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# ON BIMODULE MATRIX PROBLEBS AND ARTINIAN BIPARTITE PIECEWISE PEAK PI-RINGS OF FINITE PRINJECTIVE MODULE TYPE*

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**0. Introduction.** Our main aim of this paper is to give a characterization of a class of Drozd's bipartite bimodule matrix problems $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ of finite representation type, where $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is a category of matrices defined as follows (see [6] or [22, Section 17.9]). Let $\mathbb{K}$ and $\mathbb{L}$ be additive categories having the finite unique decomposition property. Suppose that ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ is an $\mathbb{K}$-$\mathbb{L}$-bimodule, that is,

$$\mathbb{N} : \mathbb{K}^{\mathrm{op}} \times \mathbb{L} \to \mathcal{A}b$$

is an additive functor, where $\mathcal{A}b$ is the category of abelian groups. The objects of $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ are triples $(x, y, m)$, where $x \in \mathrm{Ob}\,\mathbb{K}$, $y \in \mathrm{Ob}\,\mathbb{L}$ and $m \in \mathbb{N}(x, y)$. A morphism from $(x, y, m)$ to $(x', y', m')$ in $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is a pair $(\varphi, \psi)$, where $\varphi \in \mathbb{K}(x, x')$, $\psi \in \mathbb{L}(y, y')$ are such that $\mathbb{N}(x, \psi)m = \mathbb{N}(\varphi, y')m'$.

It is easy to check that $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is an additive category with the finite unique decomposition property. The direct sum of two objects $(x, y, m)$ and $(x', y', m')$ of $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is the object $(x \oplus x', y \oplus y', m \oplus m')$, where

$$m \oplus m' = \begin{pmatrix} m & 0 \\ 0 & m' \end{pmatrix} \in \begin{pmatrix} \mathbb{N}(x, y) & \mathbb{N}(x, y') \\ \mathbb{N}(x', y) & \mathbb{N}(x', y') \end{pmatrix} = \mathbb{N}(x \oplus x', y \oplus y').$$

under the obvious identifications. By a **bipartite bimodule matrix problem** we shall mean the classification of indecomposable objects in the category $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$. The problem is of **finite representation type** if the set of isomorphism classes of the indecomposable objects in $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is finite. In this case the sets $\mathrm{ind}(\mathbb{K})$ and $\mathrm{ind}(\mathbb{L})$ of representatives of the isomorphism classes of indecomposable objects in $\mathbb{K}$ and $\mathbb{L}$ are finite.

In the paper a class of bipartite matrix problems of finite representation type is characterized diagrammatically and the indecomposable objects are completely determined (see Remark 5.9). We mainly develop the

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## ERRATA

The title on page 89 of the preceding issue, Vol. **35** (1993), should read

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case when $\mathbb{K}$ and $\mathbb{L}$ are not semisimple. We recall that if $\mathbb{L}$ or $\mathbb{K}$ is the category $\mathrm{mod}(F)$ of all finite dimensional vector spaces over a field $F$ then the category $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is equivalent or is dual to the subspace category $\mathcal{U}(\mathbb{K}'_F)$ of a vector space category $\mathbb{K}'_F$. In this case the theory is well developed (see [9], [15], [22]).

Throughout we suppose that

$$\mathrm{ind}(\mathbb{K}) = \{X_1, \dots, X_n\}, \quad \mathrm{ind}(\mathbb{L}) = \{Y_1, \dots, Y_m\}$$

and the following two conditions are satisfied:

**(p1)** The categories $\mathbb{K}$, $\mathbb{L}$ are schurian PI-categories, that is, the endomorphism rings $A_i = \mathbb{K}(X_i)$, $F_p = \mathbb{L}(Y_p)$ are division rings which are finite dimensional over their centers for all $i$ and $p$. Moreover the following numbers are finite

$$\begin{array}{llll} d_{ij} = \dim \mathbb{K}(X_j, X_i)_{A_j}, & d_{ip} = \dim \mathbb{N}(X_i, Y_p)_{A_i} & \text{if } i, j \leq n, j \neq i; \\ d_{ip} = \dim \mathbb{N}(X_i, Y_p)_{A_i}, & d'_{ip} = \dim_{F_p} \mathbb{N}(X_i, Y_p) & \text{if } p \leq m, i \leq n; \\ d_{pq} = \dim \mathbb{L}(Y_q, Y_p)_{F_q}, & d'_{pq} = \dim_{F_p} \mathbb{L}(Y_q, Y_p) & \text{if } p, q \leq m, p \neq q. \end{array}$$

**(p2)** The bimodule ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ is perfectly faithful, that is, $N$ is non-zero and the functors $\mathbb{N}(-, Y_j)$, $\mathbb{N}(X_i, -)$ are faithful for all $i$ and $j$.

We associate with ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ a **bipartite value scheme** $(\mathbf{I}_{\mathbb{N}}, \mathbf{d})$ consisting of the set

$$\mathbf{I}_{\mathbb{N}} = I' \cup I'', \quad I' = \{1, \dots, n\}, \ I'' = \{1, \dots, m\}$$

of points connected by valued dashed arrows

$$i \overset{(d_{ij}, d'_{ij})}{- - \rightarrow} j \quad \Longleftrightarrow \quad [d_{ij} \neq 0 \quad \text{and} \quad d'_{ij} \neq 0].$$

The bipartition is defined by the sets $I'$ and $I''$. We write $i - - \rightarrow j$ instead of $i \overset{(1,1)}{- - \rightarrow} j$. Following [9] we define a **right weighted width** of $(\mathbf{I}_{\mathbb{N}}, \mathbf{d})$ and a **left weighted width** of $(\mathbf{I}_{\mathbb{N}}, \mathbf{d})$ by the formulas

$$r\omega(\mathbf{I}_{\mathbb{N}}, \mathbf{d}) = \max_{J, p \in I''} \{ \sum_{j \in J} d_{jp} d'_{jp} \},$$

$$l\omega(\mathbf{I}_{\mathbb{N}}, \mathbf{d}) = \max_{L, i \in I'} \{ \sum_{p \in L} d_{ip} d'_{ip} \}$$

where $J \subseteq I'$ and $L \subseteq I''$ run through all subsets of mutually incomparable elements with respect to the relation

$$i \prec j \iff d_{ij} \neq 0.$$

Our main results of the paper are the following two theorems.

**Theorem A.** *If ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ is a bimodule and the conditions* **(p1)**, **(p2)** *are satisfied, then the category* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ *is of finite representation type if and only if the value scheme* $(\mathbf{I}_{\mathbb{N}}, \mathbf{d})$ *is a bipartite valued partially ordered set (abbreviated: poset) with respect to the relation* $\prec$, $l\omega(\mathbf{I}_{\mathbb{N}}, \mathbf{d}) \leq 3$, $r\omega(\mathbf{I}_{\mathbb{N}}, \mathbf{d}) \leq 3$ *and* $(\mathbf{I}_{\mathbb{N}}, \mathbf{d}, \prec)$ *does not contain full bipartite valued subposets being bipartite isomorphic to one of the forms shown in Table* 1 *or to their opposite forms. If this is the case then*

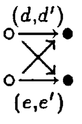
(a) *The category* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ *has Auslander-Reiten sequences and has a preprojective component which coincides with* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$.

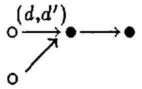
(b) *If* $(x, y, m)$ *is an indecomposable object of* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ *then the endomorphism ring of* $(x, y, m)$ *is isomorphic to one of the division rings* $A_1, \ldots, A_n$, $F_1, \ldots, F_m$. *Moreover, if* $x \cong X_1^{s_1} \oplus \cdots \oplus X_n^{s_n}$ *and* $y \cong Y_1^{t_1} \oplus \cdots \oplus Y_m^{t_m}$ *then* $s_i \leq 6$ *and* $t_j \leq 6$ *for all* $i \in I'$ *and* $j \in I''$.

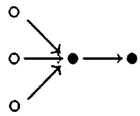
**Theorem B.** *If the category* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ *is of finite representation type and the conditions* **(p1)**, **(p2)** *are satisfied, then the category* $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ *has a sincere object if and only if either* $|\mathbf{I}_{\mathbb{N}}| = 1$ *or the bipartite valued poset* $(\mathbf{I}_{\mathbb{N}}, \mathbf{d}, \prec)$ *of the bimodule* ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ *is bipartite isomorphic to one of the forms shown in Table* 2 *or to one of their dual forms.*

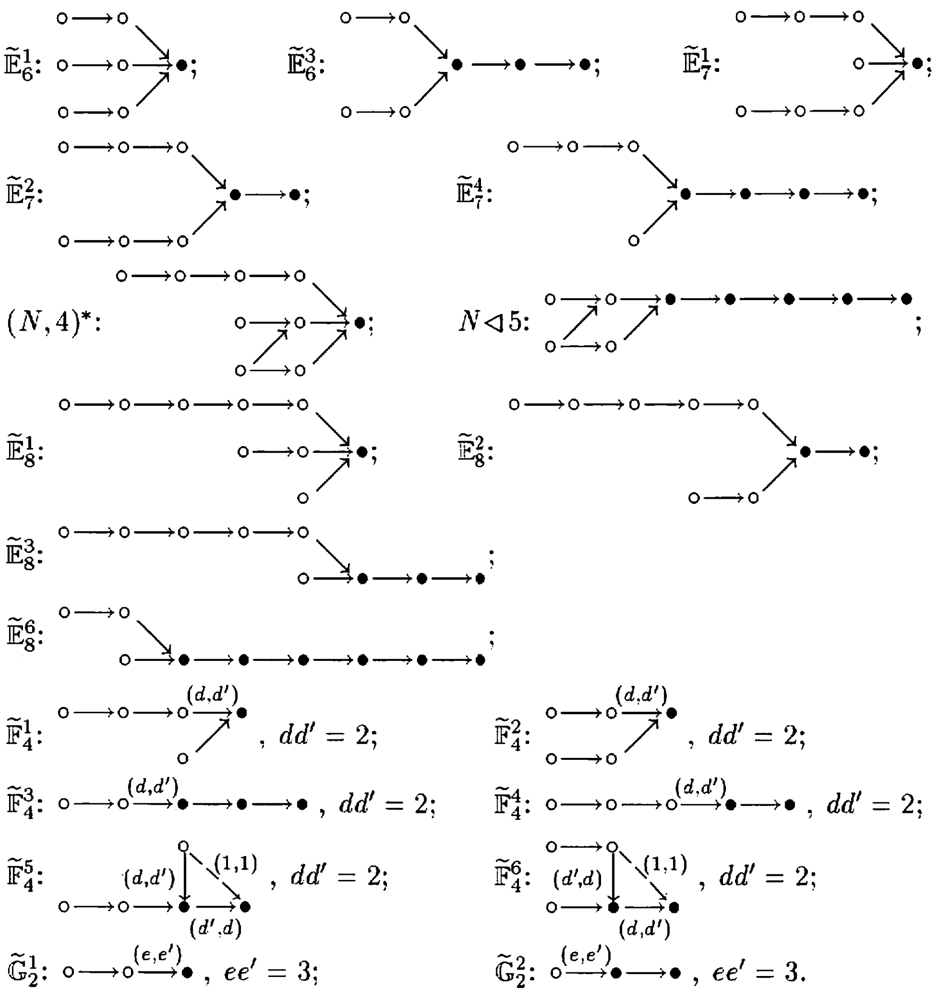
We recall that an object $(x, y, m)$ of the category $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is said to be **sincere** if it is non-zero indecomposable, all indecomposable objects of $\mathbb{K}$ are summands of $x$ and all indecomposable objects of $\mathbb{L}$ are summands of $y$.

**Table 1. Critical bipartite valued posets for piecewise artinian PI-rings of finite adjusted module type**

$\widetilde{\mathbb{A}}_{\bullet}^{\bullet}$: 

, $dd', ee' \leq 2$;  $\widetilde{\mathbb{CD}}$: 

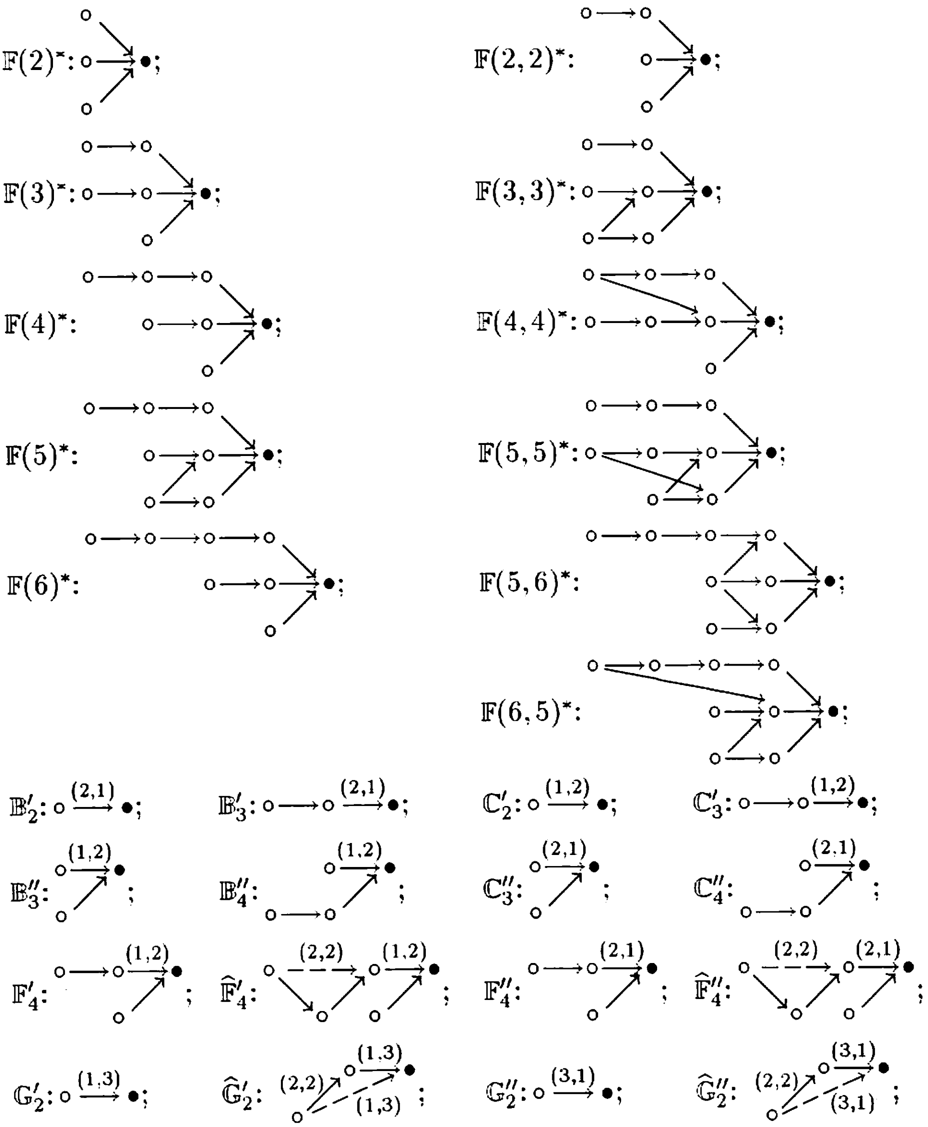
, $dd' = 2$;  $\widetilde{\mathbb{D}}_4^2$: 
;



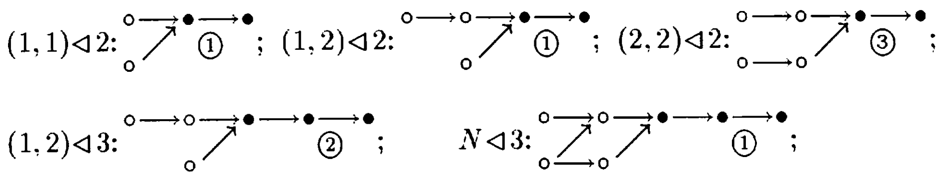

Here we draw only a generating set of valued arrows. The remaining values should be computed according to the rules $(r_0)$ – $(r_2)$ and $(r_0')$ – $(r_2')$ in Lemma 3.7. The bipartition is defined by the black and white points.

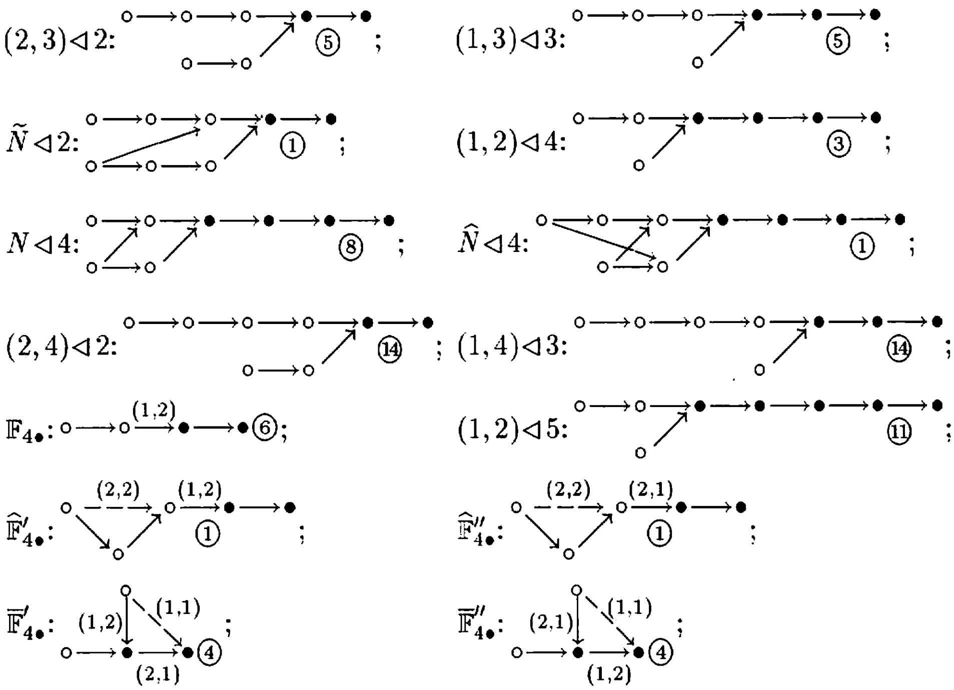
**Table 2. Sincere bipartite valued posets of piecewise peak artinian PI-rings of finite adjusted module type.**

**Part A. Sincere one-peak forms.**




**Part B. Sincere non-one-peak forms.**






In Part B the encircled numbers mean the numbers of indecomposable sincere adjusted modules. They can be determined according to a recipe given in Proposition 5.8. There is precisely 81 sincere adjusted modules over the rings of Part A and they are presented in [9, Appendix] and in [22].

Theorems A and B are proved in Section 1 by reducing the problem via the functors (see (2.8))

$$\text{(0.1)} \qquad \mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}) \cong \mathrm{prin}(R)^A_B \xrightarrow{\mathbf{ad}} \mathrm{adj}(\mathbf{R}_{\mathbb{N}})$$

defined in [13], [19] and [22, Section 17.9] to a corresponding problem for the category $\mathrm{adj}(\mathbf{R}_{\mathbb{N}})$ of adjusted modules (defined in Section 1) over the bipartite artinian PI-ring

$$\text{(0.2)} \qquad \mathbf{R}_{\mathbb{N}} = \begin{pmatrix} A & {}_A N_B \\ 0 & B \end{pmatrix}$$

associated to ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ as follows. We set $A = \mathbb{K}(X,X)$ and $F = \mathbb{L}(Y,Y)$, where $X = X_1 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_m$. It follows from our assumption (see [15], [19]) that there is a Morita duality $D\colon \mathrm{mod}(F^{\mathrm{op}}) \to \mathrm{mod}(\widetilde{F})^{\mathrm{op}}$, where $\widetilde{F}$ is an artinian PI-ring. We take for $B$ the ring $\widetilde{F}$ and for ${}_AN_B$ the image of the left $F$-module $\mathbb{N}(X,Y)$ under the duality $D$.

If $\mathbb{L} = \text{mod}(F)$ is the category of all finite dimensional vector spaces over a field $F$ then the category $\text{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is equivalent to the subspace category $\mathcal{U}(\mathbb{K}_F)$ of a vector space category $\mathbb{K}_F$ and our Theorems A and B follow from the main results in [9]. The case when $\mathbb{L} = \text{mod}(F)$ and $F$ is a product of division rings is studied in [12] and [16]. A general case is discussed in [13] and [19].

In Section 2 notation, terminology and a Morita duality type results for adjusted modules are presented (see Proposition 2.3 and (2.18)). Moreover, given two idempotents $e \in A$ and $\eta \in B$ we define a full and faithful embedding (2.17)

$$(0.3) \qquad \mathcal{I}^e_\eta : \text{adj}\begin{pmatrix} eAe & eM\eta \\ 0 & \eta B\eta \end{pmatrix} \longrightarrow \text{adj}(R)^A_B$$

induced by the idempotent induction functor (2.12). The functors $\mathcal{I}^e_\eta$ allow us to reconstruct all indecomposable modules in $\text{adj}(R)^A_B$ from the 327 sincere forms in Theorem 1.11 below in a way presented in Remark 5.9, where $R$ is a bipartite piecewise peak artinian $PI$-ring of finite adjusted module type, that is, the category $\text{adj}(R)^A_B$ is of finite representation type.

In Section 3 we give a combinatorial characterization of bipartite valued posets $(\mathbf{I}_R, \mathbf{d})$ of piecewise peak artinian $PI$-rings $R$ satisfying the following conditions:

(i) $d_{is}d'_{is} \leq 3$ for all $i \in \mathbf{I}_A$ and $s \in \mathbf{I}_B$,

(ii) $(\mathbf{I}_R, \mathbf{d})$ does not contain as a full bipartite valued subposet the critical forms $\widetilde{\mathbb{A}}^\bullet_\bullet$, $\widetilde{\mathbb{G}}^1_2$ and $\widetilde{\mathbb{G}}^2_2$ in Table 1 (see Theorem 3.10).

This result essentially depends on Lemma 3.7 and Proposition 3.9, and plays an important role in the proof of the crucial implication (b) $\Rightarrow$ (a) of Theorem 1.6.

Throughout this paper $\text{pr}(R)$ and $\text{inj}(R)$ denote full subcategories of $\text{mod}(R)$ consisting of projective and injective modules, respectively. The full subcategories of $\text{mod}(R)$ consisting of modules having a projective socle and an injective top are denoted by $\text{mod}_{\text{sp}}(R)$ and $\text{mod}_{\text{ti}}(R)$, respectively. If $R$ is schurian of the form (1.3) below we view any $R$-module $X$ as a system

$$X = (X_1, \ldots, X_{n+m}; {}_i\varphi_j)_{i \preceq j \in \mathbf{I}_R}$$

where $X_i = Xe_i$ and ${}_j\varphi_i: X_i \otimes e_iRe_j \to X_j$ is the $e_jRe_j$-linear map induced by the multiplication. The vector

$$\mathbf{dim}(X) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in \mathbb{Z}^{n+m}$$

with $x_i = \dim(X_i)_{e_iRe_i}$ is called the dimension vector of $X$. We say that $\mathrm{adj}(R)^A_B$ is of tame representation type (resp. of wide representation type) if $R$ is a finite dimensional algebra over an algebraically closed field and $\mathrm{adj}(R)^A_B$ is of tame representation type (resp. wild representation type) (see [5] and [22, Section 14.4]).

Given a family $\mathcal{C}$ of modules in $\mathrm{mod}(R)$ we denote by $[\mathcal{C}]$ the ideal in $\mathrm{mod}(R)$ consisting of all homomorphisms having a factorization through a direct sum of modules in $\mathcal{C}$ (see [15], [22]).

A preliminary version of this paper is presented in the preprint [23].

**1. Piecewise peak PI-rings of finite prinjective module type.** Throughout this paper $R$ denotes a basic artinian PI-ring which is bipartite, that is, $R$ has the form

$$R = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix} \tag{1.1}$$

where $A$, $B$ are rings and ${}_AM_B$ is an $A$-$B$-bimodule. We denote by $J(R)$ the Jacobson radical of $R$. We say that $R$ is a **PI-ring** if $R$ satisfies a polynomial identity. It is well known that a basic artinian ring $R$ is a PI-ring if $R/J(R)$ is a product of division rings each of which is finite dimensional over its center.

A right module $X$ in $\mathrm{mod}(R)$ will be identified with the system

$$X = (X'_A, X''_B, \varphi : X' \otimes {}_AM_B \to X''_B)$$

where $X'_A$ is in $\mathrm{mod}(A)$, $X''_B$ is in $\mathrm{mod}(B)$ and $\varphi$ is a $B$-homomorphism. Note that $\varphi$ is uniquely determined by the $B$-homomorphism

$$\overline{\varphi} : X'_A \longrightarrow \mathrm{Hom}_B({}_AM_B, X''_B)$$

adjoint to $\varphi$ and defined by formula $\overline{\varphi}(x)(m) = \varphi(x \otimes m)$.

Following [13] we call a module $X$ in $\mathrm{mod}(R)$ **prinjective** if $X'_A$ is $A$-projective and $X''_B$ is $B$-injective. We denote by $\mathrm{prin}(R)^A_B$ the full subcategory of the category $\mathrm{mod}(R)$ consisting of all prinjective modules. The bipartite ring $R$ is said to be of **finite prinjective module type** if the category $\text{prin}(R)_B^A$ is of finite representation type, that is, it has finitely many isoclasses of indecomposable modules.

We denote by $\text{adj}(R)_B^A$ the full subcategory of the category $\text{mod}(R)$ of finitely generated right $R$-modules consisting of **adjusted modules**, that is, $R$-modules $X$ such that $\text{soc}(X)$ is a $B$-module and $\text{top}(X)$ is an $A$-module via the natural ring surjections $A \leftarrow R \leftarrow B$. This means that the module $X = (X'_A, X''_B, \varphi)$ in $\text{mod}(R)$ is adjusted if and only if the $B$-homomorphism $\varphi$ is surjective and the $A$-homomorphism $\overline{\varphi}$ is injective. The ring $R$ is said to be of **finite adjusted module type** if the category $\text{adj}(R)$ is of finite representation type. It follows from [13] and [19] that $R$ is of finite prinjective module type if and only if $R$ is of finite adjusted module type (see also the diagram (2.8) below). The category of adjusted modules was introduced in [18] and [19] as a module theoretic form of bimodule matrix problems (see also [13] and [24]).

A ring $R$ will be called a **piecewise peak ring** if $R$ is bipartite of the form (1.1), for any primitive idempotents $e \in A$, $\eta \in B$ the rings $eAe$, $\eta B\eta$ are division rings, ${}_AM\eta$ is $A$-faithful and $eM_B$ is $B$-faithful, or equivalently, if $\begin{pmatrix} eAe & eM \\ 0 & B \end{pmatrix}$ is a left peak ring and $\begin{pmatrix} A & {}_AM\eta \\ 0 & \eta B\eta \end{pmatrix}$ is a right peak ring. In particular $R$ is schurian in the sense that $eRe$ is a division ring for any primitive idempotent $e$ of $R$.

We recall from [15] that an artinian ring $T$ is a right (resp. left) peak ring if $\text{soc}(R_R)$ (resp. $\text{soc}({}_RR)$) is a projective $R$-module and is isomorphic to a direct sum of copies of a simple $R$-module $P_*$ called a **peak module** of $R$. It is clear that right peak rings are piecewise peak. Another class of examples of piecewise peak rings is provided by the paths algebras

$$K(I \lhd J) = \begin{pmatrix} KI & M \\ 0 & KJ \end{pmatrix} \tag{1.2}$$

where $K$ is a field, $I$ and $J$ are finite posets, $I \lhd J$ is the disjoint union of posets $I$ and $J$ with additional relations $i \prec j$ for all $i \in I$ and $j \in J$, and $M = \bigoplus_{i\in I}\bigoplus_{j\in J} K(i,j)$ is considered as an $KI$-$KJ$-bimodule in a natural way. It follows from [17, Lemma B.7.16] and [19, Example 5.17] that the study of $\text{adj}(K(I \lhd J))_{KJ}^{KI}$ is equivalent to the study of matrix representations of the pair of posets $I$, $J$ in the sense of Kleiner [7] and [8] (see Chapter 16 of [22]).

Throughout this paper we fix a complete set of primitive orthogonal idempotents $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}$ of $R$ and suppose that $e_1, \ldots, e_n \in$

$A$, $e_{n+1},\ldots,e_{n+m} \in B$. It will be shown in Section 2 that if $R$ is a piecewise artinian PI-ring of finite adjusted module type, then $R$ has the upper triangular matrix form

$$(1.3)\qquad R = \begin{pmatrix} A_1 & & {}_iM_j & {}_1M_1 & \ldots & {}_1M_m \\ & \ddots & & \vdots & & \vdots \\ 0 & & A_n & {}_nM_1 & \ldots & {}_nM_m \\ & & & B_1 & & {}_pB_q \\ & 0 & & & \ddots & \\ & & & 0 & & B_m \end{pmatrix}$$

where $A_i = e_iAe_i$, $1 \leq i \leq n$, $B_p = e_{n+p}Be_{n+p}$, $1 \leq p \leq m$, are division rings, ${}_iM_p = e_iMe_{n+p} \neq 0$ and ${}_iA_j = e_iAe_j$, ${}_pB_q = e_{n+p}Be_{n+q}$ for $i,j = 1,\ldots,n$, $p,q = 1,\ldots,m$. The multiplication is given by bilinear maps $c_{ijk}\colon {}_iA_j \otimes {}_jA_k \to {}_iA_k$, $c_{ijp}\colon {}_iA_j \otimes {}_jM_p \to {}_iM_p$, $c_{ipq}\colon {}_iM_p \otimes {}_pB_q \to {}_iM_q$, $c_{pqr}\colon {}_pB_q \otimes {}_qB_r \to {}_pB_r$ induced by the multiplication in $R$.

Following [15] we associate with $R$ the **bipartite valued poset** $(\mathbf{I}_R, \mathbf{d})$, where

$$(1.4)\qquad \mathbf{I}_R = I_A \cup I_B = \{1,\ldots,n,n+1,\ldots,n+m\},\ i \preceq j \iff e_iRe_j \neq 0,$$

$I_A = \{1,\ldots,n\}$, $I_B = \{n+1,\ldots,n+m\}$, and given $i \preceq j$ we define two values

$$(1.5)\qquad d_{ij} = \dim(e_iRe_j)_{e_jRe_j},\quad d'_{ij} = \dim_{e_iRe_i}(e_iRe_j)\quad \text{for } i \neq j$$

We set $d_{ii} = d'_{ii} = 1$ and $d_{ij} = d'_{ij} = 0$ if $i$, $j$ are incomparable in $(\mathbf{I}_R, \preceq)$. Valued arrows are defined as in the Introduction. The bipartition of $\mathbf{I}_R$ is defined by the subset $I_A$ and $I_B$. We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$.

It is easy to see that if ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ is an $\mathbb{K}$-$\mathbb{L}$-bimodule then $\mathbf{R}_{\mathbb{N}}$ is a bipartite piecewise peak artinian PI-ring if and only if the conditions **(p1)** and **(p2)** are satisfied. It follows from [15, Proposition 2.5] that the bipartite value scheme $(\mathbf{I}_{\mathbb{N}}, \mathbf{d})$ of ${}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}}$ is bipartite isomorphic with the bipartite value poset (1.4) of the ring $R = \mathbf{R}_{\mathbb{N}}$. Moreover, it follows from [13] and [19] that there exist full dense additive functors shown in (0.1) preserving the indecomposability and finite representation type. Therefore the category $\mathrm{Mat}({}_{\mathbb{K}}\mathbb{N}_{\mathbb{L}})$ is of finite representation type if and only if the ring $\mathbf{R}_{\mathbb{N}}$ is of finite adjusted module type.

It follows that Theorems A and B are an immediate consequence of the following two theorems and the properties of the functor **ad** established in [13] and [19].

**Theorem 1.6.** *Let $R$ be a basic artinian piecewise peak PI-ring of the form* (1.1) *and let and let* $(\mathbf{I}_R, \mathbf{d})$ *be the bipartite value scheme of $R$. Then the following conditions are equivalent*:

(a ) *The category* $\mathrm{prin}(R)_B^A$ *is of finite representation type.*

(a′) *The category* $\mathrm{adj}(R)_B^A$ *is of finite representation type.*

(b ) *The ring $R$ is schurian of the triangular form* (1.3), $l\omega(\mathbf{I}_R, \mathbf{d}) \leq 3$, $r\omega(\mathbf{I}_R, \mathbf{d}) \leq 3$, $(\mathbf{I}_R, \mathbf{d})$ *is symmetrizable* (*i.e.* $d_{ij} f_j = f_i d'_{ij}$ *for some natural numbers* $f_i$), $(\mathbf{I}_R, \mathbf{d})$ *is a bipartite valued poset with respect to relation* $i \preceq j \Leftrightarrow i = j$ *or* $d_{ij} \neq 0$, *and does not contain a full valued bipartite subposets being bipartite isomorphic to one of the forms of Table* 1 *or to their dual forms.*

(c) *There are positive integers* $f_1, \dots, f_{n+m}$ *such that* $d_{ij} f_j = f_i d'_{ij}$ *for all* $i$, $j \in \mathbf{I}_R$ *and the rational quadratic form* $q_R \colon \mathbb{Q}^{n+m} \to \mathbb{Q}$

$$(1.7)\quad q_R(x) = \sum_{i=1}^{n+m} f_i x_i^2 + \sum_{1 \leq i \prec j \leq n} f_i d_{ij} x_i x_j + \sum_{n+1 \leq p \prec q \leq n+m} f_p d_{pq} x_p x_q - \sum_{i=1}^{n} \sum_{p=1}^{m} f_{n+p} d_{i\, n+p} x_i x_{n+p}$$

*is weakly positive, i.e.* $q_R(x) > 0$ *if the vector* $x = (x_1, \dots, x_{n+m}) \in \mathbb{N}^{n+m}$ *is not zero.*

(d) *If $X$ is an indecomposable module in* $\mathrm{adj}(R)_B^A$, *then the multiplicity of any simple module occurring in* $(\mathrm{top}\, X) \oplus (\mathrm{soc}\, X)$ *is smaller than or equal to* 6.

(e) *If* $X_1 \xrightarrow{\varphi_1} X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{\varphi_n} X_{n+1} \longrightarrow \cdots$ *is a sequence of monomorphisms between indecomposable modules in* $\mathrm{adj}(R)_B^A$, *then there is an integer $h$ such that $\varphi_j$ is an isomorphism for all* $j \geq h$.

*If, in addition, $R$ is a finite dimensional algebra over a commutative infinite field $K$, then each of the conditions* (a)–(e) *is equivalent to the following one*

(f) *For any vector* $s = (s_1, \dots, s_{n+m}) \in \mathbb{N}^{n+m}$ *the irreducible algebraic $K$-variety* $\mathcal{X}(s) = \mathrm{Hom}_R(\mathcal{P}(s), \mathcal{Q}(s))$, *with*

$$\mathcal{P}(s) = (e_1 R)^{s_1} \oplus \cdots \oplus (e_n R)^{s_n},$$
$$\mathcal{Q}(s) = E(n+1)^{s_{n+1}} \oplus \cdots \oplus E(n+m)^{s_{n+m}}$$

*has finitely many orbits with respect to the natural algebraic group action*

$$(1.8)\qquad * : \mathcal{G}(s) \times \mathcal{X}(s) \longrightarrow \mathcal{X}(s)$$

*where* $\mathcal{G}(s) = \mathrm{Aut}(\mathcal{Q}(s)) \times \mathrm{Aut}(\mathcal{P}(s))$ *and* $E(j) = E(\mathrm{top}(e_j R))$.

**Corollary 1.9.** *If* $R$ *is as above and one of the conditions* (a) – (f) *of Theorem* 1.6 *is satisfied then* (*compare with* [9, Proposition 6.2], [10] *and* [20, 5.4])

(a) *The categories* $\mathrm{adj}(R)^A_B$ *and* $\mathrm{prin}(R)^A_B$ *have Auslander-Reiten sequences.*

(b) *The Auslander-Reiten quivers* $\Gamma(\mathrm{adj}(R)^A_B)$ *and* $\Gamma(\mathrm{prin}(R)^A_B)$ *of the categories* $\mathrm{adj}(R)^A_B$ *and* $\mathrm{prin}(R)^A_B$) *have preprojective components.*

(c) *For every indecomposable module* $X$ *of* $\mathrm{adj}(R)^A_B$ (*resp. in* $\mathrm{prin}(R)^A_B$)) *the ring* $\mathrm{End}(X)$ *is isomorphic to a division ring* $e_j R e_j$ *for some* $j$, *the group* $\mathrm{Ext}^1_R(X, X)$ *is zero and* $X$ *is uniquely determined by its composition factors as well as by the coordinate vector*

$$(1.10)\qquad v = \mathbf{cdn}(X) \in \mathbb{N}^{n+m}$$

*where the coordinates* $v(1), \ldots, v(n+m)$ *of* $v$ *are defined in such a way that the projective cover* $P(X)$ *of* $X$ *and the injective envelope* $E(X)$ *of* $X$ *have the forms*

$$P(X) = (e_1 R)^{v(1)} \oplus \cdots \oplus (e_n R)^{v(n)},$$
$$E(X) = E(n+1)^{v(n+1)} \oplus \cdots \oplus E(n+m)^{v(n+m)}.$$

The structure of rings characterized in Theorem 1.6 is described in Corollary 5.5.

The module $X$ in $\mathrm{adj}(R)^A_B$ is defined to be **sincere** if $X$ is indecomposable and $\mathbf{cdn}(X)(j) \neq 0$ for all $j = 1, \ldots, n+m$. The bipartite ring $R$ is called **adj-sincere** if $R$ has a sincere adjusted module.

**Theorem 1.11.** *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a basic artinian piecewise peak PI-ring of finite prinjective module type and let and let* $(\mathbf{I}_R, \mathbf{d})$ *be the bipartite value poset of* $R$. *Then* $R$ *has a sincere adjusted module if and only if* $(\mathbf{I}_R, \mathbf{d})$ *is bipartite isomorphic or anti-isomorphic with one of the* 49 *bipartite valued posets of Table* 2.

*If* $X$ *is an indecomposable sincere module in* $\mathrm{adj}(R)^A_B$ *then* $(\mathbf{cdn}X)(j) \leq 6$ *for* $j = 1, \ldots, n+m$.

The total number of non-isomorphic bipartite valued posets of rings described in Theorem 1.11 is equal 93. The total number of sincere adjusted modules over such rings is equal to 327 (see Remark 5.9).

Proofs of Theorems 1.6 and 1.11 are presented in Section 5. They essentially depend on the results in [9]. We reduce the main problems of this paper to the right peak case (see Corollary 5.5) by the peak reduction (Theorem 4.1) and the Arrow Waist Reduction (Theorem 4.12) introduced in [20]. Corollary 1.9 is a consequence of Corollary 5.7 and the well-known properties of preprojective components (see [22, Section 11.9], [24] and the note added in proof).

## 2. Adjusted modules, bipartite valued posets and a duality.

Let us start with a useful characterization of artinian piecewise peak rings which is a simple consequence of [15, Proposition 2.2].

**Lemma 2.1.** (a) *An artinian ring* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a piecewise peak ring if and only if* $\begin{pmatrix} eAe & eM \\ 0 & B \end{pmatrix}$ *is left peak ring and* $\begin{pmatrix} A & M\eta \\ 0 & \eta B\eta \end{pmatrix}$ *is a right peak ring for all primitive idempotents* $e \in A$ *and* $\eta \in B$.

(b) *Let* $R$ *be a basic artinian ring of the form* (1.3) *and suppose that* $A_1, \dots, A_n$ *and* $B_1, \dots, B_m$ *are local rings. Then* $R$ *is a piecewise peak ring if and only if* $A_1, \dots, A_n$ *and* $B_1, \dots, B_m$ *are division rings, the bimodules* ${}_iM_s = e_iRe_{n+s}$ *are non-zero and the bimodule maps*

$$(2.2) \qquad \begin{aligned} \bar{c}_{ijs} &: {}_iA_j \longrightarrow \mathrm{Hom}_{B_s}({}_jM_s, {}_iM_s), \\ \bar{\bar{c}}_{ist} &: {}_sB_t \longrightarrow \mathrm{Hom}_{A_i}({}_iM_s, {}_iM_t) \end{aligned}$$

*adjoint to* $c_{ijs}: {}_iA_j \otimes {}_jM_s \to {}_iM_s$ *and* $c_{ist}: {}_iM_s \otimes {}_sB_t \to {}_iM_t$, *respectively, are injective for* $i, j = 1, \dots, n$ *and* $s, t = 1, \dots, m$, *where* ${}_iA_i = A_i$ *and* ${}_sB_s = B_s$.

*We shall frequently use the following Morita duality result.*

**Proposition 2.3.** *Let* $R$ *be a basic schurian artinian piecewise peak PI-ring of the form* (1.3) *and let*

$$(2.4) \qquad \widetilde{R} = \mathrm{End}(Q) = \begin{pmatrix} \widetilde{A} & \widetilde{M} \\ 0 & \widetilde{B} \end{pmatrix}$$

*where* $Q = E(1) \oplus \cdots \oplus E(n+m)$, $E(j) = E(\mathrm{top}\, e_j R)$ *for* $j = 1, \dots, n+m$, $\widetilde{A} = \mathrm{End}(E(1) \oplus \cdots \oplus E(n))$ *and* $\widetilde{B} = \mathrm{End}(E(n+1) \oplus \cdots \oplus E(n+m))$.

*Then*

(a) *The right $R$-module $Q$ is a finitely generated, $\widetilde{R}$ is a schurian artinian piecewise peak $PI$-ring of the form shown in* [15, Proposition 2.5] *and the natural projections $Q \to E(j)$ define a complete set of primitive idempotents $e'_1, \dots, e'_{n+m}$ of the bipartite ring $\widetilde{R}$.*

(b) *The valued bipartite poset $(\widetilde{\mathbf{I}}_{\widetilde{R}}, \widetilde{\mathbf{d}})$ of $\widetilde{R}$ defined with respect to the idempotents $e'_1, \dots, e'_{n+m}$ is equal to $(\mathbf{I}_R, \mathbf{d})$. The functor*

$$D = \mathrm{Hom}_R(Q, -): \ \mathrm{mod}(R) \longrightarrow (\mathrm{mod}(\widetilde{R}^{\mathrm{op}}))^{\mathrm{op}} \tag{2.5}$$

*is a Morita duality such that if $X = (X_1, \dots, X_{n+m}, {}_j\varphi_i)$ is in* mod$(R)$ *then*

$$D(X) = (X_1^*, \dots, X_{n+m}^*, {}_j\varphi_i^*) \quad \text{and} \quad \mathbf{dim}D(X) = \mathbf{dim}X \tag{2.6}$$

*where $X_j^* = \mathrm{Hom}_{e_jRe_j}(X_j, e_jRe_j)$. If, in addition, $R$ is an artin algebra then there is a ring isomorphism $\widetilde{R} \cong R$ and $D$ is the standard duality.*

*Proof.* The statement (a) and the first part of (b) are an immediate consequence of [15, Proposition 2.5] and its proof. In order to prove the second part of (b) we recall from [15, p.539] that $E(j) \cong L_j(e_jRe_j)$, where $L_j = \mathrm{Hom}_{e_jRe_j}(Re_j, -): \mathrm{mod}(e_jRe_j) \to \mathrm{mod}(R)$ is fully faithful functor which is right adjoint to the restriction functor $\mathrm{res}_j: \mathrm{mod}(R) \to \mathrm{mod}(e_jRe_j)$ (see [22, Theorem 17.46]). Hence we get

$$\begin{aligned} e'_jD(X) &\cong \mathrm{Hom}_{\widetilde{R}}(\widetilde{R}e'_j, D(X)) \cong \mathrm{Hom}_{\widetilde{R}}(DE(j), D(X)) \\ &\cong \mathrm{Hom}_R(X, E(j)) \cong \mathrm{Hom}_R(X, L_j(e_jRe_j)) \\ &= \mathrm{Hom}_{e_jRe_j}(\mathrm{res}_j(X), e_jRe_j) \cong X_j^*. \end{aligned}$$

Since $R$ is a $PI$-ring then by [4, Proposition 1.2] $\dim_{e_jRe_j} X_j^* = \dim(X_j)_{e_jRe_j}$ and the proof is complete.

**Corollary 2.7.** *The duality* (2.5) *induces a duality*

$$D: \ \mathrm{adj}(R)_B^A \longrightarrow (\mathrm{adj}(\widetilde{R}^{\mathrm{op}})_{\widetilde{B}^{\mathrm{op}}}^{\widetilde{A}^{\mathrm{op}}})^{\mathrm{op}}.$$

Let us recall from [13] and [18] that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{prin}(R)_B^A & \xrightarrow{\Theta^A} & \mathrm{mod}_{\mathrm{ic}}(R)_B \\ \big\downarrow \Theta_B & \searrow^{\mathbf{ad}} & \big\downarrow \Theta_B \\ \mathrm{mod}^{\mathrm{pg}}(R)^A & \xrightarrow[\Theta^A]{} & \mathrm{adj}(R)_B^A \end{array} \tag{2.8}$$

where $\mathbf{ad} = \Theta_B \circ \Theta^A$, and $\mathrm{prin}(R)_B^A$, $\mathrm{mod_{ic}}(R)_B$, $\mathrm{mod^{pg}}(R)^A$ are full subcategories of $\mathrm{mod}(R)$ consisting of prinjective, injectively $B$-cogenerated, projectively $A$-generated modules, respectively. The module

$$X = (X'_A, X''_B, \varphi : X' \otimes {}_AM_B \to X''_B) \tag{2.9}$$

in $\mathrm{mod}(R)$ is in $\mathrm{prin}(R)_B^A$ (resp. in $\mathrm{mod^{pg}}(R)^A$) if $X'_A \in \mathrm{pr}(A)$ and $X''_B \in \mathrm{inj}(B)$ (resp. $X'_A \in \mathrm{pr}(A)$ and $\varphi$ is surjective). Further, $X$ is in $\mathrm{mod_{ic}}(R)_B$ if $X''_B \in \mathrm{inj}(B)$ and the map

$$\overline{\varphi} : X'_A \longrightarrow \mathrm{Hom}_B({}_AM_B, X''_B) \tag{2.10}$$

adjoint to $\varphi$ is injective. Note also that $X$ is in $\mathrm{adj}(A)_B^A$ if $\varphi$ is surjective and $\overline{\varphi}$ is injective. The functors $\Theta^A$, $\Theta_B$, $\mathbf{ad}$ are full dense and $\mathrm{Ker}\,\Theta^A = [\mathrm{pr}(A)]$, $\mathrm{Ker}\,\Theta_B = [\mathrm{inj}(B)]$, $\mathrm{Ker}\,\mathbf{ad} = [\mathrm{pr}(A), \mathrm{inj}(B)]$. In particular all categories in the diagram (2.8) are of the same representation type. We recall from [18] that $\Theta_B(X) = (X'_A, \mathrm{Im}\,\varphi, \widetilde{\varphi})$ and $\Theta^A(X) = (\mathrm{Im}\,\overline{\varphi}, X'', \widetilde{\widetilde{\varphi}})$, where $\widetilde{\varphi}$ and $\widetilde{\widetilde{\varphi}}$ are maps induced by $\varphi$ in a natural way.

If $J \subseteq I_R$ we set $e(J) = \sum_{j \in J} e_j$. Given idempotents $e \in A$ and $\eta \in B$ we set

$$R_\eta^e = \begin{pmatrix} eAe & eM\eta \\ 0 & \eta T \eta \end{pmatrix} \tag{2.11}$$

If $R$ is of the form (1.3) and $e = e(J)$, $\eta = e(L)$ for some $J \subseteq \mathbf{I}_A$, $L \subseteq \mathbf{I}_B$ then $R_\eta^e$ is obtained from (1.3) by omitting the $j^{\mathrm{th}}$ row and the $j^{\mathrm{th}}$ column for all $j \in \mathbf{I}_R - (J \cup L)$. Let us define an **idempotent induction functor**

$$\mathcal{I}_\eta^e : \mathrm{mod}(R_\eta^e) \longrightarrow \mathrm{mod}(R) \tag{2.12}$$

by the formula $\mathcal{I}_\eta^e(Y', Y'', \psi) = (Y' \otimes_{eA_e} eA, \mathrm{Hom}_{\eta B_\eta}(B\eta, Y''), \psi')$, where $\psi'(y \otimes ea \otimes m)(b\eta) = \psi(y \otimes eamb\eta)$ for all $y \in Y'$, $a \in A$, $b \in B$, $m \in M$ (see [22, Theorem 17.46]). It is easy to see that $\mathcal{I}_\eta^e$ carries prinjective modules to prinjective ones and there is a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{prin}(R_\eta^e) & \xrightarrow{\mathcal{I}_\eta^e} & \mathrm{prin}(R)_B^A & & & & \\
\downarrow \Theta^{eAe} & & \downarrow \Theta^A & & & & \\
\mathrm{mod_{ic}}(R_\eta^e) & \xrightarrow{\overline{\mathcal{I}}_\eta^e} & \mathrm{mod_{ic}}(R)_B & \xrightarrow[\sim]{\nabla_-} & \mathrm{mod^{pg}}(R^\nabla)^{\widetilde{B}} & \xrightarrow[\sim]{D} & (\mathrm{mod_{ic}}(R^\bullet)_{\widetilde{B}^{\mathrm{op}}})^{\mathrm{op}} \\
\downarrow \Theta_{\eta B \eta} & & \downarrow \Theta_B & & \downarrow \Theta^{\widetilde{B}} & & \downarrow \Theta_{\widetilde{B}^{\mathrm{op}}} \\
\mathrm{adj}(R_\eta^e) & \xrightarrow{\overline{\overline{\mathcal{I}}}_\eta^e} & \mathrm{adj}(R)_B^A & & \mathrm{adj}(R^\nabla)_A^{\widetilde{B}} & \xrightarrow[\sim]{D} & (\mathrm{adj}(R^\bullet)_{\widetilde{B}^{\mathrm{op}}}^{\widetilde{A}^{\mathrm{op}}})^{\mathrm{op}}
\end{array} \tag{2.13}$$

where $\overline{\mathcal{I}}^e_\eta(Z) = \Theta^A \mathcal{I}^e_\eta(Z)$ for $Z$ in $\mathrm{mod}_{\mathrm{ic}}(R^e_\eta)$, $\overline{\overline{\mathcal{I}}}^e_\eta(Z) = \Theta_B \Theta^A \mathcal{I}^e_\eta(Z)$ for $Z$ in $\mathrm{adj}(R^e_\eta)$,

$$R^\nabla = \begin{pmatrix} B & D_B(M) \\ 0 & A \end{pmatrix}, \qquad R^\bullet = (\widetilde{R^\nabla})^{\mathrm{op}}$$

is the **reflection form of** $R$ and the **dual reflection form** of $R$, respectively, $\widetilde{R^\nabla}$ is defined by the formula in (2.4), $D_B\colon \mathrm{mod}(B) \to \mathrm{mod}(B^{\mathrm{op}})$ is the Morita duality defined as in (2.5) and $\nabla_-$ is the **reflection functor** defined in [19, Definition 2.13], which is an equivalence of categories (see [13], [19], [20]).

**Proposition 2.14.** *Let* $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ *be a basic piecewise peak artinian PI-ring of the form* (1.3), *let* $(\mathbf{I}_R, \mathbf{d})$ *be the bipartite valued poset of* $R$ *and let* $e = e(J) \in A$, $\eta = e(L) \in B$ *be the idempotents associated to some fixed sets* $J \subseteq \mathbf{I}_A$, $L \subset \mathbf{I}_B$.

(a) *The rings* $R^e_\eta$, $R^{\mathrm{op}}$, $R^\nabla$ *and* $R^\bullet$ *are piecewise peak PI-rings.*

(b) *The bipartite valued poset of* $R^e_\eta$ *is the full bipartite valued subposet of* $(\mathbf{I}_R, \mathbf{d})$ *defined by* $J \cup L$. *If* $(\overline{\mathbf{I}}, \overline{\mathbf{d}})$ *is the bipartite valued poset of* $R^\nabla$, *then*

$$\overline{\mathbf{I}} = \overline{\mathbf{I}}_{\widetilde{B}} \cup \overline{\mathbf{I}}_A, \quad (\overline{\mathbf{I}}_{\widetilde{B}}, \overline{\mathbf{d}}) = (\mathbf{I}_B, \mathbf{d}), \quad (\overline{\mathbf{I}}_A, \overline{\mathbf{d}}) = (\mathbf{I}_A, \mathbf{d})$$

*and* $(\overline{d}_{si}, \overline{d}'_{si}) = (d'_{is}, d_{is})$ *for all* $i \in I_A$, $s \in \mathbf{I}_B = \overline{\mathbf{I}}_{\widetilde{B}}$. *The bipartite valued poset of* $R^\bullet$ *is dual to* $(\overline{\mathbf{I}}, \overline{\mathbf{d}})$.

(c) *The diagram* (2.13) *is commutative and the functors* $\mathcal{I}^e_\eta$, $\overline{\mathcal{I}}^e_\eta$, $\overline{\overline{\mathcal{I}}}^e_\eta$ *are fully faithful embeddings. If* $\mathcal{I}$ *is one of the functors* $\mathcal{I}^e_\eta$, $\overline{\mathcal{I}}^e_\eta$, $\overline{\overline{\mathcal{I}}}^e_\eta$ *and* $\mathrm{res}^e_\eta\colon \mathrm{mod}(R) \to \mathrm{mod}(R^e_\eta)$ *is the restriction functor defined by* $\mathrm{res}^e_\eta(X) = (X'e, X''_\eta, \varphi|)$, *then* $\mathrm{res}^e_\eta \mathcal{I} \cong \mathrm{id}$ *and*

$$\begin{aligned} \mathbf{cdn}\,(\mathcal{I}Y)(i) &= (\mathbf{cdn}\,Y)(i) && \text{for } i \in J \cup L \\ &= 0 && \text{for } i \in \mathbf{I}_R \setminus (J \cup L). \end{aligned}$$

*The category* $\mathrm{Im}\,\mathcal{I}$ *consists of all* $R$*-modules* $X$ *such that* $(\mathbf{cdn}\,X)(i) = 0$ *for all* $i \in \mathbf{I}_R \setminus (J \cup L)$.

(d) *If the category* $\mathrm{adj}(R)^A_B$ *is of finite representation type (resp. tame representation type), then* $\mathrm{adj}(R^e_\eta)^{eAe}_{\eta B \eta}$ *and* $\mathrm{adj}(R^\bullet)^{\widetilde{A}^{\mathrm{op}}}_{\widetilde{M}^{\mathrm{op}}}$ *are of finite representation type (resp. tame representation type).*

*Proof.* (a) Since $M_{e_s}$ is $A$-faithful for any $s \in \mathbf{I}_B$ and by Proposition 2.3 we have $e'_s D(M) \cong (Me_s)^*$ then $e'_s D(M)$ is obviously $A$-faithful. Furthermore, since $e_i M$ is $B$-faithful for any $i \in \mathbf{I}_A$ then the left $\widetilde{B}$-module $D(M)e_i \cong D(E_i M)$ is faithful because it follows from the Proposition 2.3 and [15, Proposition 2.3] that the functor $D$ carries faithful modules to the faithful ones. It follows that $R^\nabla$ is a piecewise peak artinian $PI$-ring. The remaining part of (a) is obvious.

The statement (b) easily follows from Proposition 2.3 (b).

(c) The commutativity of the left hand squares in (2.13) follows from definitions of the functors $\mathcal{I}^e_\eta$, $\overline{\mathcal{I}}^e_\eta$ and $\overline{\overline{\mathcal{I}}}^e_\eta$. The right hand square is commutative by definitions of $D$, $\Theta^B$ and $\Theta_{\widetilde{B}^{\mathrm{op}}}$. It is easy to conclude from [22, Theorem 17.46] that $\mathcal{I}^e_\eta$ is fully faithful embedding with the properties required in (c). In order to prove (c) for the functor $\overline{\mathcal{I}}^e_\eta$ we note that $\overline{\mathcal{I}}^e_\eta$ is the functor $T_K L$ defined in [19, 2.22] with $K = J \cup L$. Then the required properties of $\overline{\mathcal{I}}^e_\eta$ follow from [19, Proposition 2.24] and from the proof of [16, Proposition 1.15]. Since the diagram (2.13) is commutative, $\Theta_{\eta B \eta}$ and $\Theta_B$ are full functors and obviously $\mathrm{res}^e_\eta \overline{\overline{\mathcal{I}}}^e_\eta \cong \mathrm{id}$ then $\overline{\overline{\mathcal{I}}}^e_\eta$ is full and the properties of $\overline{\mathcal{I}}^e_\eta$ proved above imply the properties of $\overline{\overline{\mathcal{I}}}^e_\eta$ required in (c). Let us show for example that $\overline{\overline{\mathcal{I}}}^e_\eta$ is faithful. For, suppose that $f: Y \to Z$ is a map in $\mathrm{adj}(R^e_\eta)$ such that $\overline{\overline{\mathcal{I}}}^e_\eta(f) = 0$. Then there exists a map $g: Y_1 \to Z_1$ in $\mathrm{mod}_{\mathrm{ic}}(R^e_\eta)$ such that $f = \Theta_{\eta B \eta}(g)$. It follows that $\overline{\mathcal{I}}^e_\eta(g) \in \mathrm{Ker}\Theta_B = [\mathrm{inj}(B)]$ and therefore $\overline{\mathcal{I}}^e_\eta(g)$ has a factorization $\overline{\mathcal{I}}^e_\eta(Y_1) \xrightarrow{h} Q \xrightarrow{t} \overline{\mathcal{I}}^e_\eta(Z_1)$ where $Q = (0, Q''_B, 0)$, and $Q''$ is the injective envelope of $Z''_1$ in $\mathrm{mod}(B)$. It follows from the properties of $\overline{\mathcal{I}}^e_\eta$ that $Q \cong \overline{\mathcal{I}}^e_\eta(0, E, 0)$, where $E$ is in $\mathrm{inj}(\eta B \eta)$. Since $\overline{\mathcal{I}}^e_\eta$ is full and faithful then $g$ has a factorization through $(0, E, 0)$ and therefore $g \in \mathrm{Ker}\Theta_{\eta B \eta}$. It follows that $f = \Theta_{\eta B \eta}(g) = 0$ and $\overline{\overline{\mathcal{I}}}^e_\eta$ is faithful as we required.

Since (d) easily follows from (c) the proof is complete.

Given an indecomposable module $X$ in $\mathrm{adj}(R)^A_B$ (resp. in $\mathrm{prin}(R)^A_B$, $\mathrm{mod}_{\mathrm{ic}}(R)_B$, $\mathrm{mod}^{\mathrm{pg}}(R)^A$) we define the **coordinate support**

$$\mathbf{csup}\,(X) = \mathbf{csup}\,(X)_A \cup \mathbf{csup}\,(X)_B \tag{2.15}$$

of $X$, where

$$\mathbf{csup}\,(X)_A = \{i \in \mathbf{I}_A : \mathbf{cdn}\,(X)(i) \neq 0\},$$
$$\mathbf{csup}\,(X)_B = \{s \in \mathbf{I}_B : \mathbf{cdn}\,(X)(s) \neq 0\}.$$

It follows that the module $X$ is **sincere** if and only if $\mathbf{csup}\,(X) = \mathbf{I}_R$.

An important consequence of Proposition 2.14 is following

**Corollary 2.16.** *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an artinian PI-ring of the form (1.3), where $A_1, \ldots, A_n, B_1, \ldots, B_m$ are local rings. Then*

(a) *If $X$ is an indecomposable module in $\mathrm{adj}(R)_B^A$, $e = e(\mathbf{csup}\,(X)_A) \in A$, $\eta = e(\mathbf{csup}\,(X)_B) \in B$ and*

$$\mathrm{adj}(R^e_\eta)^{eAe}_{\eta B \eta} \underset{\mathrm{res}^e_\eta}{\overset{\overline{\overline{T}}^e_\eta}{\rightleftarrows}} \mathrm{adj}(R)^A_B \tag{2.17}$$

*is the pair of functors defined in (2.13), then the module $Y = \mathrm{res}^e_\eta(X)$ in the category $\mathrm{adj}(R^e)^{eAe}_{\eta B \eta}$ is indecomposable sincere and $X \cong \overline{\overline{T}}^e_\eta(Y)$.*

(b) *The ring $R$ is not adj-sincere if and only if every indecomposable module $X$ in $\mathrm{adj}(R)^A_B$ is of the form $X = \overline{\overline{T}}^e_\eta(Y)$, where $Y$ is indecomposable in $\mathrm{adj}(R^e_\eta)^{eAe}_{\eta B\eta}$ and $e = e(J)$, $\eta = e(L)$ for some $J \subseteq \mathbf{I}_A$, $L \subseteq \mathbf{I}_B$ such that $J \cup L \neq \mathbf{I}_R$.*

(c) *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{mod}_{\mathrm{ic}}(R)_B & \xrightarrow{D^\bullet} & \left(\mathrm{mod}_{\mathrm{ic}}(R^\bullet_B)_{\widetilde{B}^{\mathrm{op}}}\right)^{\mathrm{op}} \\ \big\downarrow \overline{\Theta}_B & & \big\downarrow \overline{\Theta}_{\widetilde{B}^{\mathrm{op}}} \\ \widetilde{\mathrm{adj}}(R)^A_B & \xrightarrow{\overline{D}^\bullet} & \left(\widetilde{\mathrm{adj}}(R^\bullet)^{\widetilde{A}^{\mathrm{op}}}_{\widetilde{B}^{\mathrm{op}}}\right)^{\mathrm{op}} \end{array} \tag{2.18}$$

*where $D^\bullet = D\nabla_-$, $\widetilde{\mathrm{adj}}(R)^A_B$ is the factor category of $\mathrm{adj}(R)^A_B$ modulo all maps having a factorization through coproducts of $E(n+1), \ldots, E(n+m)$, the functor $\overline{\Theta}_B$ is the composition of $\Theta_B$ with the residue class functor and $\overline{D}^\bullet$ is the equivalence induced by $D^\bullet$.*

*Proof.* The statements (a) and (b) follow from Proposition 2.14. For the proof of (c) we note that $\Theta_B(E(n+t)) \cong E(n+t)$ and

$$D^\bullet(0, E_B(n+t)) \cong D(e_{n+t}R^\nabla) \cong E_{R^\bullet}(n+t),$$
$$D^\bullet(E(n+t)) \cong D(e'_{n+t}\widetilde{B}, 0) \cong (0, E_{\widetilde{B}^{\mathrm{op}}}(n+t))$$

for $t = 1, \ldots, m$. It follows that $D^\bullet$ induces a unique functor $\overline{D}^\bullet$ such that the diagram (2.18) is commutative and $\overline{D}^\bullet$ is an equivalence of categories, because we know that $\Theta_B$ is full dense and $\mathrm{Ker}\,\Theta_B = [\mathrm{inj}(B)]$.

Following [10] and [21] we call the functors $D^\bullet$ and $\overline{D}^\bullet$ **reflection duality functors.**

**3. Valued bipartite posets of piecewise peak rings.** In the proof of Theorem 1.6 we shall need the following observation.

**Proposition 3.1.** *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a basic piecewise peak artinian PI-ring and let* $e_1, \ldots, e_n \in A$, $e_{n+1}, \ldots, e_{n+m} \in B$ *be complete sets of primitive orthogonal idempotents. Then*

(a) $A_i = e_i R e_i$, $B_s = e_{s+n} R e_{s+n}$ *are division rings finitely generated over their centers and* ${}_iM_S = e_i R e_{n+s}$ *is non-zero for* $i = 1, \ldots, n$, $s = 1, \ldots, m$.

(b) *If* ${}_iM_s$ *is simple* $A_i$-$B_s$*-bimodule for all* $i$ *and* $s$*, then* $R$ *has an upper triangular form* (1.3) *and* $(\mathbf{I}_R, \mathbf{d})$ *defined by* (1.4) *and* (1.5) *is a bipartite valued poset with respect to the ordering* $i \prec j \Leftrightarrow e_i R e_j \neq 0$. *The bipartition of* $(\mathbf{I}_R, \mathbf{d})$ *is given by* $\mathbf{I}_A = \{1, \ldots, n\}$, $\mathbf{I}_B = \{n+1, \ldots, n+m\}$ *and we have* $i \prec j$ *for all* $i \in \mathbf{I}_A$, $j \in \mathbf{I}_B$.

(c) *If the category* $\mathrm{adj}(R)^A_B$ *is of finite representation type then*

$$1 \leq (\dim_{A_i}({}_iM_s))(\dim({}_iM_s)_{B_s}) \leq 3 \tag{3.2}$$

${}_iM_j$ *is a simple bimodule for* $i = 1, \ldots, n$, $s = 1, \ldots, m$, *and* (b) *applies.*

*Proof.* (a) follows from definitions and the injectivity of $\overline{c}_{ijs}$ in (2.2).

(b) By applying [15, Propositions 2.2, 2.3] to the rings $R^{1_A}_{e_{n+s}}$, $R^{e_i}_{1_B}$ we easily conclude that

(i) $e_i R e_j \neq 0$ implies $e_j R e_i = 0$ for all $i \neq j$,

(ii) $e_i R e_j \neq 0$ and $e_j R e_k \neq 0$ implies $e_i R e_k \neq 0$ for all $i, j, k$.

Hence (b) follows.

(c) Given $i \leq n$ and $s \leq m$ we consider the hereditary ring $S = R^e_\eta = \begin{pmatrix} A_i & {}_iM_s \\ 0 & B_s \end{pmatrix}$, where $e = e_i$ and $\eta = e_{n+s}$. Since $\mathrm{adj}(R)^A_B$ is of finite representation type then, by Proposition 2.14(d), $\mathrm{adj}(S)$ is of finite representation type and therefore $\mathrm{mod}(S)$ is of finite representation type, because $A_i$ and $B_s$ are division rings and $\mathrm{adj}(S)$ is cofinite in $\mathrm{mod}(S)$. Then (c) follows from the main result in [4].

If $R$ is a basic schurian piecewise peak artinian $PI$-ring of the form (1.3) and the condition (3.2) holds we associate with $R$ the bipartite valued poset $(\mathbf{I}_R, \mathbf{d})$ as in (1.4) and (1.5) satisfying the condition (b) in Proposition 2.14. We shall view $(\mathbf{I}_R, \mathbf{d})$ as a set of points $1, \ldots, n+m$ connected by valued dashed arrows

$$i \overset{(d_{ij}, d'_{ij})}{- - \rightarrow} j \tag{3.3}$$

if $i \prec j$, $i \neq j$, and $d_{ij}$, $d'_{ij}$ are defined by (1.5). We shall write $i \dashrightarrow j$ if $d_{ij} = d'_{ij} = 1$. We shall write

$$i \xrightarrow{(d_{ij}, d'_{ij})} j$$

if $i \prec j$ and there is no $s \neq i, j$ such that $i \prec s \prec j$. The bipartition is marked by writing the points $n+1, \dots, n+m$ in boldface, or by underlining them. We call $(\mathbf{I}, \mathbf{d})$ homogeneous if $d_{ij}d'_{ij} \leq 1$ for all $i, j \in \mathbf{I}_R$. In this case we consider $(\mathbf{I}, \mathbf{d})$ as a usual poset.

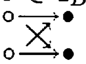
The values $d_{ij}, d'_{ij}$ form the bipartite Cartan matrix

$$(3.4)\ C(R) = \left(\begin{array}{cccc|cccc} 1 & d_{12} & \cdots & d_{1n} & d_{1\,n+1} & d_{1\,n+2} & \cdots & d_{1\,n+m} \\ d'_{12} & 1 & \cdots & d_{2n} & d_{2\,n+1} & d_{2\,n+2} & \cdots & d_{2\,n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d'_{1n} & d'_{2n} & \cdots & 1 & d_{n\,n+1} & d_{n\,n+2} & \cdots & d_{n\,n+m} \\ \hline d'_{1\,n+1} & d'_{2\,n+2} & \cdots & d'_{n\,n+1} & 1 & d_{n+1\,n+2} & \cdots & d_{n+1\,n+m} \\ d'_{1\,n+2} & d'_{2\,n+2} & \cdots & d'_{n\,n+2} & d'_{n+1\,n+2} & 1 & \cdots & d_{n+2\,n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d'_{1\,n+m} & d'_{2\,n+m} & \cdots & d'_{n\,n+m} & d'_{n+1\,n+m} & d'_{n+2\,n+m} & \cdots & 1 \end{array}\right)$$

We call $C(R)$ the **bipartite value matrix** of $R$ (compare with [9], [11] and [13]).

We recall from the introduction that the algebra $R = K(I \lhd J)$ defined by (1.2) is a homogeneous piecewise peak $K$-algebra with $\mathbf{I}_R = I \lhd J$, $\mathbf{I}_A = I$, $\mathbf{I}_B = J$. Note that $d_{ij} = d'_{ij} = 1$ if $i = j$ and $d_{ij} = d'_{ij} = 0$ if $i \not\preceq j$.

A partial converse of the observation above is given by the following

**Lemma 3.5.** *Suppose that $R$ is a basic schurian artinian piecewise peak PI-ring of the form (1.3) such that $d_{is}d'_{is} = 1$ for all $i \in \mathbf{I}_A$ and $s \in \mathbf{I}_B$. If $(\mathbf{I}_A, \mathbf{d})$ does not contain as a full bipartite subposet the poset*  *then $d_{ij}d'_{ij} \leq 1$ for all $i, j \in \mathbf{I}_R$, $L = \mathbf{I}_A$ and $J = \mathbf{I}_B$ are subposets of $\mathbf{I}_R$ such that $\mathbf{I}_R = L \lhd J$ (see 1.2) and there are ring isomorphism $F := A_1 \cong \cdots \cong A_n \cong B_1 \cong \cdots \cong B_m$,*

$$R \cong F(L \lhd J) \cong \begin{pmatrix} FL & M \\ 0 & FJ \end{pmatrix}$$

*and an $A$-$B$-bimodule isomorphism $M \cong \bigoplus_{i \in L} \bigoplus_{s \in J} F(i, j)$.*

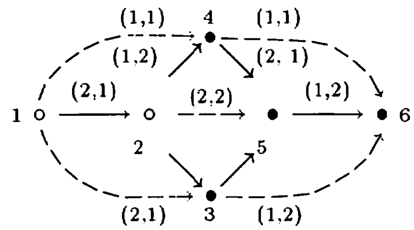
*Proof.* It follows from the assumption that $L$ or $J$ is linearly ordered. Suppose that $J$ is linearly ordered. Since $R$ is piecewise peak then [9, Lemma 2.13] yields $A_i \cong B_s$ for all $i \in L$, $s \in J$, $d_{ij}d'_{ij} = 1$ if and only if $i \prec j$ in $(\mathbf{I}_A, \mathbf{d})$, there are $F$-algebra isomorphisms $A \cong FL$, $B \cong FJ$ and the lemma follows if $m = 1$. The desired result follows by a simple induction on $m$ like in the proof of [2, Proposition 3.2].

A typical non-homogeneous piecewise peak algebra is the following.

**Example 3.6.** Let $\mathbb{R}$ and $\mathbb{C}$ be the real and the complex number field, respectively. By Lemma 2.1 the $\mathbb{R}$-subalgebra

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} = \left(\begin{array}{cc|cccc} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{R} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \hline & & \mathbb{R} & 0 & \mathbb{R} & \mathbb{C} \\ & & & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \multicolumn{2}{c|}{0} & & & \mathbb{R} & \mathbb{C} \\ & & & 0 & & \mathbb{C} \end{array}\right)$$

of the matrix algebra $\mathbb{M}_6(\mathbb{C})$ is a piecewise peak algebra and $(\mathbf{I}_R, \mathbf{d})$ has the form




completed by three arrows $1 \xrightarrow{(2,1)} 5$, $2 \xrightarrow{(1,2)} 6$, $1 \dashrightarrow 6$. Since $r\omega(\mathbf{I}_R, \mathbf{d}) = 4$ then by Theorem 1.6 the category $\mathrm{adj}(R)^A_B$ is of infinite representation type.

A basic tool for the study of piecewise peak rings is the following.

**Lemma 3.7.** *Let $R$ be a schurian artinian piecewise peak PI-ring of the form (1.3). Let $L = \{i, j, s\}$, $J = \{i, s, t\}$, $i, j \leq n$, $s, t \geq n+1$, and let $(L, \mathbf{d})$, $(J, \mathbf{d})$ be full bipartite valued subposets of $(\mathbf{I}_R, \mathbf{d})$ defined by $L$ and $J$. Suppose that $d_{is}d'_{is}$, $d_{js}d'_{js}$, $d_{it}d'_{it} \leq 3$ and $i \prec j$, $s \prec t$. Then $(L, \mathbf{d})$ and $(J, \mathbf{d})$ is of one of the forms presented in* Figures 1 *and* 2, *respectively. In particular $d_{ij}d'_{ij} \leq 9$, $d_{st}d'_{st} \leq 9$ and we have the following composition rules:*

($r_0$) $d_{ij} = d_{is}d'_{js}$ *if and only if* $d'_{ij} = d'_{is}d_{js}$.
($r'_0$) $d_{st} = d_{it}d'_{is}$ *if and only if* $d'_{st} = d'_{it}d_{is}$.
($r_1$) $d_{ij} = d'_{ij}$ *iff* $d_{ij} \leq d_{is}d'_{is} \leq 3$ *and* $(d_{is}, d'_{is}) = (d_{js}, d'_{js})$.
($r'_1$) $d_{st} = d'_{st}$ *iff* $d_{st} \leq d_{it}d'_{it} \leq 3$ *and* $(d_{is}, d'_{is}) = (d_{it}, d'_{it})$.
($r_2$) *If* $d_{ij} \neq d'_{ij}$ *then* $(d_{ij}, d'_{ij}) = (d_{is}d'_{js}, d'_{is}d_{js})$.
($r'_2$) *If* $d_{st} \neq d'_{st}$ *then* $(d_{st}, d'_{st}) = (d'_{it}, d_{is})$.

*Proof.* The lemma follows for $(L, \mathbf{d})$ from [9, Lemma 3.3]. The proof for $(J, \mathbf{d})$ reduces by duality to that one for $(L, \mathbf{d})$.

**Figure 1.**

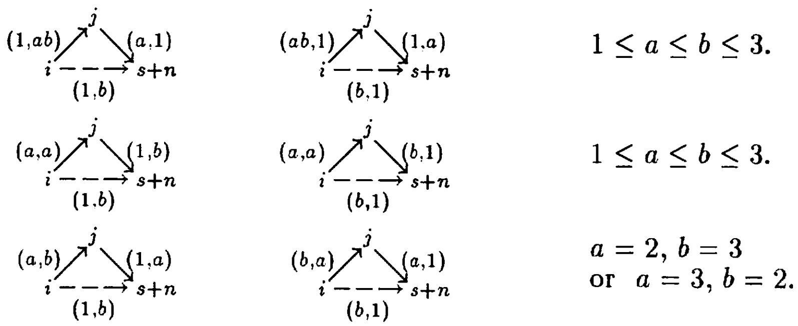
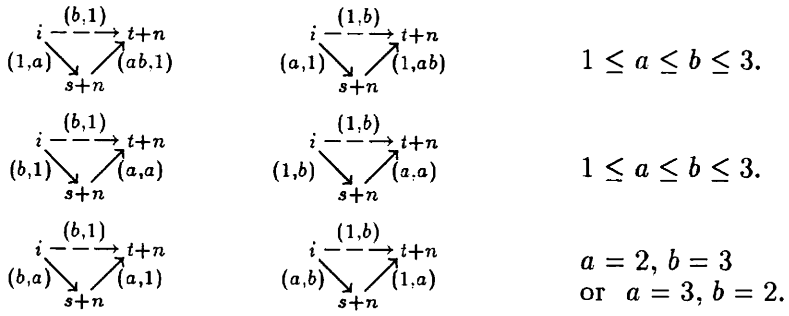



**Figure 2.**



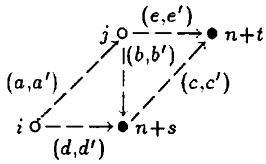

**Corollary 3.8.** *If $R$ is a schurian artinian piecewise peak PI-ring of the form (1.3) and $d_{is}d'_{is} \leq 3$ for all $i \in \mathbf{I}_A$, $s \in \mathbf{I}_B$ then $(\mathbf{I}_R, \mathbf{d})$ and $C(R)$ are symmetrizable, i.e. there are positive integers $f_1, \ldots, f_{n+m}$ such that $d_{ij}f_j = f_i d'_{ij}$ for all $i, j \in I_R$.*

*Proof.* Apply Lemma 3.7 and [9, Corollary 3.4].

Another restriction for the coefficients of $C(R)$ is given by the following result.

**Proposition 3.9.** *Let* $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ *be a basic schurian artinian piecewise peak PI-ring of the form* (1.3) *and let*

$I'$ :




*be a full bipartite valued subposet of* $(\mathbf{I}_R, \mathbf{d})$*. If* $d_{is}d'_{is} \leq 3$ *for all* $i \in \mathbf{I}_A$ *and* $s \in \mathbf{I}_B$*, then the following condition is satisfied.*

(rr) *If* $2 \leq bb' \leq 3$ *then* $(a, a') = (1, 1)$ *or* $(c, c') = (1, 1)$.

*Proof.* Without loss of generality we can suppose that $n = m = 2$, $i = s = 1$, $j = t = 2$, $(\mathbf{I}_R, \mathbf{d}) = I'$ and

$$R = \left(\begin{array}{cc|cc} A_1 & {}_1A_2 & {}_1M_1 & {}_1M_2 \\ 0 & A_2 & {}_2M_1 & {}_2M_2 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & B_1 & {}_1B_2 \\ & & 0 & B_2 \end{array}\right).$$

Let us consider the right $R$-module $Y = e_1J(R) = ({}_1A_2, {}_1M_1, {}_1M_2)$ as a right module over the left peak ring

$$S = \left(\begin{array}{c|cc} A_2 & {}_2M_1 & {}_2M_2 \\ \hline \multirow{2}{*}{0} & B_1 & {}_1B_2 \\ & 0 & B_2 \end{array}\right).$$

First we shall prove that

(†) *If* $Y$ *is indecomposable and either* $c \neq c'$ *or* $c = c' = bb'$*, then* $a = a' = 1$.

For this purpose we recall from Proposition 2.3 that there is a duality

$$D : \operatorname{mod}_{ti}(S) \longrightarrow \operatorname{mod}_{sp}(\widetilde{S}^{op}),$$

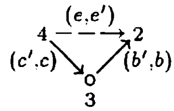
$\widetilde{S}$ has the form

$$\widetilde{S} = \left(\begin{array}{c|cc} A_2 & {}_2\widetilde{M}_1 & {}_2\widetilde{M}_2 \\ \hline 0 & B_1 & {}_1\widetilde{B}_2 \\ 0 & 0 & B_2 \end{array}\right)$$

and $(d_{ij}, d'_{ij}) = (\tilde{d}_{ij}, \tilde{d}'_{ij})$, where $\tilde{d}_{ij}$ and $\tilde{d}'_{ij}$ are entries in $C(\tilde{S})$. Moreover, the left $\tilde{S}$-module $D(Y)$ has the form

$$D(Y) = \begin{pmatrix} {}_1A_2^* \\ {}_1M_1^* \\ {}_1M_2^* \end{pmatrix}.$$

Since the bimodules ${}_1M_1$, ${}_2M_2$ are simple, the maps ${}_1A_2 \otimes {}_2M_1 \to {}_1M_1$, ${}_1M_1 \otimes {}_1B_2 \to {}_1M_2$, ${}_1A_2 \otimes {}_2M_2 \to {}_1M_2$ are surjective and therefore $Y$ is in $\mathrm{mod}_{\mathrm{ti}}(S)$. Now, if $Y$ is indecomposable the module $D(Y)$ is indecomposable and ${}_1M_2^* \neq 0$. The valued poset of $\tilde{S}^{\mathrm{op}}$ has the form

$(\tilde{\mathbf{I}}, \tilde{\mathbf{d}})$ : 


2 is a peak vertex and by our assumption $(\tilde{\mathbf{I}}, \tilde{\mathbf{d}})$ has one of the forms shown in Figure 1. Hence if either $c \neq c'$ or $c = c' = bb'$ then using the rules $(\mathrm{r}_0)$–$(\mathrm{r}_2)$ and $(\mathrm{r}'_0)$–$(\mathrm{r}'_2)$ in Lemma 3.7 one can easily check that the sets $J = \{4\}$, $J'' = \{3\}$, $J' = \emptyset$ define a splitting decomposition of $(\tilde{\mathbf{I}}, \tilde{\mathbf{d}})$ in the sense of [9, Definition 4.5]. By [9, Lemma 4.6, Theorem 4.3] the left module $\binom{{}_1A_2^*}{{}_1M_1^*} = \mathrm{res}_{J''}(D(Y))$ over the hereditary ring $\begin{pmatrix} A_2 & {}_2\widetilde{M}_1 \\ 0 & B_1 \end{pmatrix}$ is injective and therefore is a direct sum of $d_{12} = \dim_{A_2}({}_1A_2^*)$ copies of the injective module $\binom{A_2}{{}_2\widetilde{M}_1^*}$, where ${}_2\widetilde{M}_1^* = \mathrm{Hom}_{A_2}({}_2\widetilde{M}_1, A_2)$ (see [15, Proposition 2.5]). It follows from (2.6) that

$$d = d_{13} = \dim_{B_1}({}_1M_1^*) = d_{12} \dim_{B_1}({}_2\widetilde{M}_1^*) = d_{12}d_{23} = ab$$

and similarly $d' = a'b'$. Hence $a = a'$, because otherwise the rule $(\mathrm{r}_2)$ in Lemma 3.7 yields $a = db'$, $a' = d'b$ and we get $d = dbb'$, or equivalently $bb' = 1$; a contradiction. Furthermore, since $a = a'$ then $(\mathrm{r}_1)$ yields $b = d$, $b' = d'$ and consequently we get $a = a' = 1$ as we required. Then (†) follows.

In order to finish the proof suppose to the contrary that $aa' \neq 1 \neq cc'$. Since $d_{14}d'_{14} \leq 3$, in view of the duality $D\colon \mathrm{adj}(R) \to (\mathrm{adj}(\tilde{R}^{\mathrm{op}}))^{\mathrm{op}}$ and Proposition 2.3(b), we can consider only the case $d_{14} = 1$. Since the module $({}_1A_2, {}_1M_1)$ is socle projective over $\begin{pmatrix} A_2 & {}_2M_1 \\ 0 & B_1 \end{pmatrix}$. The module $D({}_1A_2, {}_1M_1) = \binom{{}_1A_2^*}{{}_1\widetilde{M}_1^*}$ (see (2.6)) is top injective, because the bimodule ${}_1M_1^*$ is simple and the map ${}_1B_2 \otimes {}_1M_2 \to {}_1M_1$ is surjective. Since $d_{14} = 1$ then (2.6) yields

$\mathbf{dim}D(Y) = \mathbf{dim}Y = (a, d, 1)$ and therefore $D(Y)$ is indecomposable, because it is top injective and has a simple top.

It follows that $Y$ is indecomposable and since $aa' \neq 1$ then according to (†) we get $c = c' < bb' \leq 3$. The assumption $cc' \neq 1$ yields $c = c' = 2$ and $bb' = 3$. Now applying ($r_1$) and ($r_2$) we get $d = d_{14} = 1$, $d' = d'_{14}$, $e = b$, $e' = b'$, $a = b'$ and $a' = bd'$. This reduces the study to the following cases:

1° $(d, d') = (1, 2)$, $(b, b') = (1, 3)$, $(a, a') = (3, 2)$.

2° $(d, d') = (1, 2)$, $(b, b') = (3, 1)$, $(a, a') = (1, 6)$.

3° $(d, d') = (1, 3)$, $(b, b') = (3, 1)$, $(a, a') = (1, 9)$.

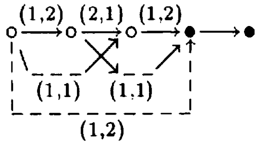
It follows that $\mathbf{dim}D(Y) = \mathbf{dim}Y = (3, 1, 1)$ in case 1°, and it is equal to (1,1,1) in cases 2° and 3°. Hence $Y$ is indecomposable. It follows that $DY$ is indecomposable in $\mathrm{mod}_{sp}(\widetilde{S}^{\mathrm{op}})$ and since $\widetilde{S}^{\mathrm{op}}$ is sp-sincere of type $\widehat{\mathbb{G}}'_2$ or $\widehat{\mathbb{G}}''_2$ (see [9, Theorem B]) then the vector obtained from the $\mathbf{dim}DY$ by inverting the order of coordinates is one of the vectors in [9, Appendix, Table II.8]. Now note that (1,1,3) does not appear in Table II.8, whereas (1,1,1) appears for the type $\widehat{\mathbb{G}}''_2$ only. Since $\widetilde{S}^{\mathrm{op}}$ is of type $\widehat{\mathbb{G}}'_2$ in the cases 2° and 3°, we get a contradiction and the proof is complete.

Now we are able to give a combinatorial characterization of bipartite valued posets of artinian piecewise peak $PI$-rings.

**Theorem 3.10.** *Let* $(\mathbf{I}, \mathbf{d})$ *be a bipartite valued poset with the bipartition* $I' \cup I''$ *such that* $i \prec s$ *and* $1 \leq d_{is}d'_{is} \leq 3$ *for all* $i \in I'$, $s \in I''$. *Suppose that* $d_{ii} = d'_{ii} = 1$ *for all* $i \in I$ *and* $(\mathbf{I}, \mathbf{d})$ *does not contain as a full subposet one of the critical posets* $\widetilde{\mathbb{A}}^{\bullet}_{\bullet}$, $\widetilde{\mathbb{G}}^1_2$ *and* $\widetilde{\mathbb{G}}^2_2$ *in Table* 1. *Then* $(\mathbf{I}, \mathbf{d})$ *is a valued bipartite poset of a piecewise artinian PI-ring if and only if* $(\mathbf{I}, \mathbf{d})$ *is symmetrizable, the composition rules* ($r_0$)–($r_2$), ($r'_0$)–($r'_2$), (rr) *hold for* $(\mathbf{I}, \mathbf{d})$, *and* $(\mathbf{I}, \mathbf{d})$ *as well as* $(\mathbf{I}, \mathbf{d})^{\mathrm{op}}$ *has the properties* ($r_3$)–($r_5$) *listed in* [9, Theorem 3.5]. *In this case there exists a finite dimensional piecewise peak algebra* $R$ *over an infinite field* $K$ *such that* $(\mathbf{I}, \mathbf{d}) = (\mathbf{I}_R, \mathbf{d})$.

*Proof.* The "if" part follows from Lemma 3.7, Corollary 3.8 and Proposition 3.9. The converse implication follows from the final statement in the theorem, which can be proved by repeating the arguments in the proof of [9, Theorem 3.8].

**Remark 3.11.** It follows from Lemma 3.7 and [9, Theorem 3.8] that the bipartite valued poset




is not of the form $(\mathbf{I}_R, \mathbf{d})$, where $R$ is a piecewise peak artinian $PI$-ring.

**Remark 3.12.** It follows from the results above that most of the dashed valued arrows in $(\mathbf{I}_R, \mathbf{d})$ are uniquely determined by the continuous ones according to the composition rules above. In this case we will mark in the picture of the valued poset $(\mathbf{I}_R, \mathbf{d})$ of $R$ only a minimal set of valued arrows and we presume that the reader is able to reconstruct the remaining ones according to the rules presented in this section.

**4. A peak reduction and an arrow waist reflection functor.** The aim of this section is to prove Theorems 4.1 and 4.12 which reduce the study of $\operatorname{adj}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with arbitrary ring $B$ to the case when $B$ is a division ring, under some assumption of the valued poset of the ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. This will play a key role in the proof of the main results of this paper.

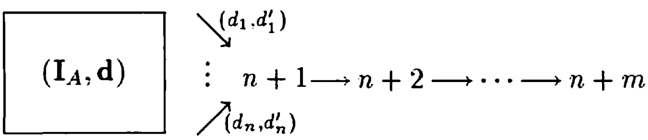
We recall from Proposition 2.3 that if $B$ is an artin algebra then we have defined the Nakayama equivalence

$$\mathfrak{N}_B : \operatorname{pr}(B) \longrightarrow \operatorname{inj}(B)$$

by the formula $\mathfrak{N}_B(-) = D\mathrm{Hom}_B(-, B)$, where $D\colon \operatorname{pr}(B^{\mathrm{op}}) \to \operatorname{inj}(B)$ is the duality induced by (2.5) with $R$ and $B^{\mathrm{op}}$ interchanged.

Let us start with a peak reduction introduced in [20, Section 2].

**Theorem 4.1.** *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a basic artinian piecewise peak $PI$-ring of the form (1.3) such that the bimodules ${}_iM_s$ are simple for $i = 1, \ldots, n$, $s = 1, \ldots, m$ and suppose that the valued poset $(\mathbf{I}_R, \mathbf{d})$ of $R$ has the form*



*where* $d_j = d_{jn+1}$, $d'_j = d'_{jn+1}$ *for* $j \in \mathbf{I}_A$ *and* $d_{st} = d'_{st} = 1$ *for all* $n+1 \leq s \leq t \leq n+m$. *Then the following statements hold.*

$$(4.2) \qquad \Omega(R) = \begin{pmatrix} A & \mathfrak{N}_B^{-1}(M) \\ 0 & B \end{pmatrix}$$

*is a schurian artinian right peak PI-ring and the functor*

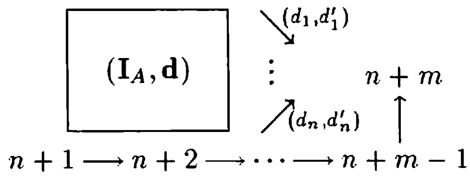
$$(4.3) \qquad \mathfrak{N}_+ : \operatorname{mod}_{\mathrm{ic}}(R)_B \longrightarrow \operatorname{mod}_{\mathrm{sp}}(\Omega(R))$$

*defined by the formula* $\mathfrak{N}_+(X'_A, X''_B, \varphi) = (X'_A, \mathfrak{N}_B^{-1}(X''), \varphi')$ *is an equivalence of categories, where* $\varphi'$ *is the map adjoint to the composed one*

$$X'_A \xrightarrow{\overline{\varphi}} \operatorname{Hom}_B({}_AM_B, X''_B) \cong \operatorname{Hom}_B(\mathfrak{N}_B^{-1}({}_AM_B), \mathfrak{N}_B^{-1}(X''_B)).$$

(b) *The valued poset of* $\Omega(R)$ *has the form*

(*)



*The elements* $i \in \mathbf{I}_A$ *and* $n+s$ *are unrelated in* $(\mathbf{I}_{\Omega(R)}, \mathbf{d})$ *for* $s = 1, \ldots, m-1$.

*Proof.* Note that $e_iM_B = ({}_iM_1, \ldots, {}_iM_m; c_{i,n+s,n+t})$ for $i \in \mathbf{I}_A$. We know from Proposition 3.1 that ${}_iM_s$ is non-zero for $s \geq 1$. Since ${}_iM_s$ is a simple and ${}_sB_t \neq 0$ for $s \leq t$, it follows from Lemma 2.1 that the map $\overline{\overline{c}}_{ist}$ in (2.2) is non-zero and therefore the map

$$c_{i,n+s,n+t} : {}_iM_s \otimes {}_sB_t \longrightarrow {}_iM_t$$

is surjective. Since $d_{n+s,n+t} = d'_{n+s,n+t} = 1$ for $s \leq t$ then the bimodule ${}_iM_s \otimes {}_sB_t$ is simple and therefore $c_{i,n+s,n+t}$ is injective for all $s \leq t \leq m$. It follows that $e_jM_B \cong (e_{n+1}B)^{d_j}$ and therefore it is projective-injective $B$-module as we required. Then the definition of $\Omega(R)$ makes sense. Since $B$ is a hereditary PI-ring of Dynkin type $\bullet \to \bullet \to \cdots \to \bullet \to \bullet$ then $B$ is an artin algebra, there exists the Nakayama equivalence $\mathfrak{N}_B: \mathrm{pr}(B) \to \mathrm{inj}(B)$ and it follows from [20, Corollary 2.11] that the isomorphism $e_jM_B \cong (e_{n+1}B)^{d_j}$ yields the isomorphism $e_i\mathfrak{N}_B^{-1}(M_B) \cong (e_{n+m}B)^{d_i}$ for any $i \in \mathbf{I}_A$. Consequently, $\Omega(R)$ is an artinian right peak $PI$-ring and (b) holds. Since by [20, Theorem 2.9] the functor (4.3) is an equivalence of categories the theorem is proved.

**Corollary 4.4.** *In the notation and assumption of Theorem* 4.1 *the following statements hold.*

(a) *If* $X = (X_1, \dots, X_{n+m}, {}_i\varphi_j)$ *is in* $\mathrm{mod_{ic}}(R)_B$, *then* $\mathfrak{N}_+(X) = (X'_i, {}_i\varphi'_j)$, *where* $X'_j = X_j$ *for* $j \in \mathbf{I}_A$, $X'_{n+m} = X_{n+1}$ *and (see* (*) *in Theorem* 4.1)

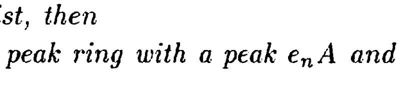
$$X'_{n+s} = \mathrm{Ker}\,(X_{n+1} \cong X_{n+1} \otimes {}_{n+1}B_{n+s+1} \xrightarrow{{}_{n+s+1}\varphi_{n+1}} X_{n+s+1}).$$

(b) *The* $R$*-module* $X$ *in* $\mathrm{mod_{ic}}(R)_B$ *has* $(\mathbf{cdn}\,X)(j) \neq 0$ *for any* $j \in \mathbf{I}_R$ *if and only if the* $\Omega(R)$*-module* $\mathfrak{N}_+(X)$ *is* sp*-sincere and* $X'_{n+m-1} \neq X'_{n+m}$.

(c) *The category* $\mathrm{adj}(R)^A_B$ *is of finite representation type (resp. tame, wild) if and only if* $\mathrm{mod_{sp}}(\Omega(R))$ *is of finite representation type (resp. tame, wild).*

*Proof.* (a) Since $B$ is a hereditary PI-ring of Dynkin type $\bullet \to \bullet \to \cdots \to \bullet$ and $X''_B = (X_{n+1}, \dots, X_{n+m}, {}_i\varphi_j)$ is an injective $B$-module then $X''_B$ is a direct sum of modules of the form $E_B(j) = (F, \dots, F, 0, \dots, 0)$ with $j$ copies of the division ring $F \cong B_{n+1}$, where $j = 1, \dots, m$. Then (a) follows from [20, Corollary 2.11]. The statement (b) is a consequence of (a). The proof of (c) is left as an exercise.

Following [20, Definition 4.2] we say that the ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ of the form (1.3) has an **arrow waist** if $R$ is schurian, $(\mathbf{I}_R, \mathbf{d})$ is a valued poset, $n$ is a unique maximal element in the poset $(\mathbf{I}_A, \mathbf{d})$, $n+1$ is a unique minimal element in the poset $(\mathbf{I}_B, \mathbf{d})$ and $d_{nn+1} = d'_{nn+1} = 1$. In this case $(\mathbf{I}_R, \mathbf{d})$ has the form

(4.5)  .

**Lemma 4.6.** *If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a piecewise peak artinian PI-ring of the form* (1.3), *the bimodules* ${}_iM_s$ *are simple for* $i \in \mathbf{I}_A$, $s = 1, \dots, m$ *and* $R$ *has an arrow waist, then*

(a) $A$ *is a right peak ring with a peak* $e_nA$ *and* ${}_AM$ *is a top injective* $A$*-module.*

(b) *$B$ is a left peak ring with a peak $Be_{n+1}$ and $M_B$ is a top injective $B$-module.*

*Proof.* We know from [15, Proposition 2.2] that $B$ is a left peak ring if and only if for every $b \in {}_sB_t$, $b \neq 0$, there is $a \in {}_1B_s$ such that $c_{n+1,n+s,n+t}(a \otimes b) \neq 0$. Consider the commutative diagram

$$\begin{array}{ccc} {}_nM_1 \otimes {}_1B_s \otimes {}_sB_t & \xrightarrow{1\otimes c_{n+1,n+s,n+t}} & {}_nM_1 \otimes {}_1B_t \\ \big\downarrow{\scriptstyle c_{n,n+1,n+s}\otimes 1} & & \big\downarrow{\scriptstyle c_{n,n+1,n+t}} \\ {}_nM_s \otimes {}_sB_t & \xrightarrow{c_{n,n+s,n+t}} & {}_sM_t \end{array}$$

Since $R_{1_B}^{e_n}$ is a left peak ring and the element $b \in {}_sB_t$ is non-zero there is $m \in {}_nM_s$ such that $c_{n,n+s,n+t}(m \otimes b) \neq 0$. Since the bimodule ${}_nM_s$ is simple and $d_{n,n+1} = d'_{n,n+1} = 1$ then $c_{n,n+1,n+s}$ is surjective and therefore $m = c_{n,n+1,n+s}(x \otimes a)$ for some $x \in {}_nM_1$ and $a \in {}_1B_s$. Then the commutativity of the diagram yields $c_{n+1,n+s,n+t}(a \otimes b) \neq 0$ and therefore $B$ is a left peak ring.

Since $e_jM_B = ({}_jM_1, \ldots, {}_jM_m, c_{j,n+s,n+t})$, ${}_jM_s \neq 0$ and ${}_1B_s \neq 0$ for $j \in \mathbf{I}_A$, $s = 1, \ldots, m$, then by Lemma 2.1 the map $c_{j,n+1,n+s}: {}_jM_1 \otimes {}_1B_s \to {}_jM_s$ is non-zero and therefore it is surjective, because ${}_jM_s$ is a simple bimodule. It follows that $\mathrm{top}(e_jM_B) \cong (e_{n+1}B)^{d_{j,n+1}}$ and it is injective, because $Be_{n+1}$ is left peak of $B$. Hence (b) follows. The statement (a) follows from (b) in view of duality (2.5).

If the ring $B$ in (1.3) is a left peak ring we define a reflection form

$$(4.7)\quad B^{\Delta} = \begin{pmatrix} (1_B - e_{n+1})B(1_B - e_{n+1}) & (e_{n+1}B(1_B - e_{n+1}))^* \\ 0 & B_{n+1} \end{pmatrix}$$

of $B$ (see [15, 2.6], [19, 2.13]), where $(-)^* = \mathrm{Hom}_{B_{n+1}}(-, B_{n+1})$, and a reflection functor

$$(4.8)\qquad \nabla_+ : \mathrm{mod}_{\mathrm{ti}}(B) \longrightarrow \mathrm{mod}_{\mathrm{sp}}(B^{\Delta})$$

by the formula $\nabla_+(Y) = (Y', Ye_{n+1}, h)$, where

$$Y' = \mathrm{Ker}\,(Ye_{n+1} \otimes e_{n+1}B(1_B - e_{n+1}) \longrightarrow Y(1_B - e_{n+1}))$$

and $h: Y' \otimes (e_{n+1}B(1_B - e_{n+1}))^* \to Ye_{n+1}$ is such that the map $\overline{h}$ adjoint to $h$ is the composed one

$$\begin{aligned} Y' &\hookrightarrow Ye_{n+1} \otimes e_{n+1}B(1_B - e_{n+1}) \\ &\cong \mathrm{Hom}_{B_{n+1}}\Big((e_{n+1}B(1_B - e_{n+1}))^*, Ye_{n+1}\Big). \end{aligned}$$

It follows form [15, Proposition 2.6] that $\nabla_+$ is an equivalence of categories such that if $\mathbf{dim}\, Y = (y_{n+1}, \ldots, y_{n+m})$ and $\mathbf{dim}\, \nabla_+(Y) = (y'_{n+1}, \ldots, y'_{n+m})$ then

(4.9) $y'_{n+1} = y_{n+1}, y'_{n+s} = -y_{n+s} + y_{n+1} d_{n+1,n+s}$ for $s = 2, \ldots, m$.

If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a piecewise peak ring of the form (1.3) and $R$ has an arrow waist then by Lemma 3.6, $B$ is a left peak ring and $M_B$ is in $\mathrm{mod}_{\mathrm{ti}}(B)$. Then following [20, 4.5] we define an **arrow waist reflection form**

$$(4.10) \qquad \delta R = \begin{pmatrix} (1_A - e_n)A(1_A - e_n) & (1_A - e_n)\nabla_+(M_B) \\ 0 & B^{\Delta} \end{pmatrix}$$

of $R$ as well as an **arrow waist reflection functor**

$$(4.11) \qquad \boldsymbol{\delta} : \mathrm{adj}(R)_B^A \longrightarrow \mathrm{mod}_{\mathrm{sp}}(\delta R)$$

by the formula

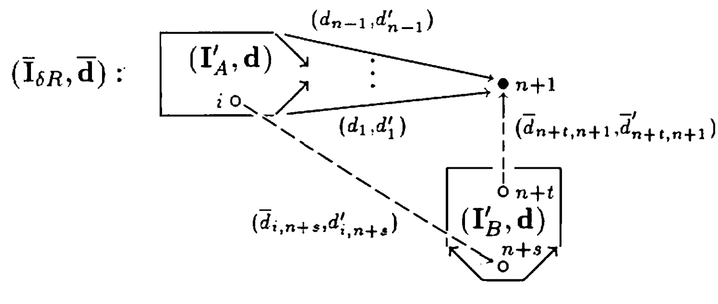
$$\boldsymbol{\delta}(X'_A, X''_B, \varphi) = (X'(1_A - e_n), \nabla_+(X''_B), \widetilde{\varphi})$$

where $\widetilde{\varphi}$ is the map adjoint to the composed monomorphism

$$\begin{array}{c} X'(1_A - e_n) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_B({}_AM_B, X''_B)(1_A - e_n) \\ \downarrow \nabla_+(1_A - e_n) \\ \mathrm{Hom}_{B^{\Delta}}((1_A - e_n)\nabla_+(M_B), \nabla_+(X'')). \end{array}$$

Note that the valued poset $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ is obtained from (4.5) by removing the vertex $n$ and by "reflecting" the part $(\mathbf{I}_B - \{n+1\}, \mathbf{d})$ at the vertex $n+1$. This is presented in the following Figure 3.

**Figure 3.**



where $\mathbf{I}'_A = \mathbf{I}_A \backslash \{n\}$, $\mathbf{I}'_B = \mathbf{I}_B \backslash \{n+1\}$, $d_j = d_{j,n+1}$ and $d'_j = d'_{j,n+1}$.

Now we are able to prove our arrow waist reflection theorem.

**Theorem 4.12.** *Let $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be a basic piecewise peak artinian PI-ring of the form (1.3) such that the bimodules ${}_iM_s$ are simple for $i = 1, \dots, n$ and $s = 1, \dots, m$. If $R$ has an arrow waist then*

(a) *The waist reflection form $\delta R$ of $R$ is a schurian artinian right peak PI-ring, $e_{n+1}\delta R$ is a peak of $\delta R$ and the functor (4.11) is an equivalence of categories.*

(b) *The valued poset $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ of $R$ has the form shown in Figure 3, i.e. $n+1$ is a unique maximal element in $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$, $\overline{\mathbf{I}}_{\delta R} = \{1, \dots, n-1, n+1, \dots, n+m\}$ and*

(i) $\overline{d}_{ij} = d_{ij}$, $\overline{d}'_{ij} = d'_{ij}$ *if* $1 \le i, j \le n-1$ *or* $n+2 \le i, j \le n+m$,

(ii) $(\overline{d}_{i,n+1}, \overline{d}'_{i,n+1}) = (d_{i,n+1}, d'_{i,n+1})$ *for* $i = 1, \dots, n-1$,

(iii) $(\overline{d}_{t,n+1}, \overline{d}'_{t,n+1}) = (d'_{n+1,t}, d_{n+1,t})$ *for* $t = n+2, \dots, n+m$,

(iv) $\overline{d}_{i,n+s} = -d_{i,n+s} + d_{i,n+1}d_{n+1,n+s}$,
$\overline{d}'_{i,n+s} = -d'_{i,n+s} + d_{i,n+1}d_{n+1,n+s}$ *for* $i = 1, \dots, n-1$ *and* $s = 2, \dots, m$.

(v) *There is a relation $i \prec j$ in $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ if and only if $d_{ij} \ne 0$.*

(c) *If $n \ge 2$, the ring $R$ is not* adj*-sincere.*

*Proof.* (a) By Lemma 4.6, $B^{\Delta}$ is a right peak ring and $e_{n+1}B^{\Delta}$ is a peak of $B^{\Delta}$. Moreover, since ${}_AM$ is faithful then given $i \le n-1$ the socle of the right $R_{1_B}^{e_n}$-module $e_iR(1-e_n)$ is a $B$-module and therefore $e_i(\delta R) = (e_iR)(1-e_n)$ is socle projective. It follows that $\delta R$ is schurian artinian right peak $PI$-ring. Moreover, the Lemma 4.6 implies that $R$ has an arrow waist $n \to n+1$, and according to [20, Theorem 4.12] the functor (4.11) is an equivalence of categories.

(b) The statement (i) follows because $e_i(\delta R)e_j \cong e_iRe_j$ if $i$ and $j$ run as required in (i). For the statement (ii) we note that $\overline{d}_{i,n+1} = \dim(e_i(\delta R)e_{n+1})_{B_1} = \dim(e_i\nabla_+(M)e_{n+1})_{B_1} = \dim(\nabla_+(e_iM)e_{n+1})_{B_1} \overset{(4.9)}{=} \dim(e_iMe_{n+1})_{B_1} = d_{i,n+1}$ if $i = 1, \dots, n-1$. The equality $\overline{d}'_{i,n+1} = d'_{i,n+1}$ follows in a similar way.

(iii) If $n+2 \le t \le n+m$ we have $\overline{d}_{t,n+1} = \dim(e_t(\delta R)e_{n+1})_{B_1} = \dim(e_tB^{\Delta}e_{n+1})_{B_1} + \dim(e_{n+1}Be_t)^{*}_{B_1} \overset{(+)}{=} \dim_{B_1}(e_{n+1}Be_t) = d'_{n+1,t}$ and similarly we show that $\overline{d}'_{t,n+1} = d_{n+1,t}$. The equality (+) above follows from [4, Proposition 1.3], because the $PI$-property of $R$ implies that $B_1$ is finite dimensional over its center.

(iv) $\overline{d}_{i,n+s} = \dim(e_i(\delta R)e_{n+s})_{B_s} = \dim(e_i\nabla_+(M)e_{n+s})_{B_s}$ $= \dim(\nabla_+(e_iM)e_{n+s})_{B_s} \overset{(4.9)}{=} -d_{i,n+s} + d_{i,n+1}d_{n+1,n+s}$ for $i = 1,\ldots,n-1$, $s = 2,\ldots,m$. Since there is an exact sequence

$$0 \longrightarrow \nabla_+(e_iM)e_{n+s} \longrightarrow e_iMe_{n+1} \otimes e_{n+1}Me_{n+s} \longrightarrow e_iMe_{n+s} \longrightarrow 0$$

of $A_i$-$B_s$-bimodules then the second equality in (iv) can be proved in a similar way as the first one.

The remaining statements in (b) follow immediately from definitions.

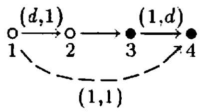
(c) Suppose that $X$ is an indecomposable module in $\mathrm{adj}(R)_B^A$ and $X_i = Xe_i \neq 0$ for all $i = 1,\ldots,n-1$. Since $d_{n,n+1} = d'_{n,n+1} = 1$, then the multiplication map ${}_{n+1}\varphi_n : X_n \otimes {}_nM_{n+1} \to X_{n+1}$ is bijective and therefore $(\mathrm{top}X)e_n = 0$, because ${}_{n+1}\varphi_n : X_n \otimes_n M_{n+1} \to X_{n+1}$ is surjective. Hence $e_nR$ is not a summand of $P(X)$ and therefore $X$ is not sincere. Consequently, the statement (c) follows and the proof is complete.

**Example 4.13.** Let $G \subset F$ be commutative fields such that $d = \dim_G F \leq 3$. Then

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} = \left(\begin{array}{cc|cc} F & F & F & F \\ 0 & G & F & F \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & G & F \\ & & 0 & F \end{array}\right)$$

is a piecewise peak $G$-algebra and

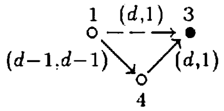
$(\mathbf{I}_R, \mathbf{d})$ :




where $(d_{13}, d'_{13}) = (d, 1)$ and $(d_{24}, d'_{24}) = (1, d)$ (see Remark 3.12). The arrow $\circ \longrightarrow \bullet$ is a waist of $R$ and $\delta R$ has the form

$$\delta R = \begin{pmatrix} F & U & V \\ 0 & F & F^* \\ 0 & 0 & G \end{pmatrix}$$

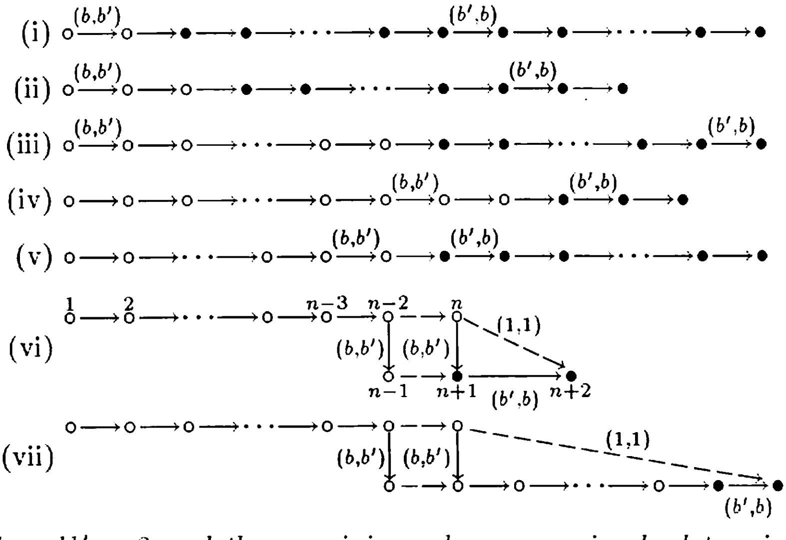
where $F^* = \mathrm{Hom}_G(F, G)$, and $(U, V) = \nabla_+(F, F)$ is the reflection of the right $\left(\begin{smallmatrix} G & F \\ 0 & F \end{smallmatrix}\right)$-module with respect to the vertex 4 (see [3]). By the definition of $\nabla_+$ we have $V = F$ and $U$ is the kernel of the multiplication map $F \otimes_G F \to F$. Hence ${}_F U_F \cong {}_F F_F^{d-1}$. It follows that $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ has the form




Theorem 4.12 (a) yields $\mathrm{adj}(R)_B^A \cong \mathrm{mod}_{\mathrm{sp}}(\delta R)$ and applying [9, Theorem A] we conclude that $\mathrm{adj}(R)_B^A$ is of finite representation type. Note that the poset $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ can be easily described by applying Theorem 4.12 (b) without determining the algebra $\delta R$.

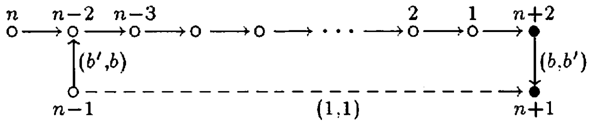
**5. Main results.** In the proof of Theorem 1.6 we shall need the following result.

**Lemma 5.1.** *Suppose that* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a basic artinian piecewise peak PI-ring of the form* (1.3). *Then* $\mathrm{adj}(R)_B^A$ *is of finite representation type if the valued bipartite poset* $(\mathbf{I}_R, \mathbf{d})$ *of* $R$ *is a full bipartite valued subposet of one of the following forms:*




*where* $bb' = 2$ *and the remaining values are uniquely determined by the composition rules in Lemma* 3.7 (*see Remark* 3.12).

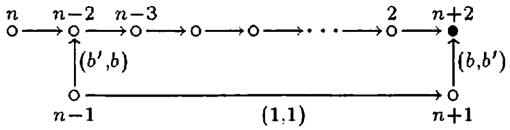
*Proof.* Suppose that $(\mathbf{I}_R, \mathbf{d})$ has the form (vi) such that the chain $1 \to 2 \to \cdots \to n-3$ has at least one point. In view of Proposition 2.14 it is sufficient to prove that $\mathrm{adj}(R^\bullet)$ is of finite representation type, where $R^\bullet$ is the reflection dual form of $R$ (see 2.13). It follows from Proposition 2.14(b) that $(\widetilde{\mathbf{I}}_{R^\bullet}, \widetilde{\mathbf{d}})$ has the form




It follows that $R^\bullet$ has an arrow waist ${}_1\circ\longrightarrow\bullet_{n+2}$ and according to Theorem 4.12 there is an equivalence of categories

$$\delta : \operatorname{adj}(R^\bullet) \longrightarrow \operatorname{mod}_{\mathrm{sp}}(\delta R^\bullet).$$

Applying Theorem 4.12(b) we show that the bipartite valued poset $(\overline{\mathbf{I}}_{R^\bullet}, \overline{\mathbf{d}})$ of the right peak ring $\delta R^\bullet$ has the form




because we easily calculate that

$$\begin{aligned} \overline{d}_{i,n+1} &= -\widetilde{d}_{i,n+1} + \widetilde{d}_{i,n+2}\widetilde{d}_{n+2,n+1} \\ &= -b + 1b = 0, \quad \text{if } i = 2, \dots, n-2, n, \\ &= -1 + bb' = 1, \quad \text{if } i = n-1 \end{aligned}$$

and similarly $\overline{d}'_{i,n+1} = 1$ if $i = n-1$, and $\overline{d}'_{i,n+1} = 0$ if $i = 2, \dots, n-2, n$.

It follows that the point $n+1$ is incomparable with $2, \dots, n-2, n$, whereas there is an arrow ${}_{n-1}\circ\longrightarrow\circ_{n+1}$ as indicated above. Since the valued poset $(\overline{\mathbf{I}}_{R^\bullet}, \overline{\mathbf{d}})$ does not contain as a full one peak valued subposet any of the critical forms in [9, Theorem A] then $\operatorname{mod}_{\mathrm{sp}}(\delta R^\bullet)$ is of finite representation type and $\operatorname{adj}(R)^A_B$ is of finite representation type if $(\mathbf{I}_R, \mathbf{d})$ has the form (vi).

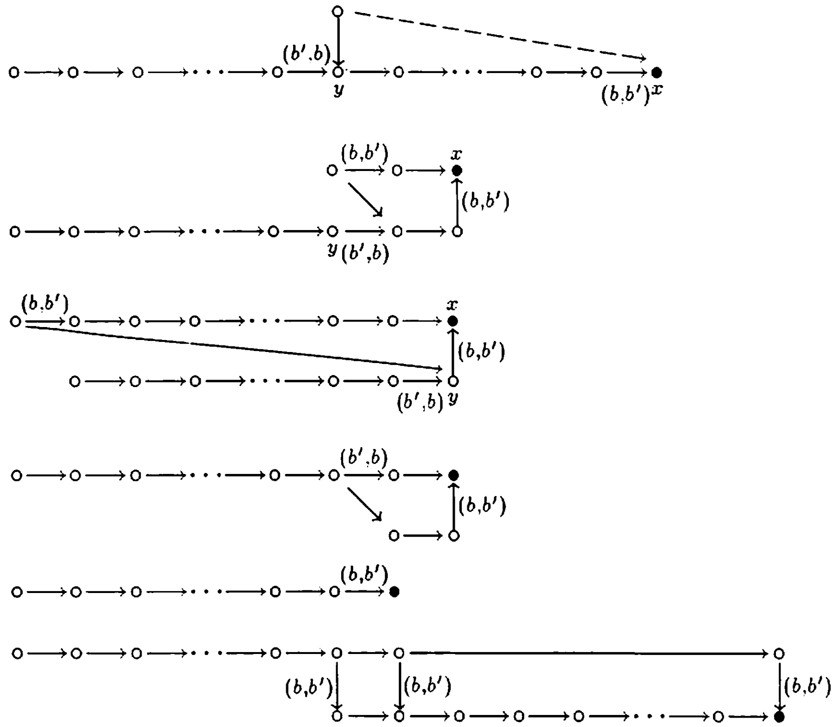
Now suppose that $(\mathbf{I}_R, \mathbf{d})$ is a full bipartite valued poset of (vi). Similarly as in [9, Theorem 3.8] one can easily construct a piecewise peak artinian $PI$-ring

$$S = \begin{pmatrix} A' & M' \\ 0 & B' \end{pmatrix}$$

such that $(\mathbf{I}_S, \mathbf{d})$ has the form (vi) with nontrivial chain $1 \to 2 \to \cdots \to n-3$ and $R$ is isomorphic to $S^e_\eta$ for some idempotents $e \in A'$ and $\eta \in B'$

(see 2.11). It follows from the first part of the proof that $\mathrm{adj}(S)^{A'}_{B'}$ is of finite representation type and by Proposition 2.14(d), the category $\mathrm{adj}(R)^{A}_{B}$ is of finite representation type. This proves the lemma in the case (vi).

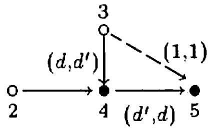
Now suppose that $(\mathbf{I}_R, \mathbf{d})$ is one of the forms (i) – (v), (vii). It follows that $R$ has an arrow waist and Theorem 4.12 applies to $R$. Then similarly as in the proof above we show that $(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ has the form




in the cases (i) – (v) and (vii), respectively, where $bb' = 2$ and $\overline{d}_{jx} = \overline{d}'_{jx} = 1$ for $j \leq y$. Then by [9, Theorem A], the ring $R$ is sp-representation-finite and in view of Theorem 4.12 the categories $\mathrm{adj}(R)^{A}_{B} \cong \mathrm{mod}_{\mathrm{sp}}(\delta R)$ are of finite representation type. This finishes the proof.

**Lemma 5.2.** *If* $(\mathbf{I}_R, \mathbf{d})$ *is of one of the forms* $\widetilde{\mathbb{F}}^5_4$, $\widetilde{\mathbb{F}}^6_4$ *presented in Table* 1 *then the category* $\mathrm{adj}(R)^{A}_{B}$ *is of infinite representation type.*

*Proof.* Suppose that $(\mathbf{I}_R, \mathbf{d})$ is of type $\widetilde{\mathbb{F}}^6_4$. Then $R$ has the form $R = \begin{pmatrix} F & {}_F N_T \\ 0 & T \end{pmatrix}$ where $F$ is the division ring $F_1$, $T$ is a piecewise peak ring, $(\mathbf{I}_T, \mathbf{d})$ has the form




, $dd' = 2,$

$N_T$ is in $\operatorname{mod}_{sp}(T)_B$ and $\mathbf{cdn}(N_T) = \left(\begin{smallmatrix} & 1 & \\ 0 & 0 & 1\end{smallmatrix}\right)$. Consider the vector space category

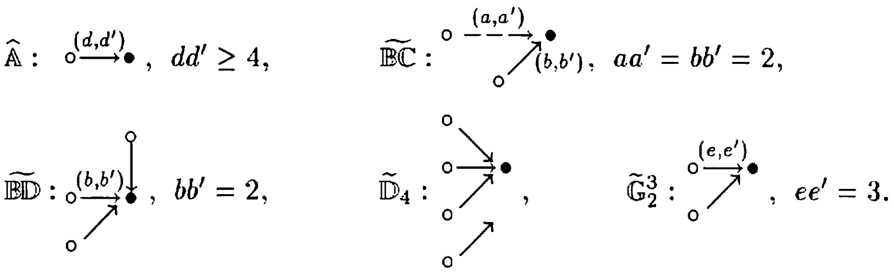
$$\mathbb{H}_F = \operatorname{Hom}_T({}_FN_T, \operatorname{mod}_{sp}(T)_B)$$

over $F$ and let $\mathbb{S}(\mathbb{H}_F)$ be the full subcategory of the subspace category $\mathbb{U}(\mathbb{H}_F)$ of $\mathbb{H}_F$ consisting of objects without summand of the form $(F, 0, 0)$. It follows from [19, 4.18 and 4.20] that there is a full dense functor $\Phi: \operatorname{mod}_{sp}(R)_B \to \mathbb{S}(\mathbb{H}_F)$ which establishes a representation equivalence of the $\mathbb{S}(\mathbb{H}_F)$ and a full subcategory of $\operatorname{mod}_{sp}(R)_B$. From Proposition 5.8(b) and the results in [13, Section 3] we easily conclude that the Auslander-Reiten quiver $\Gamma(\operatorname{mod}_{sp}(T)_B)$ of $\operatorname{mod}_{sp}(T)_B$ is obtained from the quivers in Figures 4 and 5 by deleting the final vertices $u = 2$ and $v = 3$. Then a simple analysis of the quivers in Figures 4 and 5 shows that if $Y$ is an indecomposable in $\operatorname{mod}_{ic}(T)_B$ such that $\mathbf{cdn}(Y)$ is $uv^2xy$ and $u^2v^2x^2y$ respectively (for the convention see [22, 11.88]), then $\operatorname{End}(Y) \cong F$ and $\dim|\overline{Y}|_F = 2$, where $\overline{Y} = \operatorname{Hom}_T({}_FN_T, Y)$ is considered as an object in $\mathbb{H}_F$. Moreover the left dimension of the $F$-space $|\overline{Y}|_F$ over $\mathbb{H}(\overline{Y}, \overline{Y}) \cong \operatorname{End}(Y) \cong F$ is also two. It follows that $\mathbb{S}(\mathbb{H}_F)$ is of infinite representation type (see [9] and therefore $\operatorname{mod}_{ic}(R)_B$ is of infinite representation type. Since the functor $\Theta_B$ in (2.8) preserves finite representation type the category $\operatorname{adj}(R)^A_B$ is of infinite representation type as we required.

Suppose that the poset $(\mathbf{I}_R, \mathbf{d})$ is of type $\widetilde{\mathbb{F}}^5_4$. It follows from Proposition 2.14(b) that the bipartite valued poset of the reflection dual form $R^\bullet$ of $R$ has the form $\widetilde{\mathbb{F}}^6_4$ and therefore $\operatorname{adj}(R^\bullet)$ is of infinite representation type. In view of the reflection duality (2.18), $\operatorname{adj}(R)^A_B$ is of infinite representation type and the lemma follows.

One can easily prove the following

**Lemma 5.3.** *Let $R = \left(\begin{smallmatrix} A & M \\ 0 & B\end{smallmatrix}\right)$ be an artinian piecewise peak PI-ring. Then $r\omega(R) \leq 3$ and $l\omega(R) \leq 3$ if and only if the bipartite valued poset $(\mathbf{I}_R, \mathbf{d})$ of $R$ does not contain as a full bipartite valued subposet one of the following posets and of their duals:*




*Proof of Theorem 1.6.* (a) $\Longrightarrow$ (b) It follows from Proposition 3.1 that $R$ is schurian of the triangular from (1.3) and $(\mathbf{I}_R, \mathbf{d})$ is a bipartite valued poset such that $d_{is}d'_{is} \leq 3$, for all $i \in \mathbf{I}_A$, $s \in \mathbf{I}_B$. By Corollary 3.8, the poset $(\mathbf{I}_R, \mathbf{d})$ is symmetrizable. Then in view of Proposition 2.12 it is sufficient to show that $\mathrm{adj}(R)^A_B$ is of infinite representation type if $(\mathbf{I}_R, \mathbf{d})$ is one of the critical forms presented in Table 1 or the forms in Lemma 5.3.

If $(\mathbf{I}_R, \mathbf{d})$ is either $\widetilde{\mathbb{A}}_\bullet^\bullet$ or is of one of the forms in Lemma 5.3 we easily conclude from [4] that $\mathrm{adj}(R)^A_B$ is of infinite representation type. By [9, Theorem A], $\mathrm{adj}(R)^A_B$ is of infinite representation type if $(\mathbf{I}_R, \mathbf{d})$ is one of the forms in Table 1 having a unique black vertex. If $(\mathbf{I}_R, \mathbf{d})$ is one of the forms in Table 1 and the full subposet of $(\mathbf{I}_R, \mathbf{d})$ consisting of black points is a homogeneous chain of length $\geq 2$ then according to Theorem 4.1 there is an equivalence of categories $\mathrm{mod}_{\mathrm{ic}}(R)_B \cong \mathrm{mod}_{\mathrm{sp}}(\Omega(R))$ and by [9, Theorem A], the schurian right peak $PI$-ring $\Omega(R)$ is sp-representation-infinite. Since the functor $\Theta_B$ in the diagram (2.8) preserves the representation type the category $\mathrm{adj}(R)^A_B$ is of infinite representation type. The proof in the remaining cases $\widetilde{\mathbb{F}}^i_4$, $i = 5, 6$ presented in Table 1 follows from Lemma 5.2.

(b) $\Longrightarrow$ (a) Since $r\omega(R) \leq 3$ then $d_{it}d'_{it} \leq 3$, then the bimodule ${}_iM_t$ is simple and Proposition 3.1 yields $i \prec n + t$ for $i \in \mathbf{I}_A$ and $t = 1, \ldots, m$. It follows that the rules $(\mathrm{r}_0) - (\mathrm{r}_2)$, $(\mathrm{r}'_0) - (\mathrm{r}'_2)$ in Lemma 3.7 and (rr) in Proposition 3.9 apply to the bipartite valued poset $(\mathbf{I}_R, \mathbf{d})$ of $R$.

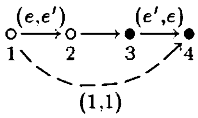
If $m = 1$, $B$ is division ring, $R$ is a right peak ring (by [15, Proposition 2.2]) and according to [9, Theorem A] the category $\mathrm{mod}_{\mathrm{sp}}(R) = \mathrm{mod}_{\mathrm{ic}}(R)_B$ is of finite representation type. Consequently, in view of (2.8) the statement (a) follows.

If $n = 1$, $A$ is a division ring, $R$ is a left peak ring, $\widetilde{R}^{\mathrm{op}}$ is a right peak ring and the valued poset of $\widetilde{R}^{\mathrm{op}}$ is equal to $(\mathbf{I}_R, \mathbf{d})$. Then the case $m = 1$ applies to $\widetilde{R}^{\mathrm{op}}$ and in view of Corollary 2.7 we conclude that (b) implies (a).

Suppose that $n \geq 2$ and $m \geq 2$. Since $(\mathbf{I}_A, \mathbf{d})$ does not contain the poset $\widetilde{\mathbb{A}}^{\bullet}_{\bullet}$ presented in Table 1, then $(\mathbf{I}_A, \mathbf{d})$ or $(\mathbf{I}_B, \mathbf{d})$ is linearly ordered and $i \prec m+1$ for all $i \in \mathbf{I}_A$.

Suppose that $(\mathbf{I}_B, \mathbf{d})$ is linearly ordered and that (b) holds. First we prove that if, in addition, $d_{i,n+t}d'_{i,n+t} = 3$ for some $i \in \mathbf{I}_A$ and $n+t \in \mathbf{I}_B$ then $n = m = 2$ and $(\mathbf{I}_R, \mathbf{d})$ has the form

(5.4)




where $ee' = 3$, $(d_{13}, d'_{13}) = (e, e')$ and $(d_{24}, d'_{24}) = (e', e)$. For, suppose that $i$ and $t$ are as above. Then $(\mathbf{I}_A, \mathbf{d})$ is linearly ordered, because $r\omega(R) \le 3$. It follows from Proposition 3.9 that $i = 1$ or $t = m$, because otherwise $(\mathbf{I}_R, \mathbf{d})$ contains $\widetilde{\mathbb{G}}^1_2$ or $\widetilde{\mathbb{G}}^2_2$; a contradiction. In view of the duality in Corollary 3.7 we can consider only the case $i = 1$.

We shall show that $t = 1$. Assume, to the contrary, that $t \ge 2$. Applying Proposition 3.9 to $i = 1$, $j = n$, $s = 1$ and $t$ we conclude that $d_{n,n+1} = d'_{n,n+1} = 1$, because otherwise $d_{1n} = d'_{1n} = 1$ and $(d_{n,n+t}, d'_{n,n+t}) = (d_{1,n+t}, d'_{n+t})$ or $d_{n+1,n+t} = d'_{n+1,n+t} = 1$ and $(d_{1,n+1}, d'_{1,n+1}) = (d_{1,n+t}, d'_{1,n+t})$ (apply $(\mathrm{r}_1)$ and $(\mathrm{r}'_1)$). It follows that $\widetilde{\mathbb{G}}^1_2$ or $\widetilde{\mathbb{G}}^2_2$ is contained in $(\mathbf{I}_R, \mathbf{d})$ and we get a contradiction. Then $d_{n,n+1} = d'_{n,n+1} = 1$ and by $(\mathrm{r}_1)$, $(\mathrm{r}'_1)$ we get $d_{1n}d'_{1n} = d_{1,n+1}d'_{1,n+1} \le 3$. If $d_{1n}d'_{1n} = 1$ or $d_{n,n+t}d'_{n,n+t} = 1$ then $\widetilde{\mathbb{G}}^1_2$ or $\widetilde{\mathbb{G}}^2_2$ is contained in $(\mathbf{I}_R, \mathbf{d})$, a contradiction. Thus $d_{1n}d'_{1n} \ne 1$ and $d_{n,n+t}d'_{n,n+t} \ne 1$. It follows that $d_{1n} \ne d'_{1n}$, $d_{n,n+t} \ne d'_{n,n+t}$ and in view of $(\mathrm{r}_2)$, $(\mathrm{r}'_2)$ we get

$$
\begin{aligned}
d_{1n} &= d_{1,n+t}d'_{n,n+t} \quad \text{and} \\
d'_{n,n+t} &= d'_{n+1,n+t} = d'_{1,n+t}d_{1,n+t} = d'_{1,n+t}d_{1n}.
\end{aligned}
$$

It follows that $d_{1,n+t}d'_{1,n+t} = 1$; a contradiction. This proves that $t = 1$ as we claimed.

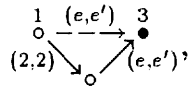
Since $m \ge 2$ then applying Proposition 3.9 to $i = 1$, $j = n$, $s = 1$, $t \ge 2$ we conclude as above that $d_{n,n+1} = d'_{n,n+1} = 1$ and by Lemma 3.7 we get

$$
\begin{aligned}
(d_{1n}, d'_{1n}) &= (d_{1,n+1}d'_{1,n+1}) \quad \text{and} \\
(d_{n+1,n+t}, d'_{n+1,n+t}) &= (d_{n,n+t}, d'_{n,n+t}).
\end{aligned}
$$

Since $\widetilde{\mathbb{G}}^2_2$ is not contained in $(\mathbf{I}_R, \mathbf{d})$, we have $(d_{n+1,n+t}, d'_{n+1,n+t}) \ne (1,1)$ and therefore $d_{n+1,n+t} \ne d'_{n+1,n+t}$, because otherwise $d_{n,n+t}d'_{n,n+t} \ge 4$;

a contradiction. It then follows as above that $d_{1,n+t} = d'_{1,n+t} = 1$ and applying Lemma 3.7 to $L' = \{1, n+1, n+t\}$ we conclude that $(d_{n+1,n+t}, d'_{n+1,n+t}) = (d'_{1,n+1}, d_{1,n+1})$. It follows from $(r'_1)$ and $(r'_2)$ that $m = 2$, because otherwise $d_{m+2,m+3} = d'_{m+2,m+3} = 1$ and $\widetilde{\mathbb{G}}_2^2$ is contained in $(\mathbf{I}_R, \mathbf{d})$. Furthermore, applying the arguments above to $(\mathbf{I}_R, \mathbf{d})^{\mathrm{op}}$ we conclude that $n = 2$ and therefore $(\mathbf{I}_R, \mathbf{d})$ is of the form (5.4) as we claimed.

It follows that $R$ has an arrow waist and applying Theorem 4.12 we can show as in Example 4.13 that

$(\overline{\mathbf{I}}_{\delta R}, \overline{\mathbf{d}})$ : 

, $ee' = 3$.

Now we conclude from [9, Theorem 6.10] that $\mathrm{adj}(R)^A_B \cong \mathrm{mod}_{\mathrm{sp}}(\delta R)$ is of finite representation type. Then (b) $\Rightarrow$ (a) is proved under the assumption above.

Next we suppose that $n, m \geq 2$, the poset $(\mathbf{I}_B, \mathbf{d})$ is linearly ordered and $d_{is}d'_{is} \leq 2$ for all $i \in \mathbf{I}_A$ and $s \in \mathbf{I}_B$. If $d_{is}d'_{is} \leq 1$ for all $i \in \mathbf{I}_A$, $s \in \mathbf{I}_B$ then in view of Lemma 3.5, $R$ is the path algebra $F\mathbf{I}_R$ of the poset $\mathbf{I}_R = \mathbf{I}_A \lhd \mathbf{I}_B$ over a division ring $F$, $(\mathbf{I}_B, \mathbf{d})$ is a homogeneous chain and Theorem 4.1 applies. It then follows that there is an equivalence of categories

$$\mathrm{mod}_{\mathrm{ic}}(R)_B \cong \mathrm{mod}_{\mathrm{sp}}(\Omega(R)) \cong T\text{-sp}$$

where $T = I_A \cup \{n+1 \longrightarrow n+2 \longrightarrow \cdots \longrightarrow n+m-1\}$ is a disjoint union of posets, that is, the elements $i$ and $n+s$ are incomparable in $T$ for all $i \in \mathbf{I}_A$ and $s = 1, \ldots, m-1$. Since $(\mathbf{I}_R, \mathbf{d})$ does not contain the critical forms of Table 1, the poset $T$ does not contain as a full subposet the critical posets $(1,1,1,1)$, $(2,2,2)$, $(1,3,3)$, $(N,4)$, $(1,2,5)$ of Kleiner [7] and therefore the category $T$-sp is of finite representation type (see [22, Theorem 10.1]). Consequently the category $\mathrm{adj}(R)^A_B$ is of finite representation type and (a) follows if $d_{is}d'_{is} \leq 1$ for all $i \in \mathbf{I}_A$, $s \in \mathbf{I}_B$.

It remains to prove the implication (a) $\Rightarrow$ (b) in the case there exist elements $i \in \mathbf{I}_A$ and $t \in \mathbf{I}_B$ such that $d_{it}d'_{it} = 2$ and the assumptions above are satisfied.

First we consider the case when the poset $(\mathbf{I}_A, \mathbf{d})$ is linearly ordered. If $d_{n,n+1}d'_{n,n+1} = 2$ then by Proposition 3.9 the poset $(\mathbf{I}_A, \mathbf{d})$ or $(\mathbf{I}_B, \mathbf{d})$ is a homogeneous chain. In view of the duality in Corollary 2.7 we can suppose without loss of generality that $(\mathbf{I}_B, \mathbf{d})$ is a homogeneous chain and the Theorem 4.1 applies to $R$. It follows that the categories $\mathrm{mod}_{\mathrm{ic}}(R)_B \cong \mathrm{mod}_{\mathrm{sp}}(\Omega(R))$ are of finite representation type, because (b) implies that the valued poset $(*)$ in Theorem 4.1 of the right peak ring $\Omega(R)$ does not contain any of the critical forms listed in [9, Theorem A]. Hence (a) follows, because the functor $\Theta_B$ in (2.8) preserves the finite representation type.

Now suppose that $d_{n,n+1} = d'_{n,n+1} = 1$. If one of the posets $(\mathbf{I}_A, \mathbf{d})$ or $(\mathbf{I}_B, \mathbf{d})$ is homogeneous, the statement (a) follows by the arguments applied above. Suppose that $d_{i,i+1}d'_{i,i+1} \neq 1$ and $d_{s,s+1}d'_{s,s+1} \neq 1$ for some $i < n$ and $s > n+1$. Let $i$ be a maximal element and let $s$ be a minimal element with the respect to this property. It follows from Lemma 3.7 that $d_{i,i+1}d'_{i,i+1} = d_{i,n+1}d'_{i,n+1} = 2$, $d_{s,s+1}d'_{s,s+1} = d_{n,s+1}d'_{n,n+1} = 2$, $d_{i,s+1} = d'_{i,s+1} = 1$ and $(d_{i,i+1}, d'_{i,i+1}) = (d'_{s,s+1}, d_{s,s+1})$. Moreover, it follows from Proposition 3.9 that $d_{j,j+1} = d'_{j,j+1} = 1$ for all $j \neq i, s$. Since $(\mathbf{I}_R, \mathbf{d})$ does not contain $\widetilde{\mathbb{F}}_4^3$ and $\widetilde{\mathbb{F}}_4^4$, a simple combinatorial analysis shows that one of the posets $(\mathbf{I}_R, \mathbf{d})$ or $(\mathbf{I}_R, \mathbf{d})^{\mathrm{op}}$ is a full bipartite valued subposet of one of the posets (i) – (v) listed in Lemma 5.1, and (a) follows.

Next consider the case when $(\mathbf{I}_A, \mathbf{d})$ is not linearly ordered. We can suppose that $(\mathbf{I}_B, \mathbf{d})$ is not homogeneous, because otherwise (a) follows from Theorem 4.1 and the arguments used above. Let $s \geq n+1$ be a minimal element such that $d_{s,s+1}d'_{s,s+1} \neq 1$. It follows from Lemma 3.7 that $d_{s,s+1} = d_{n+1,s+1}$, $d'_{s,s+1} = d'_{n+1,s+1}$ and since $r\omega(R) \leq 3$, there is no triple of incomparable elements in $\mathbf{I}_A$. Moreover, if $i, j \in \mathbf{I}_A$ are incomparable and $d_{i,n+1}d'_{i,n+1} \neq 1$ then $d_{j,n+1} = d'_{j,n+1} = 1$. Choose $i$ and $j$ maximal with respect to this property. It follows from Lemma 3.7 that $d_{i,n+1}d'_{i,n+1} = 2$, $(d_{s,s+1}d'_{s,s+1}) = (d'_{i,n+1}d_{i,n+1})$ and $d_{i,s+1} = d'_{i,s+1} = 1$.

Since $\widetilde{\mathbb{CD}}$ is not contained in $(\mathbf{I}_R, \mathbf{d})$, there is no $t \in \mathbf{I}_B$ such that $d_{t,t+1} = d'_{t,t+1} = 1$, because otherwise the set $\{i, j, t, t+1\}$ generates a bipartite valued subposet of $(\mathbf{I}_R, \mathbf{d})$ of type $\widetilde{\mathbb{CD}}$. It follows that $s = n+1$ and Proposition 3.9 yields $m = 2$. Moreover, Proposition 3.9 together with the discussion above implies that the full bipartite valued subposet of $(\mathbf{I}_A, \mathbf{d})$ consisting of all points $h$ such that $h \prec i$ is a homogeneous chain.

Suppose that $\mathbf{I}_A$ has a unique maximal element $n$. It follows from Lemma 3.7 and Proposition 3.9 that $d_{k,n+1} = d'_{k,n+1}$, if $i, j \prec k \prec n+1$. Since $(\mathbf{I}_R, \mathbf{d})$ does not contain the critical forms $\widetilde{\mathbb{F}}_4^5$ and $\widetilde{\mathbb{F}}_4^6$ then a simple analysis shows that $(\mathbf{I}_R, \mathbf{d})$ is a full bipartite valued subposet of the poset (vii) in Lemma 5.1 and (a) follows.

Finally we suppose that $\mathbf{I}_A$ has two maximal elements $n-1$ and $n$. Since $r\omega(R) \leq 3$ and $\widetilde{\mathbb{F}}_4^6, \widetilde{\mathbb{F}}_4^5$ are not contained in $(\mathbf{I}_R, \mathbf{d})$, a simple combinatorial analysis shows that $(\mathbf{I}_R, \mathbf{d})$ is a full bipartite valued subposet of the poset (vi) in Lemma 5.1 and (a) follows. This finishes the proof of (b) $\Longrightarrow$ (a).

(a) $\Longrightarrow$ (c) It follows from Proposition 3.1 and Corollary 3.8 that $(\mathbf{I}_R, \mathbf{d})$ is a symmetrizable bipartite valued poset. Then Lemma 3.7, [9, Theorem 3.5] and its dual form apply to $R$ and to the rings $R^{e_i}_{1_B}$, $R^{1_A}_{e_{n+t}}$, respectively. It follows from Theorem 3.10 that there exists a basic finite dimensional algebra $\overline{R}$ over an infinite field $K$ such that $C(\overline{R}) = C(R)$, $(\overline{\mathbf{I}}_R, \overline{\mathbf{d}}) = (\mathbf{I}_R, \mathbf{d})$ and $q_{\overline{R}} = q_R$. By the equivalence of (a) and (b), the category $\operatorname{adj}(\overline{R})^{\overline{A}}_{\overline{B}}$ is of finite representation type and therefore $\operatorname{prin}(\overline{R})^{\overline{A}}_{\overline{B}}$ is also of finite representation type, because the functor $\Theta_{\overline{B}}$ in (2.8) preserves finite representation type. Then, by [13, Proposition 4.2] and the discussion in [13, Section 4], the quadratic form $\chi_{\overline{R}} = q_{\overline{R}} = q_R$ is weakly positive and (c) follows.

(a) $\Longrightarrow$ (f) $\Longrightarrow$ (c) Apply the arguments used in [13, Section 4] and in the proof of [15, Theorem 3.11].

(c) $\Longrightarrow$ (b) Applying the arguments in the proof of Lemma 4.12 in [13] we show that $d_{is}d'_{is} \leq 3$ for all $i \in \mathbf{I}_A$ and $s \in \mathbf{I}_B$. Then Proposition 3.1 yields the first part of (b). In order to prove the second part it is sufficient to show that the quadratic form $q_R$ of $R$ is not weakly positive if $(\mathbf{I}_R, \mathbf{d})$ is one of the critical forms in Table 1 or one of the forms in Lemma 5.3. For the one peak valued posets apply [9, Theorem A]. The proof in remaining cases is an easy exercise and we leave it to the reader.

(a) $\Longrightarrow$ (d) Suppose that $\operatorname{adj}(R)^A_B$ is of finite representation type and $X$ is an indecomposable module in $\operatorname{adj}(R)^A_B$. Then (d) holds if and only if $\mathbf{cdn}(X)(j) \leq 6$ for all $j \in \mathbf{I}_R$. In view of Proposition 2.14(c) and Corollary 2.16, the problem reduces to the case when $R$ is adj-sincere and $X$ is sincere. Consequently, (d) follows from Theorem 1.11 proved below.

(d) $\Longrightarrow$ (e) It follows from (d) that $\operatorname{length}(X) \leq 6(n+m)\operatorname{length}(R_R)$ for all indecomposable modules $X$ in $\operatorname{adj}(R)^A_B$. This implies (e).

(e) $\Longrightarrow$ (b) If $R$ has the property (e) then by Proposition 2.14 the rings $\begin{pmatrix} A_i & {}_iM_s \\ 0 & B_s \end{pmatrix}$ also have the property (e) for all $i \in \mathbf{I}_A$, $s \in \mathbf{I}_B$ and we conclude from [9, Theorem A], [11, Theorem 2.7] and from [4] that $d_{is}d'_{is} \leq 3$ for all $i \in \mathbf{I}_A$ and $s \in \mathbf{I}_B$. Hence Proposition 3.1 applies. It follows that $R$ has the triangular form (1.3) and $(\mathbf{I}_R, \mathbf{d})$ is a bipartite valued poset. In view of Proposition 2.14 it is sufficient to show that if $(\mathbf{I}_R, \mathbf{d})$ is of one of the forms in Table 1 and in Lemma 5.3 then (e) does not hold in $\mathrm{adj}(R)^A_B$. If the valued poset $(\mathbf{I}_R, \mathbf{d})$ is of type $\widetilde{\mathbb{A}}^\bullet_\bullet$, the ring $R$ is hereditary and our result follows from [4] and [14, Corollary 3.4]. If $(\mathbf{I}_R, \mathbf{d})$ has a unique black vertex, our result follows from [9, Theorem A]. If $(\mathbf{I}_R, \mathbf{d})$ is of one of the types $\widetilde{\mathbb{F}}^5_4$, $\widetilde{\mathbb{F}}^6_4$ we reduce the problem to the right peak case as in the proof of Lemma 5.2 and our result follows from [9, Theorem A]. If $(\mathbf{I}_R, \mathbf{d})$ is one of the remaining types in Table 1, the black vertices of $\mathbf{I}_R$ form a homogeneous chain and according to Theorem 4.1 there is an equivalence of categories

$$\mathrm{mod}_{\mathrm{ic}}(R)_B \cong \mathrm{mod}_{\mathrm{sp}}(\Omega(R))$$

where $\Omega(R)$ is a schurian artinian right peak $PI$-ring. It follows from [9, Theorem A] that there exists a sequence

$$\mathbb{Y}: Y_1 \xrightarrow{h_1} Y_2 \longrightarrow \cdots \longrightarrow Y_t \xrightarrow{h_t} Y_{t+1} \longrightarrow \cdots$$

of proper monomorphisms between indecomposable modules in $\mathrm{mod}_{\mathrm{ic}}(R)_B$. It follows from the definition of the functor $\Theta_B$ that the sequence $\Theta_B(\mathbb{Y})$ contains a subsequence of proper monomorphisms between indecomposable modules in $\mathrm{adj}(R)^A_B$. This finishes the proof of (e) $\Longrightarrow$ (b). Since (a) $\Longleftrightarrow$ (a′) follows from (2.8) then the statements (a) – (f) of Theorem 1.6 are equivalent

As a consequence of the proof of (a) $\Longrightarrow$ (b) above we get the following.

**Corollary 5.5.** *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be an artinian piecewise peak PI-ring of the form* (1.3) *such that* $\mathrm{adj}(R)^A_B$ *is of finite representation type and* $|\mathbf{I}_A| \geq 2$, $|\mathbf{I}_B| \geq 2$. *Then*

(1) *Either* $(\mathbf{I}_R, \mathbf{d})$ *is a homogeneous poset,* $\mathbf{I}_R = \mathbf{I}_A \lhd \mathbf{I}_B$ *(see* 1.2), $R \cong F\mathbf{I}_R$ *for some division ring* $F$ *and one of the posets* $\mathbf{I}_A$, $\mathbf{I}_B$ *is linearly ordered; or* $(\mathbf{I}_R, \mathbf{d})$ *is not homogeneous and has one of the following forms up to duality:*

(a) $(\mathbf{I}_B, \mathbf{d})$ *is a homogeneous chain and* $r\omega(\mathbf{I}_A \cup \{n+1\}, \mathbf{d}) = 2$.

(b) $(\mathbf{I}_R, \mathbf{d})$ *is of the form* (5.4) *or of one of the forms* (i) – (v), (vii) *listed in Lemma* 5.1 *with* $d_{n,n+1} = d'_{n,n+1} = 1$.

(c) $(\mathbf{I}_R, \mathbf{d})$ *has no arrow waist and is a full bipartite valued subposet of the poset* (vi) *in Lemma* 5.1.

(2) *The properties of* $(\mathbf{I}_R, \mathbf{d})$ *listed in* (1) *together with the results in* [9, Sections 5-7] *completely determine the structure of the ring R.*

(3) *There are equivalences of the categories*

$$(5.6) \qquad \begin{aligned} &\delta : \operatorname{adj}(R)_B^A \longrightarrow \operatorname{mod}_{\mathrm{sp}}(\delta R), \\ &\widetilde{\operatorname{adj}}(R)_B^A \xrightarrow[\cong]{\overline{D}^\bullet} (\widetilde{\operatorname{adj}}(R^\bullet))^{\mathrm{op}} \xrightarrow{\delta^{\mathrm{op}}} (\widetilde{\operatorname{mod}}_{\mathrm{sp}}(\delta R^\bullet))^{\mathrm{op}} \end{aligned}$$

*in the case* (b) *and* (c), *respectively, and an equivalence*

$$\mathfrak{N}_+ : \operatorname{mod}_{\mathrm{ic}}(R)_B \longrightarrow \operatorname{mod}_{\mathrm{sp}}(\Omega(R))$$

*in remaining cases, where* $\delta R$, $\delta R^\bullet$ *and* $\Omega(R)$ *are* sp*-representation-finite schurian artinian right peak PI-rings and*

$$\widetilde{\operatorname{mod}}_{\mathrm{sp}}(\delta R^\bullet) = \operatorname{mod}_{\mathrm{sp}}(\delta R^\bullet)/[\delta E(1), \ldots, \delta E(n)]$$

**Corollary 5.7.** *If R is as in Corollary* 5.5 *then the Auslander-Reiten quivers* $\Gamma(\operatorname{adj}(R))$ *and* $\Gamma(\operatorname{prin}(R))$ *of* $\operatorname{adj}(R)_B^A$ *and of* $\operatorname{prin}(R)_B^A$ *have preprojective components which are equal to* $\Gamma(\operatorname{adj}(R))$ *and to* $\Gamma(\operatorname{prin}(R))$ *respectively, and can be constructed in the way presented in* [10, Section 3]. *Every indecomposable module in* $\operatorname{adj}(R)_B^A$ *(resp. in* $\operatorname{prin}(R)_B^A$, $\operatorname{mod}_{\mathrm{ic}}(R)_B$, $\operatorname{mod}^{\mathrm{pg}}(R)^A$) *is uniquely determined by its dimension vector as well as by its coordinate vector.*

*Proof.* If $|\mathbf{I}_A| = 1$ or $|\mathbf{I}_B| = 1$ then $R$ is a left peak ring or a right peak ring and the corollary follows from [9, Theorem A]. If $|\mathbf{I}_A| \geq 2$ or $|\mathbf{I}_B \geq 2$ then by Corollary 5.5 the problem for the category $\operatorname{adj}(R)_B^A$ reduces to the schurian right peak case and we are done. The remaining part of the corollary reduces to the above one by applying the adjustment functors $\Theta^A$ and $\Theta_B$ [13] in a similar way as in [22, Section 11.12].

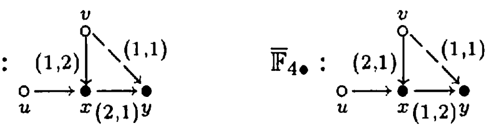
*Proof of Theorem* 1.11. The existence of Auslander-Reiten sequences in $\operatorname{mod}_{\mathrm{ic}}(R)_B$ can be proved by a slight modification of Auslander's arguments applied in [9, Proposition 6.3]. It then follows similarly as in [13, Section 3] that the categories $\operatorname{adj}(R)_B^A$ and $\operatorname{prin}(R)_B^A$ have Auslander-Reiten sequences.

If $n = 1$ or $m = 1$, $R$ is a left peak ring or a right peak ring, and in view of the duality in Proposition 2.3 our result is a consequence of [9, Theorem B]. Suppose that $n \geq 2$ and $m \geq 2$. If $R$ has an arrow waist then according to Theorem 4.2(c) $R$ is not adj-sincere. Hence, in view of Corollary 5.5, we easily conclude that $(\mathbf{I}_A, \mathbf{d})$ or $(\mathbf{I}_B, \mathbf{d})$ is homogeneous chain, or else $(\mathbf{I}_R, \mathbf{d})$ has the form (vi) in Lemma 5.1. Then our theorem is a consequence of [9, Theorem B] and the following result.

**Proposition 5.8.** *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be as in* Theorem 1.6 *and suppose that* $|\mathbf{I}_A| \geq 2$, $|\mathbf{I}_B| \geq 2$.

(a) *If* $(\mathbf{I}_B, \mathbf{d})$ *is a homogeneous chain then* $R$ *is* adj-*sincere if and only if* $(\mathbf{I}_B, \mathbf{d})$ *has one of the form in* Table 2 *except from* $\overline{\mathbb{F}}'_{4\bullet}$ *and* $\overline{\mathbb{F}}''_{4\bullet}$. *In this case the right peak ring* $\Omega(R)$ *in* Theorem 4.1 *is of one of the* sp-*sincere types listed in* [9, Theorem B] *and an indecomposable module* $X$ *in* $\mathrm{adj}(R)^A_B$ *is sincere if and only if the socle projective* $\Omega(R)$-*module* $\mathfrak{N}_+(X) = (X'_i, {}_i\varphi_j)$ *is of one of the* sp-*sincere forms in* [9, Appendix] *with* $X'_{n+m-1} \cong X'_{n+m}$ (*see* Corollary 4.4).

(b) *If* $R$ *has the form* (vi) *in* Lemma 5.1 *then* $R$ *is* adj-*sincere if and only if* $n = 2$ *and* $(\mathbf{I}_B, \mathbf{d})$ *has one of the forms*




*If this is the case the Auslander-Reiten quiver* $\Gamma(\mathrm{prin}(R)^A_B)$ *of* $\mathrm{prin}(R)^A_B$ *has one of the forms shown in* Figures 4 *and* 5, *if* $(\mathbf{I}_R, \mathbf{d})$ *is of type* $\overline{\mathbb{F}}'_{4\bullet}$ *and* $\overline{\mathbb{F}}''_{4\bullet}$, *respectively, where instead of the module* $X$ *we write* $\mathbf{cdn}(X) = (s_1, s_2, s_3, s_4)$ *in the exponential form* $(s_1, s_2, s_3, s_4) = u^{s_1} v^{s_2} x^{s_3} y^{s_4}$ (*see* [22, 11.88]), *where given a vertex* $\epsilon \in \{u, v, x, y\}$ *the power* $\epsilon^0$ *will be omitted and we put* $\epsilon^1 = \epsilon$. *The quiver* $\Gamma(\mathrm{adj}(R)^A_B)$ *is obtained from* $\Gamma(\mathrm{prin}(R)^A_B)$ *by removing the vertices* $u, v, x, y$.

**Figure 4.** *Auslander-Reiten valued quiver* $\Gamma(\mathrm{prin}(R)^A_B)$ *for* $R$ *of type* $\overline{\mathbb{F}}'_{4\bullet}$

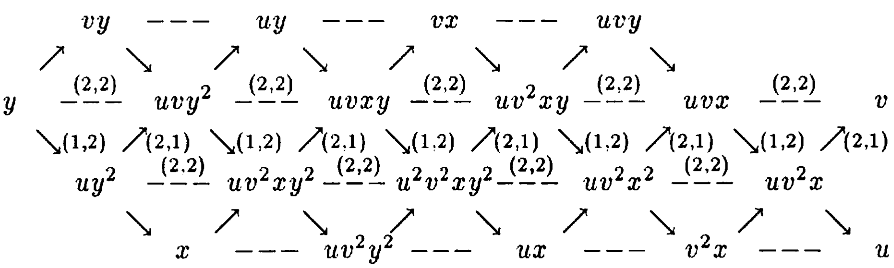


**Figure 5.** *Auslander-Reiten valued quiver* $\Gamma(\mathrm{prin}(R)_B^A)$ *for* $R$ *of type* $\mathbb{F}_4''$.

$$\begin{array}{ccccccccccc}
 & vy & --- & u^2y & --- & vx^2 & --- & u^2vy & & & \\
\nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow & & \\
y & \overset{(2,2)}{---} & u^2vy^2 & \overset{(2,2)}{---} & u^2vx^2y & \overset{(2,2)}{---} & u^2v^2x^2y & \overset{(2,2)}{---} & u^2vx^2 & \overset{(2,2)}{---} & v \\
\searrow^{(2,1)} & \nearrow^{(1,2)} & \searrow^{(2,1)} & \nearrow^{(1,2)} & \searrow^{(2,1)} & \nearrow^{(1,2)} & \searrow^{(2,1)} & \nearrow^{(1,2)} & \searrow^{(2,1)} & \nearrow^{(1,2)} & \\
uy & \overset{(2,2)}{---} & uvxy & \overset{(2,2)}{---} & u^2vxy & \overset{(2,2)}{---} & uvx^2 & \overset{(2,2)}{---} & uvx & & \\
\searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow \ \nearrow & & \searrow & & \\
x & --- & uvy & --- & ux & --- & vx & --- & u & &
\end{array}$$

*Proof.* (a) It follows from our assumption that any adjusted $R$-module is in $\mathrm{mod}_{\mathrm{ic}}(R)_B$. Then (a) follows immediately from Corollary 4.4.

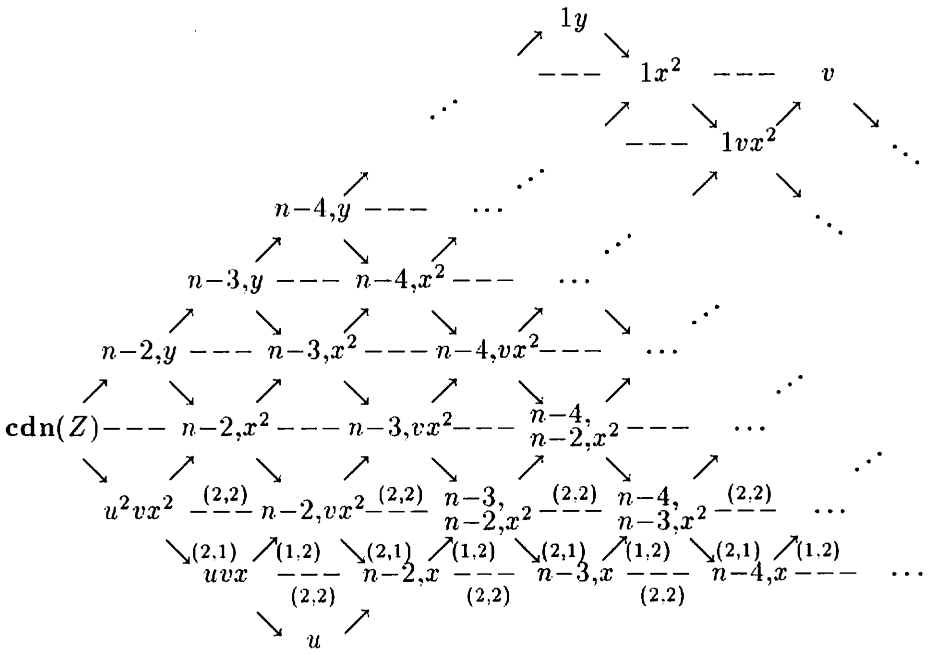
(b) Suppose that $(\mathbf{I}_R, \mathbf{d})$ is of the form (vi) in Lemma 5.1. It follows from [13, Section 3] and Corollary 5.7 that $\Gamma(\mathrm{prin}(R)_B^A)$ coincides with its preprojective component and therefore it can be described by a slight modification of the construction given in [10, Section 3] for socle projective modules over right peak rings. The construction will start with indecomposable relatively projectives $P$ in $\mathrm{prin}(R)_B^A$ with the property that $0 \to P$ is a sink map in $\mathrm{prin}(R)_B^A$. Here we shall use freely the terminology and notation introduced in [13]. Since $(\mathbf{I}_R, \mathbf{d})$ is of the form (vi) in Lemma 5.1, we easily conclude from [13, (2.5) and (2.6)] that $^\circ I_y = (0, E_B(y))$ is the unique relatively projective in $\mathrm{prin}(R)_B^A$ with the sink property above, where $y = n + 2$. Moreover the sink maps ending at indecomposable projectives in $\mathrm{prin}(R)_B^A$ are the following

$$\begin{array}{ccc}
 & & \widehat{P}_v \\
 & \nearrow & \\
{}^\circ I_y & & \\
 & \searrow^{(b,b')} & \\
 & & \widehat{P}_u \\
 & & \searrow \\
 & & {}^\circ I_x
\end{array}, \quad bb' = 2 \qquad Z \longrightarrow \widehat{P}_{n-2} \longrightarrow \cdots \longrightarrow \widehat{P}_2 \longrightarrow \widehat{P}_1$$

where $u = n - 1$, $v = n$, $x = n + 1$, $y = n + 2$, $\widehat{P}_j = (e_jA, E_B(e_jM_B), \overline{\mathrm{id}})$ and $Z$ is the unique indecomposable in $\mathrm{prin}(R)_B^A$ such that $\mathbf{cdn}(Z) = u^b vy$. It follows that $\Gamma(\mathrm{prin}(R)_B^A)$ begins with coordinate vectors of modules in the left hand sink map section above and since the function $\mathbf{cdn}(-)\colon \mathrm{prin}(R)_B^A \to \mathbb{Z}^{n+m}$ is additive on short exact sequences in $\mathrm{prin}(R)_B^A$ we construct $\Gamma(\mathrm{prin}(R)_B^A)$ in the forms shown in Figures 4 and 5 if $n = 2$ and $(b, b') = (1, 2)$ or $(b, b') = (2, 1)$, respectively.

If $n \geq 3$ the procedure will continue, because of the right sink map section starting from $Z$. It follows that in the case $(b, b') = (2, 1)$, the quiver $\Gamma(\mathrm{prin}(R)^A_B)$ is obtained from the quiver in Figure 5 by removing the vertex $v$, enlarging it by the quiver in Figure 6 and by the identification of the four point sections containing $\mathbf{cdn}(Z) = u^2vy$. Note that $\mathbf{cdn}(X)$ has at most four non-zero coordinates for any indecomposable module in $\mathrm{prin}(R)^A_B$. It follows that if $n \geq 3$ and $(b, b') = (2, 1)$ then $R$ is not adj-sincere because one can easily conclude from [13, Section 3] that $\Gamma(\mathrm{adj}(R)^A_B)$ is obtained from $\Gamma(\mathrm{prin}(R)^A_B)$ by omitting the vertices $u, v, x, y, 1, 2, \ldots, n-2$. In the case $(b, b') = (1, 2)$ the result above can be proved in a similar way. This finishes the proof.

**Figure 6.**




**Remark 5.9.** (a) There exists precisely 327 types of sincere adjusted modules over piecewice peak bipartite artinian PI-rings $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ of finite prinjective type and they can be described by applying Proposition 5.8 as follows.

i) There are precisely 155 forms of sincere adjusted modules for bipartite rings $R$ with $|\mathbf{I}_A| = 1$ or $|\mathbf{I}_B| = 1$.

For this purpose we note that if $|\mathbf{I}_B| = 1$ then $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is a right peak ring (that is $B$ is a division ring). According to [9, Theorem B] there are precisely 81 types of sp-sincere socle projective modules over right peak PI-rings and they are listed in Appendix of [9]. It is obvious that they are adjusted and sincere. Note that the ring $R$ of type $\mathcal{F}(0)^*$ has no sincere adjusted module.

Dually, if $|\mathbf{I}_A| = 1$ then $R$ is a left peak ring (i.e. $A$ is a division ring), there exists precisely 81 sincere types of adjusted modules over left peak PI-rings and they are described by the dual forms to those listed in Appendix of [9].

Since the bipartite poset $\mathcal{F}(1)^*\colon \circ \longrightarrow \bullet$ is self-dual then the incidence algebra of $\mathcal{F}(1)^*$ is a left peak algebra and a right peak algebra and therefore its unique sincere adjusted module $F \xrightarrow{\text{id}} F$ was counted twice, where $F = A = B$.

Similarly, the bipartite poset $\mathbb{E}_2'$ is dual to $\mathbb{C}_2'$, and $\mathbb{G}_2'$ is dual to $\mathbb{G}_2''$. Then the types $S_1$, $S_2$, $S_{15}$-$S_{18}$ (see [9, Appendix: Tables II]) of sincere adjusted modules over the bipartite rings corresponding to these posets were counted twice.

Consequently, there are precisely $155\,(= 81 + 81 - 7)$ forms of sincere adjusted modules over bipartite rings which are left peak rings or right peak rings.

ii) There are precisely 172 forms of sincere adjusted modules for bipartite rings $R$ with $|\mathbf{I}_A| \geq 2$ or $|\mathbf{I}_B| \geq 2$.

In this case we construct sincere adjusted modules according to the recipe given in Proposition 5.8. On this way we construct 86 sincere adjusted modules over the bipartite rings corresponding to the bipartite posets listed in Part B of Table 2 and 86 sincere adjusted modules over the bipartite rings corresponding to the dual of the bipartite posets listed in Part B of Table 2.

(b) Suppose that $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is as in Theorem 1.6 and $\text{adj}(R)_B^A$ is of finite representation type. Then all indecomposable modules in $\text{adj}(R)_B^A$ can be reconstructed from the 327 sincere adjusted forms described above via the idempotent induction functors $\mathcal{I}_\eta^e$ (see (2.13)) as follows.

Suppose that $X$ is an indecomposable module in $\text{adj}(R)_B^A$. Then by Corollary 2.16 there exist idempotents $e \in A$, $\eta \in B$, an adj-sincere piecewise peak bipartite PI-ring $T$ of one of the types listed in Theorem 1.6 and an indecomposable sincere $T$-module $Y$ in $\text{adj}(T)$ such that $T \cong R_\eta^e$ and the composed functor

$$\mathrm{adj}(T) \cong \mathrm{adj}(R_\eta^e) \xrightarrow{\overline{\overline{\mathcal{I}}}{}_\eta^e} \mathrm{adj}(R)_B^A$$

carries the module $Y$ to $X$. Here $\overline{\overline{\mathcal{I}}}{}_\eta^e$ is the idempotent induction functor induced by the functor (2.12). Note that $Y$ is one of the sincere modules in our collection of 327 sincere adjusted modules.

(c) It follows from our discussion in the introduction that if ${}_{\mathbb{K}}N_{\mathbb{L}}$ is a non-zero bimodule satisfying the conditions **(p1)** and **(p2)** of Introduction such that the matrix category $\mathrm{Mat}({}_{\mathbb{K}}N_{\mathbb{L}})$ is of finite representation type and contains a sincere object $W$, then the image $\Theta(W)$ of $W$ under the composed functor (0.1) is a sincere $\mathbf{R}_N$-module, the valued poset of $\mathbf{R}_N$ is, up to duality, of one of the forms shown in Table 2 and $\Theta(W)$ is isomorphic to one of the 327 sincere forms described above and in Proposition 5.8.

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**Note added in proof.** Since Corollary 1.9 was essentially used in Section 5 and was not proved in the paper we give here some arguments for the proof of its statements (a) and (b). The statement (c) of Corollary 1.9 follows from (a) and (b) by standard arguments (see Section 11.9 of [22]).

For this purpose we suppose that $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is an artinian bipartite piecewise peak PI-ring of finite prinjective module type.

If $R$ is an artin algebra the statement (a) (that is, the existence of Auslander-Reiten sequences in $\mathrm{adj}(R)_B^A$ and in $\mathrm{prin}(R)_B^A$) follows from the results of [13]. If $R$ is arbitrary we get the proof of (a) by a slight modification of the Auslander's arguments applied in Proposition 6.2 of [9].

According to Corollary 5.5 we split the proof of (b) into three parts.

(i) Assume that the bipartite valued poset $(\mathbf{I}_R, \mathbf{d})$ of $R$ has an arrow waist. By Theorem 4.12 there is an equivalence of categories $\mathrm{adj}(R)_B^A \cong \mathrm{mod}_{\mathrm{sp}}(\delta R)$, $\delta R$ is an artinian right peak sp-representation-finite PI-ring and according to Theorem 3.7 of [10] the statement (b) holds for $\mathrm{mod}_{\mathrm{sp}}(\delta R) \cong \mathrm{adj}(R)_B^A$. Moreover, it follows from the construction of Auslander-Reiten sequences given in Section 3 of [13] (see also Section 11.12 of [22]) that (b) holds for the category $\mathrm{prin}(R)_B^A$.

(ii) Assume that one of the valued posets $(\mathbf{I}_A, \mathbf{d})$ or $(\mathbf{I}_A, \mathbf{d})$ is a homogeneous chain. It follows from Theorem 4.1 that the bipartite artinian PI-ring $\Omega(R)$ (see 4.2) is a right peak ring and according to Theorem 4.1 the category $\mathrm{mod}_{\mathrm{ic}}(R)_B^A$ is equivalent with the category $\mathrm{mod}_{\mathrm{sp}}(\Omega(R))$ up to duality. Then by Theorem 3.7 of [10] the statement (b) holds for $\mathrm{mod}_{\mathrm{sp}}(\Omega(R)) \cong \mathrm{mod}_{\mathrm{ic}}(R)_B^A$ and again by the arguments used in Section 3 of [13] (see also Section 11.12 of [22]) the statement (b) holds also for the categories $\mathrm{prin}(R)_B^A$ and $\mathrm{adj}(R)_B^A$.

(iii) If $R$ is such that (i) or (ii) does not hold then according to Corollary 5.5 the bipartite poset $(\mathbf{I}_R, \mathbf{d})$ is a bipartite subposet of the poset (vi) shown in Lemma 5.1. For any such a ring $R$ a preprojective component in $\Gamma(\mathrm{prin}(R)_B^A)$ was explicitly constructed in the proof of Proposition 5.8. Analogous construction produces a preprojective component in $\Gamma(\mathrm{adj}(R)_B^A)$.