copies of P. For any R-module P, we use $E(P)$ to denote the injective hull of P.

Definition. An R-module P is *directly finite* provided that P is not isomophic to a proper direct summand of itself. If P is not directly finite, then P is said to be *directly infinite.*

Definition A regular ring R is said that *primitive factors are artinian* if R/P is artinian for all left primitive ideals P of R.

Definition A regular ring R is *abelian* provided all idempotents in R are central.

We shall recall the following basic properties.

(1) If P is a projective module over a regular ring, then all finitely generated submodules of P are direct summands of P ([1, Theorem 1.11]).

(2) Every projective modules over regular rings have the exchange property (see [4] and [6]), and every injective modules have this property.

(3) If R is a regular ring whose primitive factor rings are artinian, then it is unit-regular, and so all finitely generated projective R-modules have the cancellation property ([1, Theorem 4.14 and 6.10]).

(4) Let R be a regular ring whose primitive factor rings are artinian, and P be a finitely generated projective R-module. Then $End_R(P)$ is a regular ring whose primitive factor rings are artinian ([1, Corollary 6.4]).

(5) Every abelian regular rings are regular rings whose primitive factor rings are artinian ([1, Theorems 3.2 and 6.2]).

Definition Let P be a finitely generated projective module over a regular ring R. We use $L(P)$ to denote the lattice of all finitely generated submodules of P, partially ordered by inclusion.

Lemma 1.1 (cf. [1, Proposition 2.4] and [5, Lemma 5]). *Let P be a finitely generated projective module over a regular ring R, and set $T = End_R(P)$.*

(a) *There is a lattice isomorphism F: $L(T_T) \to L(P)$, given by the rule $F(J) = JP$. For $A \in L(P)$, we have $F^{-1}(A) = \{f \in T | fP \leqq A\}$.*

(b) *For J, $K \in L(T_T)$, we have $J \simeq K$ if and only if $F(J) \simeq F(K)$.*

(c) *For J, $K \in L(T_T)$, we have $J \lesssim K$ if and only if $F(J) \lesssim F(K)$.*

(d) *For J, $K \in L(T_T)$ such that $J \oplus K \in L(T_T)$, we have that $F(J \oplus K) = F(J) \oplus F(K)$. For A, $B \in L(P)$ such that $A \oplus B \in L(P)$, we have that $F^{-1}(A \oplus B) = F^{-1}(A) \oplus F^{-1}(B)$.*

Lemma 1.2. *Let R be a regular ring whose primitive factor rings are artinian, and P be a nonzero finitely generated projective R-module. Then P can not contain a chain $J_1 \geqq J_2 \geqq \dots$ of nonzero finitely generated submodules such that $nJ_n \lesssim P$ for all $n = 1, 2, \dots$.*

Proof. Put $T = End_R(P)$, and note that T is a regular ring whose primitive factor rings are artinian. It is well-known from the proof of [1, Theorem 6.6] that T can not contain a chain $J_1 \geqq J_2 \geqq \dots$ of nonzero cyclic right ideals such that $nJ_n \lesssim T_T$ for all n. Using Lemma 1.1, we conclude that this lemma holds.

Now we shall give the criterion of directly infiniteness of a projective module over a regular ring whose primitive factor rings are artinian.

Theorem 1.3. *Let R be a regular ring whose primitive factor rings are artinian. For a projective R-module P with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$, the following conditions are equivalent:*

(a) *P is directly infinite.*

(b) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$.*

(c) *There exists a nonzero cyclic projective R-module X such that $X \lesssim \bigoplus_{i \in I - \{i_1, i_2, \dots, i_n\}} P_i$ for all finite subsets $\{i_1, i_2, \dots, i_n\}$ of I.*

(d) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim_{\oplus} P$.*

Proof. It is obvious that (a) → (b) and (c) → (d) → (a) hold. (b) → (c). Assume that (b), i.e., there exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$. Let $\{i_1, i_2, \dots, i_n\}$ be a subset of I, and set $I' = I - \{i_1, i_2, \dots, i_n\}$. Since $X \lesssim_{\oplus} P = \bigoplus_{i \in I} P_i$, there exist decompositions $P_i = P_i^1 \oplus P_i^{(1)}$ for each $i \in I$ such that $X \simeq P_{i_1}^1 \oplus \dots \oplus P_{i_n}^1 \oplus (\bigoplus_{i \in I'} P_i^1)$. Note that $2X \lesssim_{\oplus} P = \bigoplus_{i \in I} P_i$ and X has the cancellation property. Then there exist decompositions $P_i^{(1)} = P_i^2 \oplus P_i^{(2)}$ for each $i \in I$ such that $X \simeq P_{i_1}^2 \oplus \dots \oplus P_{i_n}^2 \oplus (\bigoplus_{i \in I'} P_i^2)$. We continue this procedure. For each $m = 1, 2, \dots$, there exist decompositions $P_i^{(m)} = P_i^{m+1} \oplus P_i^{(m+1)}$ for each $i \in I$ such that $X \simeq P_{i_1}^{m+1} \oplus \dots \oplus P_{i_n}^{m+1} \oplus (\bigoplus_{i \in I'} P_i^{m+1})$. Put $A_m = P_{i_1}^m \oplus \dots \oplus P_{i_n}^m$ for each m. Note that $A_m <_{\oplus} P_{i_1} \oplus \dots \oplus P_{i_n}$. Since $A_1 \lesssim_{\oplus} X \simeq A_2 \oplus (\bigoplus_{i \in I'} P_i^2)$, we have R-submodules A_2' of A_2 and $\overline{P}_i^2$ of P_i^2 for each $i \in I$ such that $A_1 \simeq A_2' \oplus (\bigoplus_{i \in I'} \overline{P}_i^2)$. And since $A_2' \lesssim_{\oplus} X \simeq A_3 \oplus (\bigoplus_{i \in I'} P_i^3)$, we have R-submodules A_3' of A_3 and $\overline{P}_i^3$ of P_i^3 for each $i \in I'$ such that $A_2' \simeq A_3' \oplus (\bigoplus_{i \in I'} \overline{P}^3)$. Continuing this procedure, we have an independent family $\{A_1', A_2', \dots\}$ of finitely generated submodules of $P_{i_1} \oplus \dots \oplus P_{i_n}$ such that $A_m' \gtrsim A_{m+1}'$ and $mA_m' \lesssim P_{i_1} \oplus \dots \oplus P_{i_n}$ for each m, where $A_1' = A_1$. From Lemma 1.2, there exists a positive integer

k such that $A'_k = 0$, and hence

$$\begin{aligned} X &\simeq (\bigoplus_{i\in I'} P_i^1) \oplus A'_1 \\ &\simeq (\bigoplus_{i\in I'} P_i^1) \oplus (\bigoplus_{i\in I'} \overline{P}_i^2) \oplus A'_2 \\ &\quad \cdots\cdots\cdots \\ &\simeq (\bigoplus_{i\in I'} P_i^1) \oplus (\bigoplus_{i\in I'} \overline{P}_i^2) \oplus \ldots \oplus (\bigoplus_{i\in I'} \overline{P}_i^k) \oplus A'_k \\ &= (\bigoplus_{i\in I'} P_i^1) \oplus (\bigoplus_{i\in I'} \overline{P}_i^2) \oplus \ldots \oplus (\bigoplus_{i\in I'} \overline{P}_i^k) \\ &\leqq \bigoplus_{i\in I'} P_i. \end{aligned}$$

Therefore we conclude that (b) $\to$ (c) holds.

Let R be a regular ring. For a given nonzero finitely generated projective R-module P, we consider the following condition:

($\sharp$) For each nonzero finitely generated submodule Q of P and each family $\{A_1, B_1, A_2, B_2, \ldots\}$ of submodules of Q with decompositions

$$\begin{aligned} Q &= A_1 \oplus B_1, \\ A_i &= A_{2i} \oplus B_{2i} \text{ and} \\ B_i &= A_{2i+1} \oplus B_{2i+1} \text{ for each } i = 1, 2, \ldots, \end{aligned}$$

there exists a nonzero finitely generated projective R-module X such that $X \lesssim \bigoplus_{i=m}^{\infty} A_i$ or $X \lesssim \bigoplus_{i=m}^{\infty} B_i$ for each positive integer m.

For a given regular ring R, we consider the following property:

(DF) The direct sum of two directly finite projective R-modules is directly finite.

This property (DF) for a regular ring whose primitive factor rings are artinian is characterized as following, using Theorem 1.3 and the proof of [5, Lemmas 5, 6, 7 and Theorem 8].

Theorem 1.4. *Let R be a regular ring whose primitive factor rings are artinian. Then the following conditions are equivalent:*

(a) *R has the property* (DF).

(b) *R satisfies the condition* ($\sharp$) *as an R-module.*

(c) *For any nonzero finitely generated projective R-module P, $End_R(P)$ has the property* (DF).

(d) *For any positive integer k, $M_k(R)$ has the property* (DF).

(e) *There exists a positive integer k such that $M_k(R)$ satisfies the property* (DF).

§2. The property (DF) for the maximal right quotient ring

Proposition 2.1. *Let R be an abelian regular ring and $Q(R)$ be the maximal right quotient ring of R. If $Q(R)$ has the property* (DF), *then so does R.*

Proof. We assume that R does not have the property (DF). By Theorem 1.4, we have a nonzero cyclic right ideal I of R and a family $\{A_1, B_1, A_2, B_2, \ldots\}$ of submodules of I with decompositions

$$\begin{aligned} I &= A_1 \oplus B_1, \\ A_i &= A_{2i} \oplus B_{2i} \text{ and} \\ B_i &= A_{2i+1} \oplus B_{2i+1} \text{ for each } i = 1, 2, \ldots \end{aligned}$$

such that there exists a positive integer m with $X \not\lesssim \bigoplus_{i=m}^{\infty} A_i$ and $X \not\lesssim \bigoplus_{i=m}^{\infty} B_i$ for each nonzero cyclic projective R-module X. Then we have a nonzero cyclic right ideal $E(I)$ of $Q(R)$ and a family $\{E(A_1), E(B_1), E(A_2), E(B_2), \ldots\}$ of submodules of $E(I)$ with decompositions

$$\begin{aligned} E(I) &= E(A_1) \oplus E(B_1), \\ E(A_i) &= E(A_{2i}) \oplus E(B_{2i}) \text{ and} \\ E(B_i) &= E(A_{2i+1}) \oplus E(B_{2i+1}) \text{ for each } i = 1, 2, \ldots. \end{aligned}$$

To prove this Theorem, it is enough from Theorem 1.4 to show that for each nonzero cyclic right ideal Y of $Q(R)$, there exists a positive integer n such that $Y \not\lesssim \bigoplus_{i=n}^{\infty} E(A_i)$ and $Y \not\lesssim \bigoplus_{i=n}^{\infty} E(B_i)$. Since $Q(R)$ is abelian ([1, Theorem 3.8]), there is no loss of generality in assuming from [1, Theorem 3.4] that $Y \leqq E(I)$. Since $0 \neq Y \cap I \leqq_e Y_R$, we have a nonzero cyclic submodule X of an R-module $Y \cap I$ and hence there exists a positive integer m such that $X \not\lesssim \bigoplus_{i=m}^{\infty} A_i$ and $X \not\lesssim \bigoplus_{i=m}^{\infty} B_i$. Now let k_1 be the smallest positive integer satisfying $m \leqq 2^{k_1}$, and so $X \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} A_i$ and $X \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} B_i$. Assume that $E(X) \lesssim \bigoplus_{i=2^{k_1}}^{\infty} E(A_i)$. Note that

$$E(X) \lesssim E(A_{2^{k_1}}) \oplus \ldots \oplus E(A_{2^{k_{n+1}}-1}) \qquad \ldots\ldots\ldots(*)$$

for some positive integer n, where $k_{n+1} = k_n + 1$. Since $X <\oplus I = A_{2^{k_1}} \oplus B_{2^{k_1}} \oplus \ldots \oplus A_{2^{k_2}-1} \oplus B_{2^{k_2}-1}$, we have a decomposition $X = V^1 \oplus W^1$ such that

$$V^1 \simeq A'_{2^{k_1}} \oplus \ldots \oplus A'_{2^{k_2}-1} \text{ and } W^1 \simeq B'_{2^{k_1}} \oplus \ldots \oplus B'_{2^{k_2}-1},$$

where $A'_i <\oplus A_i$ and $B'_i <\oplus B_i$. Noting that $X \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} A_i$ and R is abelian, we have $W^1 \neq 0$ and $B_1 \not\lesssim A_{2^{k_1}} \oplus \ldots \oplus A_{2^{k_2}-1}$ for each nonzero direct summand B_1 of W^1 from [1, Theorem 3.4]. Next, since $0 \neq W^1 < \oplus I = A_{2^{k_2}} \oplus B_{2^{k_2}} \oplus \ldots \oplus A_{2^{k_3}-1} \oplus B_{2^{k_3}-1}$, using the same discussion as above, we have a decomposition $W^1 = V^2 \oplus W^2$ such that

$$V^2 \simeq A'_{2^{k_2}} \oplus \ldots \oplus A'_{2^{k_3}-1} \text{ and } W^2 \simeq B'_{2^{k_2}} \oplus \ldots \oplus B'_{2^{k_3}-1},$$

where $A'_i < \oplus A_i$ and $B'_i < \oplus B_i$, and that $W^2 \neq 0$ and $B_2 \not\lesssim A_{2^{k_2}} \oplus \ldots \oplus A_{2^{k_3}-1}$ for each nonzero direct summand B_2 of W^2. Continuing this procedure, we have that $X \oplus\!\!> W^1 \oplus\!\!> W^2 \oplus\!\!> \ldots \oplus\!\!> W^n \neq 0$, and

$$B_n \not\lesssim A_{2^{k_n}} \oplus \ldots \oplus A_{2^{k_{n+1}}-1} \qquad \ldots\ldots\ldots(**)$$

for each nonzero direct summand B_n of W^n and each n. By $(*)$, we see that $0 \neq W^n \leqq E(X) \lesssim \oplus E(A_{2^{k_1}}) \oplus \ldots \oplus E(A_{2^{k_{n+1}}-1})$ and $E(X) \simeq E_{2^{k_1}} \oplus \ldots \oplus E_{2^{k_{n+1}}-1}$, where $E_i < \oplus E(A_i)$, and hence $0 \neq X' \simeq (E_{2^{k_1}} \cap A_{2^{k_1}}) \oplus \ldots \oplus (E_{2^{k_{n+1}}-1} \cap A_{2^{k_{n+1}}-1})$ for some $X' \leqq_e E(X)$. As a result, we have that $0 \neq W^n \cap X' \lesssim (E_{2^{k_1}} \cap A_{2^{k_1}}) \oplus \ldots \oplus (E_{2^{k_{n+1}}-1} \cap A_{2^{k_{n+1}}-1})$. Therefore there exists a nonzero direct summand Z of W^n such that $Z \lesssim A_{2^{k_1}} \oplus \ldots \oplus A_{2^{k_{n+1}}-1}$, which contradicts from $(**)$. Therefore we have that $E(X) \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} E(A_i)$ as a $Q(R)$-module, and similarly $E(X) \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} E(B_i)$. Noting that $E(X) \leqq Y$, we have that $Y \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} E(A_i)$ and $Y \not\lesssim \bigoplus_{i=2^{k_1}}^{\infty} E(B_i)$. As a result, $Q(R)$ does not have the property (DF). The proof is complete.

Lemma 2.2 ([1, Theorem 6.6]). *Let R be a regular ring whose primitive factor rings are artinian. If J is any nonzero two-sided ideal of R, then J contains a nonzero central idempotent e such that eR is isomorphic to a full matrix ring over an abelian regular ring.*

Lemma 2.3 ([2, Theorem 2]). *Let R be a regular ring and $Q(R)$ be the maximal right quotient ring of R. Then the following conditions are equivalent:*

(a) *$Q(R) \simeq \prod M_{n(t)}(S_t)$, where each S_t is an abelian regular, self-injective ring.*

(b) *There exist orthogonal central idempotents $\{e_{t_\alpha}\}$ such that $e_{t_\alpha}R \simeq M_{n(t)}(A_{t_\alpha})$, where each A_t is an abelian regular ring and $\sum \oplus_{t,\alpha \in K_t} e_{t_\alpha}R \leqq_e R_R$.*

(c) *For any nonzero two-sided ideal J of R, J contains a nonzero central idempotent e such that eR is isomorphic to a full matrix ring over an abelian regular ring.*

(d) *R is right bounded and every nonzero two-sided ideal of R contains a nonzero central idempotent of R.*

Remark 1. Let R be a regular ring whose primitive factor rings are artinian and $Q(R)$ be the maximal right quotient ring of R. Using Lemmas 2.2 and 2.3, for each nonzero element x in $Q(R)$, there exists an element r in R such that $0 \neq xr \in \sum \oplus e_{t_\alpha}R$, and so $0 \neq xre_{t_\alpha} \in e_{t_\alpha}R$ for some central idempotent e_{t_α} in R. For this e_{t_α}, we have the ring decomposition $R = e_{t_\alpha}R \times (1 - e_{t_\alpha})R$ such that $e_{t_\alpha}R \simeq M_{n(t)}(A_{t_\alpha})$, where A_{t_α} is an abelian regular ring.

Lemma 2.4. *Let R be a regular ring with a ring decomposition $R = S \times T$, where S is a regular ring whose primitive factor rings are artinian. Let P be a projective R-module, and $P \simeq \bigoplus_{i \in I} x_iR$ be a direct sum decomposition of cyclic right ideals of R. We put a projective S-module $\overline{P} = P \otimes_R S$.*

(a) *If P is a directly infinite R-module, and there exists a nonzero element x in R such that $xS \neq 0$ and $\aleph_0(xR) \lesssim P$, then $\overline{P}$ is a directly infinite S-module.*

(b) *If $\overline{P}$ is a directly infinite S-module, then P is a directly infinite R-module.*

Proof. Note that $\overline{P} \simeq \bigoplus_{i \in I}(x_iR \otimes_R S_S) \simeq \bigoplus_{i \in I} x_iS$. (a) Since $\aleph_0(xR) \lesssim P$ and a left R-module S is flat, we have that $0 \neq \aleph_0(xS) \simeq \aleph_0(xR \otimes_R S_S) \lesssim \overline{P}$, and hence $\overline{P}$ is a directly infinite S-module from Theorem 1.3. (b) Assume that $\overline{P}$ is a directly infinite S-module. Then there exists a nonzero element x in S and a sequence $i_1 < i_2 < \ldots$ of positive integers such that

$$xS \lesssim (x_{i_1}R \otimes_R S) \oplus \ldots \oplus (x_{i_2}R \otimes_R S)$$

$$xS \lesssim (x_{i_2+1}R \otimes_R S) \oplus \ldots \oplus (x_{i_3}R \otimes_R S)$$
$$\cdots\cdots\cdots$$

from Theorem 1.3. Nothing that a left R-module S is flat,

$$0 \neq xR \simeq xS \otimes_S R_R \lesssim ((x_{i_1}R \otimes_R S_S) \otimes_S R_R) \oplus \ldots \oplus ((x_{i_2}R \otimes_R S_S) \otimes_S R_R) \simeq x_{i_1}R \oplus \ldots \oplus x_{i_2}R,$$
$$0 \neq xR \simeq xS \otimes_S R_R \lesssim ((x_{i_2+1}R \otimes_R S_S) \otimes_S R_R) \oplus \ldots \oplus ((x_{i_3}R \otimes_R S_S) \otimes_S R_R) \simeq x_{i_2+1}R \oplus \ldots \oplus x_{i_3}R,$$
$$\cdots\cdots\cdots$$

Therefore we have that $\aleph_0(xR) \lesssim \oplus P_R$, where P is a directly infinite R-module.

Let R be a ring with a ring decomposition $R = S \times T$, and let U and V be a S-module and a T-module respectively. Then we can define an additive group $U \oplus V$ as an R-module by a natural way, which is defined the product (u, v) and (s, t) as following: $(u, v)(s, t) = (us, vt)$ for each $u \in U$, $v \in V$, $s \in S$ and $t \in T$.

Now, we shall show our main theorem.

Theorem 2.5. *Let R be a regular ring whose primitive factor rings are artinian. If the maximal right quotient ring $Q(R)$ of R has the property* (DF), *then so does R.*

Proof. Assume that R does not have the property (DF). Then there exist directly finite projective R-modules P and Q such that $P \oplus Q$ is directly infinite, and so we can take a nonzero element x in R satisfying $\aleph_0(xR) \lesssim \oplus P \oplus Q$ from Theorem 1.3. By Remark 1, we have the ring decomposition $R = S \times T$ such that $xrS \neq 0$ for some nonzero element r in R and $S \simeq M_n(L)$ for some n, where L is an abelian regular ring. Note that $\aleph_0(xrR) \lesssim \oplus P \oplus Q$. Using Lemma 2.4, we have directly finite projective S-modules $\overline{P}$ and $\overline{Q}$ such that $\overline{P} \oplus \overline{Q}$ is directly infinite. Therefore S does not have the property (DF), and so L, $Q(L)$ and $M_n(Q(L))$ do not have (DF) from Proposition 2.1 and Theorem 1.4. Note that $Q(R) = Q(S) \times Q(T)$ and $Q(S) \simeq M_n(Q(L))$, where $Q(S)$ and $Q(T)$ are the maximal right quotient ring of S and T respectively. As a result, $Q(S)$ does not have the property (DF), and hence we have directly finite projective $Q(S)$-modules

U and V such that $U\oplus V$ is directly infinite. Adding the $Q(T)$-module 0, we see that $U\oplus 0$ and $V\oplus 0$ are directly finite $Q(R)$-modules, and that $(U\oplus 0)\oplus (V\oplus 0)$ is directly infinite. Thus we can conclude that $Q(R)$ does not have the property (DF). The proof is complete.

Remark 2. Combining Theorem 2.5 with [5, Example], we see that there exists an example of a right self-injective regular ring whose primitive factor rings are artinian, which does not have the property (DF).

Now we shall consider about the converse of Theorem 2.5. For this purpose, we need the following definitions.

Definition. A regular ring R is said to be *right continuous* if every right ideal of R is essential in a cyclic right ideal of R.

Definition. The *index* of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. (In particular, 0 is nilpotence of index 1.) The *index* of a regular ring R is the supremum of the indices of all nipotent elements of R. If this supremum is finite, then R is said to *have bounded index.*

Note that an abelian regular ring has bounded index 1. It is well-known that if a regular ring R has bounded index, then (1) R is a regular rings whose primitive factors are artinian and (2) the maximal right quotient ring $Q(R)$ of R is a regular ring of bounded index ([1, Corollaries 7.4, 7.10 and Theorem 7.20]).

Proposition 2.6 *Let R be a right continuous regular ring of bounded index. If R has the property* (DF), *then so does the maximal right quotient ring $Q(R)$ of R.*

Proof. Assume that $Q(R)$ does not have the property (DF). From Thorem 1.4, we have a nonzero cyclic right ideal I of $Q(R)$ and direct sum decompositions

$$\begin{aligned} I &= A_1\oplus B_1, \\ A_i &= A_{2i}\oplus B_{2i} \text{ and} \\ B_i &= A_{2i+1}\oplus B_{2i+1} \text{ for } i = 1,2,\ldots \end{aligned}$$

such that, for each nonzero cyclic projective right ideal X of $Q(R)$, there exists a positive integer m satisfying $X \not\lesssim \bigoplus_{i=m}^{\infty} A_i$ and $X \not\lesssim \bigoplus_{i=m}^{\infty} B_i$. Then we have from [1, Theorem 13.13] that $I = eQ(R)$, $A_i = a_iQ(R)$ and $B_i = b_iQ(R)$ for some idempotents e, a_i and b_i in R. Put I', A'_i and B'_i as following:

$$I' = R \cap (eQ(R)) = eR <\oplus R,$$
$$A'_i = R \cap (a_iQ(R)) = a_iR <\oplus I,$$
$$B'_i = R \cap (b_iQ(R)) = b_iR <\oplus I.$$

Then $I' = A'_1 \oplus B'_1$, $A'_i = A'_{2i} \oplus B'_{2i}$ and $B'_i = A'_{2i+1} \oplus B'_{2i+1}$ for each i. Let Y be a nonzero cyclic right ideal of R. We claim that $Y \not\lesssim \bigoplus_{i=n}^{\infty} A'_i$ and $Y \not\lesssim \bigoplus_{i=n}^{\infty} B'_i$ for some positive integer n. Otherwise $Y \lesssim \bigoplus_{i=n}^{\infty} A'_i$ or $Y \lesssim \bigoplus_{i=n}^{\infty} B'_i$ for each positive integer n, then $E(Y) \lesssim \bigoplus_{i=n}^{\infty} E(A'_i) = \bigoplus_{i=n}^{\infty} A'_i$ or $E(Y) \lesssim \bigoplus_{i=n}^{\infty} E(B'_i) = \bigoplus_{i=n}^{\infty} B_i$ for each positive integer n, which contradicts to above. Therefore we conclude that R does not have the property (DF).

Theorem 2.7. *Let R be a right continuous regular ring whose primitive factor rings are artinian. If R has the property* (DF), *so does the maximal right quotient ring $Q(R)$ of R.*

Proof. Assume that $Q(R)$ does not have the property (DF), and so we have directly finite projective $Q(R)$-module P and Q such that $P \oplus Q$ is directly infinite. It is obvious that there exists a nonzero element x in $Q(R)$ such that $\aleph_0(xQ(R)) \lesssim P \oplus Q$. From Remark 1, we have a ring decomposition $R = S \times T$ such that $xrS \neq 0$ and $xr \in R$ for some x in R and that $S \simeq M_n(A)$, where A is an abelian regular ring. Note that S and $Q(R)$ are regular rings of bounded index ([1, Theorem 7.12]), $\aleph_0(xrQ(R)) \lesssim P \oplus Q$ and that $Q(R) = Q(S) \times Q(T)$, where $Q(S)$ and $Q(T)$ are the maximal right quotient rings of S and T respectively. Therefore we see from Lemma 2.4 that $P \otimes_{Q(R)} Q(S)$ and $Q \otimes_{Q(R)} Q(S)$ are directly finite projective $Q(S)$-module, and that $(P \oplus Q) \otimes_{Q(R)} Q(S)$ is directly infinite. Thus $Q(S)$ does not have the property (DF), and hence so does not have S from Proposition 2.6 and the right continuity of S ([1, Proposition 13.7]). Using the same arguement of the last part's proof of Theorem 2.5, we conclude that R does not have the property (DF). The proof is complete.

We do not know whether the assumption "the right continuity" can be dropped from Theorem 2.7.

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