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ON RELATION BETWEEN GRADABILITY AND GALOIS GROUPS OF LOCAL COMMUTATIVE ARTINIAN RINGS

Dedicated to professor M. Harada on his 60th birthday

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1. Introduction. Let A be a finite dimensional local commutative artinian algebra over an algebraically closed field K. In this paper we aim to study the condition that A becomes a positively graded algebra in terms of Galois covering. The main result is that the following conditions are equivalent;

(1) A is a positively graded algebra.

(2) Any Galois covering of A is a positively graded category.

(3) $\operatorname{Rank} G \geq 1$ for a Galois group G of some Galois covering of A.

The relation between coverings of algebras and group graded algebras, whose quivers have no oriented cycles, was first treated in [4,5]. In this paper we treat more difficult cases that every arrow in a quiver of an algebra is a loop (that is, algebras are local commutative) and a grading means a positive grading. From our main theorem we know that some correspondences between Galois group and Galois covering stated in [4,5] holds for local commutative artinian algebras one of whose Galois groups has a torsion-free part. But we must notice that in our situations there are algebras with non-trivial univrersal covering but no positive grading in the case the Galois group is torsion abelian group. i.e., Any Galois groups are torsion abelian groups from our result. As we prove in Theorem 3.1, a universal Galois covering is not determined uniquely even up to isomorphism. In fact, a universal Gaois covering is determined by taking regular system of parameters of an algebra. Indeed we can get an example of a local commutative artinian algebra with two non-trivial universal Galois coverings such that one of their Galois groups is a finitely generated torsion abelian group and the other is a finitely generated torsion-free abelian group .

An algebra R is called a *positively graded algebra* if there is a direct decomposition

$$R = \sum_{i \geq 0} \oplus R_i \quad \text{and} \quad \mathrm{Rad}(R) = \sum_{i \geq 1} \oplus R_i$$

as a vector space such that $R_i \cdot R_j \subseteq R_{i+j}$ for any $i, j \geq 0$. Here $\mathrm{Rad}(R)$ means the Jacobson radical of R.

Recently a positively graded algebra which is not group graded has been studied actively by many authors, for instance see [1,7,8,9].

Positively graded algebras must satisfy more restricted condition $K = R_0$ than that of group graded algebras. Our investigation to study relations between covering and having positive grading becomes more difficut for positively graded algebras than for group graded algebras. In fact it is treated in [5] that a grading is indexed by group which becomes covering group. On the other hand our case is that a grading is indexed by 0 and the set $\mathbb{N}$ of natural numbers, but a Galois group is a finitely generated abelian group, i.e., a subgroup of a product of the set $\mathbb{Z}$ of integers.

Since we restrict algebras to commutative algebras, it is natural to consider abelian Galois covering. In this situation we can not apply the results in [5] since it was done for universal covering. Also the construction of covering in [5] is given in abstract way by using topological method. But in this paper we construct a universal abelian Galois covering for a local commutative artinian algebra in a concrete way. Also this gives a concrete method to calculate Galois group of covering.

2. Construction of a Galois covering of a factor algebra. A commutative artinian algebra is a direct sum of local artinian algebras. So we may assume that algebras are local when we treat their gradings.

Let A be a finite dimensional local artinian algebra over an algebraically closed field $K = A/\mathrm{Rad}A$. Assume $F: \overline{A} \to A$ is a Galois covering [3] of A with Galois Group G. i.e., There are some expressions $\overline{A} = (\bar{\Delta}, \bar{\rho})$ and $A = (\Delta, \rho)$ by locally finite quivers $\bar{\Delta}$, Δ and sets $\bar{\rho}$, ρ of relations such that there exists a subgroup $G < \mathrm{Aut}(\bar{\Delta}, \bar{\rho})$ and a K-linear functor $F: \overline{A} \to A$ which satisfy the following conditions;

(1) $Fg = F$ for any $g \in G$.

(2) G acts freely on $\bar{\Delta}$ and $\bar{\rho}$.

(3) $(\bar{\Delta}, \bar{\rho})/G = (\Delta, \rho)$.

(4) $\sum_{F(x)=a} x\bar{A}y = aAF(y)$, $\sum_{F(x)=a} y\bar{A}x = F(y)Aa$ for any $a \in A$ and

$y \in \bar{A}$.

We call that a Galois covering is abelian if the corresponding Galois group is abelian. Abelian covering has been studied in [2]. Also for a covering of a quiver with relations there is a paper [6].
The purpose of this section is to prove the following theorem.

Theorem 2.1 (Construction Theorem). *Let A be a finite dimensional local commutative Artinian algebra over algebraically closed field K with the radical N such that $n = \dim_K N/N^2$. Then each regular system of parameters determines a universal abelian Galois covering whose Galois group is a factor group of $\mathbb{Z}^n$.*

Before we give the proof, we explain some notations.

Let $a_1, \ldots, a_n$ be a regular system of parameters of A, t a nilpotency index of N. Since A is artinian, we have a natural ring homomorphism $\varphi: K[x_1, \ldots, x_n]_{\mathfrak{M}}/\mathfrak{M}^t \to A$ via $\varphi(x_i) = a_i$, here $\mathfrak{M}$ is a maximal ideal $(x_1, \ldots, x_n)$. We put $\ker\varphi = I/\mathfrak{M}^t$ and $I = (f_1, \ldots, f_p)$. We may assume $f_i \in \mathfrak{M}$ but $f_i \notin \mathfrak{M}^t$ and $1 \leq \deg f_i < t$ for each i.

First we give a universal abelian Galois covering Λ of $K[x_1, \ldots, x_n]_{\mathfrak{M}}/\mathfrak{M}^t$. We only state the results since it is verified easily. The set of generators of Λ is

$$\left\{ e_{(i_1,\ldots,i_n)}, x^{(j)}_{(i_1,\ldots,i_n)} \,\middle|\, (i_1, \ldots, i_n) \in \mathbb{Z}^n, j = 1, \ldots, n \right\}.$$

The relations of Λ are as following.

(1) $$e_{(i_1,\ldots,i_n)} \cdot e_{(j_1,\ldots,j_n)} = \begin{cases} e_{(i_1,\ldots,i_n)} & \text{if} \quad (i_1, \ldots, i_n) = (j_1, \ldots, j_n), \\ 0 & \text{otherwise.} \end{cases}$$

(2) $$e_{(i_1,\ldots,i_j+1,\ldots,i_n)} \cdot x^{(j)}_{(i_1,\ldots,i_n)} \cdot e_{(i_1,\ldots,i_j,\ldots,i_n)} = x^{(j)}_{(i_1,\ldots,i_n)}.$$

(3) $$x^{(k)}_{(i_1,\ldots,i_j+1,\ldots,i_n)} \cdot x^{(j)}_{(i_1,\ldots,i_n)} = x^{(j)}_{(i_1,\ldots,i_k+1,\ldots,i_n)} \cdot x^{(k)}_{(i_1,\ldots,i_n)}.$$

(4) $$\left\{ \sum_{(i_1,\ldots,i_n) \in S} \sum_{j=1,\ldots,n} K \cdot x^{(j)}_{(i_1,\ldots,i_n)} \right\}^t = 0 \text{ for any finite subset } S \subset \mathbb{Z}^n.$$

The covering functor $F: \Lambda \to K[x_1, \ldots, x_n]_{\mathfrak{M}}/\mathfrak{M}^t$ is given by

$$F(e_{(i_1,\ldots,i_n)}) = 1 + \mathfrak{M}^t \quad \text{and} \quad F(x^{(j)}_{(i_1,\ldots,i_n)}) = x_j + \mathfrak{M}^t.$$

The Galois group is a free group $\mathbb{Z}g_1 \oplus \cdots \oplus \mathbb{Z}g_n$ such that $g_s (s = 1, \ldots, n)$ maps $e_{(i_1,\ldots,i_n)}$ and $x^{(j)}_{(i_1,\ldots,i_n)}$ to $e_{(i_1,\ldots,i_s+1,\ldots,i_n)}$ and $x^{(j)}_{(i_1,\ldots,i_s+1,\ldots,i_n)}$ respectively.

Proof of theorem 2.1. From a regular system of parameters $a_1,\ldots,a_n$, we construct a universal abelian Galois covering of A using Λ constructed above. For the expression $f_i(x_1,\ldots,x_n)=\sum_s k_s^{(i)}x_1^{i_1}\cdots x_n^{i_n}$ $(i=1,\cdots,p)$, we put a vector $\mathbf{d}_s^{(i)}=(i_1,\ldots,i_n)\in\mathbb{Z}^n$ and make a subgroup H generated by $\mathbf{d}s^{(i)}-\mathbf{d}_v^{(i)}$ for s-th and v-th terms in $f_i(x_1,\ldots,x_n)$ $(i=1,\ldots,p)$. We get a universal abelian Galois covering Λ/H of A with Galois group $\mathbb{Z}^n/H$.

Next we show a universal abelian Galois covering $F{:}\,\overline{A}\rightarrow A$ with Galois group G corresponds to a regular system of parameters. We fix $e\in\bar{A}$ such that $F(e)=1$, then we have $A\cong\sum_{g\in G}\oplus e\bar{A}g(e)$. We show that there is a regular system of parameters $a_1,\ldots,a_n$ such that $a_i=F(c_{g_i})$ for some $c_{g_i}\in\bar{A}g_i(e)$, $g_i\in G$. Choose $a\in N\backslash N^2$, then $a=F(\sum c_g)$ for some $c_g\in e\bar{A}g(e)$. Since $a\notin N^2$, some $F(c_g)\notin N^2$, so we set $g_1=g$ and $a_1=F(c_g)$. Assume $a_1,\ldots,a_i$ $(i<n)$ are already chosen. Take $a\in N$ such that $\bar{a}_1,\ldots,\bar{a}_i$, $\bar{a}$ are linearly independent over A/N. Here $\bar{a}$ is a coset with respect to N. By the same argument as above, we can get some $g_{i+1}\in G$ which appears in some term of $a=\sum F(c_g)$ such that $\bar{a}_1,\ldots,\bar{a}_i$, $\overline{F(c_{g_{i+1}})}$ are linearly independent. So we put $a_{i+1}=F(c_{g_{i+1}})$. Repeating this step n-times, we get $N=(a_1,\ldots,a_n)$.

Next we prove $G=\langle g_1,\ldots,g_n\rangle$. Let $1\neq g\in G$, then $0\neq F(e\bar{a}g(e))\in N$ for some $\bar{a}\in\bar{A}$. Decompose $F(e\bar{a}g(e))=\sum k_{(i_1,\ldots,i_n)}a_1^{i_1}\cdots a_n^{i_n}$. Since

$$a_1^{i_1}\cdots a_n^{i_n}=F(ec_{g_1}g_1(e)g_1(e)c_{g_1}g_1^2(e)\cdots c_{g_n}g_1^{i_1}\cdots g_n^{i_n}(e)),$$

and G acts freely, we get $g=g_1^{i_1}\cdots g_n^{i_n}$ for some $i_1,\ldots,i_n$.

3. Main theorem.

Theorem 3.1. *Let A be a finite dimensional local commutative artinian algebra over algebraically closed field K. Then A is a positively graded algebra if and only if any abelian Galois covering $\bar{A}$ of A is a positively graded category.*

In this case a covering functor is a graded homomorphism which preserve their gradings.

Proof. Assume A is a positively graded algebra. There is a direct decomposition $A=\sum_{i\geq 0}\oplus A_i$ and $N=\operatorname{Rad}A=\sum_{i\geq 1}\oplus A_i$ such that $A_i\cdot A_j\subseteq A_{i+j}$. By [7] Lemma 2.1 we can choose homogeneous elements $a_1,\ldots a_n$ such that $a_i\in A_{q_i}$ for some q_i and $N=(a_1,\ldots,a_n)$, here $n=$

$\dim_K N/N^2$. We put $q_i = g(a_i)$. We use the same notations as defined in section 2 in the following.

$K[x_1,\dots x_n]_{\mathfrak{M}}/\mathfrak{M}^t$ becomes a graded algebra by giving a grading q_i to x_i $(1 \le i \le n)$, so we set $K[x_1,\dots x_n]_{\mathfrak{M}}/\mathfrak{M}^t = \sum_{i\ge 0} \oplus B_i$. Clearly φ is a graded homomorphism, i.e., $\varphi(B_i) \subseteq A_i$ for any i, so $\ker \varphi = I/\mathfrak{M}^t = \sum_{i\ge 0} \oplus \left\{ (I/\mathfrak{M}^t) \cap B_i \right\}$ is a graded ideal. Let Λ be a universal abelian Galois covering of $K[x_1,\dots x_n]_{\mathfrak{M}}/\mathfrak{M}^t$ constructed in section 2. Define their gradings by $g(e_{(i_1,\dots,i_n)}) = 0$, $g(x^{(j)}_{(i_1,\dots,i_n)}) = q_j$, then clealy Λ becomes a positively graded category. We set $\Lambda = \sum_{i\ge 0} \oplus \Lambda_i$, then the covering functor $F: \Lambda \to K[x_1,\dots x_n]_{\mathfrak{M}}/\mathfrak{M}^t$ satisfies $F(\Lambda_i) \subseteq B_i$. We fix a set S of representatives of cosets of G by H. Hence we have the following isomorphism L;

$$\begin{aligned}\sum_{i\ge 0} \oplus B_i &= K[x_1,\dots x_n]_{\mathfrak{M}}/\mathfrak{M}^t \\ &\cong \sum_{g\in S} \oplus \sum_{h\in H} \oplus e\Lambda gh(e) \\ &= \sum_{g\in S} \oplus \sum_{h\in H} \oplus \sum_{i\ge 0} \oplus e\Lambda_i gh(e) \\ &= \sum_{i\ge 0} \oplus \sum_{g\in S} \oplus \sum_{h\in H} \oplus e\Lambda_i gh(e).\end{aligned}$$

Since L is a graded homomorphism, L induces the graded isomorphism

$$\begin{aligned}\sum_{i\ge 0} \oplus (I/\mathfrak{M}^t \cap B_i) &= I/\mathfrak{M}^t \\ &\cong \sum_{i\ge 0} \oplus \sum_{g\in S} \oplus \sum_{h\in H} \oplus L(I/\mathfrak{M}^t \cap B_i) \cap e\Lambda_i gh(e).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{i\ge 0} \oplus \sum_{g\in S} \oplus \frac{\sum_{h\in H} \oplus \Lambda_i h(e)}{\sum_{h\in H} \oplus L(I/\mathfrak{M}^t \cap B_i) \cap e\Lambda_i gh(e)} \\ &\cong \frac{\sum_{i\ge 0} \oplus \sum_{g\in S} \oplus \sum_{h\in H} \oplus e\Lambda_i h(e)}{\sum_{i\ge 0} \oplus \sum_{g\in S} \oplus \sum_{h\in H} \oplus L(I/\mathfrak{M}^t \cap B_i) \cap e\Lambda_i gh(e)} \\ &\cong \sum_{g\in S} \oplus \left\{ \frac{\sum_{h\in H} \oplus e\Lambda h(e)}{\sum_{h\in H} \oplus L(I/\mathfrak{M}^t \cap B_i) \cap e\Lambda gh(e)} \right\} = \overline{A}.\end{aligned}$$

The last equality comes from the construction of abelian Galois covering in section 2. Thus $\overline{A}$ is a graded category and the covering functor $F: \overline{A} \to A$ satisfies $F(\bar{A}_i) \subseteq A_i$, which means F is a graded homomorphism.

By Theorem 2.1, any abelian Galois covering is realized as a factor category of the above category Λ by a subgroup, so any abelian Galois covering is graded.

Next we prove the converse. Let $\bar{A}$ be a graded category and $\overline{A} = \sum_{i\geq 0} \oplus \bar{A}_i$ a direct decomposition into homogeneous components. We put $A_i = F\Big(\sum_{g\in G} \oplus e\bar{A}_i g(e)\Big)$. Since

$$A \cong \sum_{g\in G} \oplus e\bar{A}g(e) \cong \sum_{g\in G} \oplus \sum_{i\geq 0} \oplus e\bar{A}_i g(e) = \sum_{i\geq 0} \oplus \sum_{g\in G} \oplus e\bar{A}_i g(e),$$

we have $A = \sum_{i\geq 0} \oplus A_i$ and $\operatorname{Rad} A = \sum_{i\geq 1} \oplus A_i$. Hence A is a positively graded algebra.

Theorem 3.2. *Let A be a commutative local artinian algebra satisfying $A/\operatorname{Rad} A = K$. Then A is a positively graded algebra if and only if there is a universal abelian Galois covering with a Galois group G such that* $\operatorname{Rank} G \geq 1$.

Proof. Assume $A = \sum_{i\geq 0} \oplus A_i$ is a graded algebra. It is enough to show that we construct an abelian Galois covering with Galois group $\mathbb{Z}$. We consider a locally bounded K-category $\bar{A}$ denoted as the following form;

$$\overline{A} = \begin{pmatrix} \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \cdots & 0 & A_0 & A_1 & \cdots & A_n & 0 & \cdots & \cdots \\ \cdots & \cdots & 0 & A_0 & A_1 & \cdots & A_n & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots \end{pmatrix}$$

That is, $\overline{A} = \sum_{i\in\mathbb{Z}} B_i$, here $B_i = A$. A multiplication is defined as same as the one of row finite matrices. More precisely, we put $B_i = \sum_{j=i}^{i+n} \oplus A_{ij}$, here $A_{ij} = A_{j-i}$ and we define a multiplication of two elements $(a_{i,j}), (b_{i,j}) \in \bar{A}$ by

$$(a_{i,j}) \cdot (b_{i,j}) = \Big(\sum_{k\in\mathbb{Z}} a_{i,k} \cdot b_{k,j}\Big).$$

The shift functor $t: \overline{A} \rightarrow \bar{A}$ is defined by componentwise as $t: A_{ij} \rightarrow A_{i+1,j+1}$ via $t(a) = a$ for any $a \in A_{ij} = A_{j-i}$.

Since the group generated by t is isomorphic to $\mathbb{Z}$. The covering functor $F: \overline{A} \rightarrow A$ is defined by $F(x) = x$ for any $x \in B_i = A$.

Next we assume there is a universal Galois covering $\bar{A}$ of which Galois group has a rank larger than 1. Decompose $G = g_1^{\mathbb{Z}} \times \cdots \times g_p^{\mathbb{Z}} \times tor(G)$ into the torsion-free part generated by free generators $g_1, \ldots, g_p$ and the torsion part $tor(G)$. We fix e an object of $\bar{A}$ such that $F(e) = 1$, here F is a covering functor. We may assume $e\bar{A}g_1(e) \neq 0$. In this case $e\bar{A}g_1^{-1}(e) = 0$, otherwise we have a cyclic path $e \rightarrow \cdots \rightarrow g_1(e) \rightarrow \cdots \rightarrow e$ in $\bar{Q}$ by composing two paths corresponding to $e\bar{A}g_1(e)$ and $g(e\bar{A}g_1^{-1}(e)) = g_1(e)\overline{A}e \neq 0$.

We put $G_1 = g_2^{\mathbb{Z}} \times \cdots \times g_p^{\mathbb{Z}} \times tor(G)$. Since G acts freely on $\bar{A}$, we have

$$A \cong \sum_{g \in G} e\bar{A}g(e) = \sum_{i \geq 0} \sum_{g \in G_1} e(\bar{A}g)g_1^i(e) = \sum_{i \geq 0} e\bar{A}g_1^i(e)$$

and the last sum is a direct sum. Thus A is a positively graded algebra by setting $A_i = F(e\bar{A}g_1^i(e))$.

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