Math. J. Okayama Univ. 35(1993), 63-88

AN EXACT SEQUENCE FOR THE BRAUER GROUP OF DIMODULE AZUMAYA ALGEBRAS

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Let k be a commutative ring, H a finite commutative and cocommutative Hopf algebra over k, and let $\theta : H \to H^*$ be a Hopf algebra homomorphism of H into its linear dual H^*. There is a canonical embedding of the Brauer group $\mathrm{Br}(k)$ into the centre of the group $\mathrm{BD}(\theta, H)$ of θ-Azumaya algebras over k, which sends the class of an Azumaya k-algebra A to the class of A with trivial H-dimodule action. For $\theta = \varepsilon$, the counit of H, the embedding splits and a well-known result of Beattie [2] states that there is a split exact sequence

$$1 \to \mathrm{Br}(k) \to \mathrm{BD}(\varepsilon, H) \to E(H^*) \to 1$$

where $E(H^*)$ denotes the abelian group of isomorphism classes of H^*-Galois extensions of k, cf. Chase [8]. It is shown in this paper that Beattie's exact sequence is a special case of an exact sequence of the form

$$(1) \qquad 1 \to \mathrm{Br}(k) \to \mathrm{BD}(\theta, H) \xrightarrow{\pi_\theta} D(\theta, H^*)$$

where the latter is a (non-abelian) group of isomorphism classes of certain H^*-Galois biextensions of k. For $H = k[G]$, G a finite abelian group, such a sequence has been obtained by Childs, Garfinkel, and Orzech [11] and by Childs [10], and for $G = \mathbf{Z}/2\mathbf{Z}$ by Bass [1] and Small [18]. We shall investigate the structure of $D(\theta, H^*)$ and show that any element of this group is uniquely determined by a pair (S, v) with $S \in E(H^*)$ and $v \in \mathrm{Hopf}_k(H^*, H)$. We obtain that (S, v) admits a preimage under π_θ if it satisfies certain additional conditions. In particular, any (S, v) with $v = \varepsilon$ and S commutative belongs to the image of π_θ. This is consistent with recent results of Caenepeel [6].

Throughout this paper, k denotes a commutative ring, and H a finitely generated and projective Hopf algebra over k, which is assumed to be commutative and cocommutative; $H^* = \mathrm{Hom}_k(H, k)$ denotes the dual Hopf algebra. The antipode and counit of H (or H^*) are denoted by λ and ε, respectively. We fix an element $\theta \in \mathrm{Hopf}_k(H, H^*)$. Since $H^{**} \cong H$ we may regard the dual of any $\psi : H \to H^*$ again as a map $\psi^* : H \to H^*$. Let

A be an H-comodule (algebra) with structure map $\alpha: A \to A \otimes H$, $a \mapsto \Sigma a_{(0)} \otimes a_{(1)}$. We say that A is an (H', H)-dimodule (algebra) for another Hopf k-algebra H' if A is also an H'-module (algebra) such that

$$\alpha(h'a) = \Sigma h'a_{(0)} \otimes a_{(1)}, \quad h' \in H', \ a \in A.$$

For $H = H'$ this means that A is an H-dimodule (algebra) in the sense of Long [15]. Recall from [15] that an H-dimodule algebra A is said to be H-Azumaya over k if A is faithfully projective over k and if the k-linear maps $F, F': A \otimes A \to \mathrm{End}_k A$ defined by

$$F(a \otimes b)(x) = \Sigma a(b_{(1)}x)b_{(0)}, \text{ and } F'(a \otimes b)(x) = \Sigma(x_{(1)}a)x_{(0)}b,$$

for $a, b, x \in A$, are bijective.

0. In this section we recall some basic facts. Let P be an H-dimodule which is faithfully projective over k. Then $E(P) = \mathrm{End}_k P$ is an H-Azumaya algebra over k; the corresponding H^*-module action is given by

$$(2) \qquad (g \cdot f)(x) = \Sigma g_{(1)} f(\lambda(g_{(2)})x), \quad g \in H^*, \ f \in E(P), \ x \in P,$$

[15], Prop. 2.8. Let A and B be H-dimodule algebras. The smash product algebra $A \# B$ has underlying H-dimodule $A \otimes B$ and product defined by

$$(a \otimes b)\#(a' \otimes b') = \Sigma a(b_{(1)}a') \otimes b_{(0)}b', \quad a, a' \in A, \ b, b' \in B.$$

The opposite algebra $\bar{A}$ has underlying H-dimodule A and product

$$(3) \qquad a * b = \Sigma(a_{(1)}b)a_{(0)}, \quad a, b \in A.$$

If A and B are H-Azumaya, then so are $A \# B$ and $\bar{A}$. The maps F and F' defined above are homomorphisms of H-dimodule algebras

$$(4) \qquad F: A\#\bar{A} \to \mathrm{End}_k A, \quad \text{and} \quad F': \bar{A}\#A \to (\mathrm{End}_k A)^{\mathrm{op}}$$

Let A be an H-comodule (algebra) or, equivalently, an H^*-module (algebra). Then A is naturally an (H^*, H)-dimodule (algebra) such that $ga = \Sigma a_{(0)}\langle g, a_{(1)}\rangle$ for all $g \in H^*$, $a \in A$, and hence also an H-dimodule (algebra) with

$$ha = \Sigma a_{(0)}\langle \theta(h), a_{(1)}\rangle, \quad h \in H, \ a \in A.$$

We shall mainly consider dimodules and dimodule algebras of this type using then the prefix "$\theta-$" instead of "$H-$" (the corresponding notions in [17] are defined with θ^* instead of θ). Two θ-Azumaya algebras A and B over k are said to be equivalent if $A\#\mathrm{End}_k(P) \cong B\#\mathrm{End}_k(Q)$ for θ-dimodules P and Q which are faithfully projective over k. The equivalence classes of θ-Azumaya algebras over k form the group $\mathrm{BD}(\theta,\ H)$ with multiplication induced by the smash product, [17], [19]. This group can be regarded as a subgroup of the Brauer-Long group $\mathrm{BD}(H)$ of [15]. On the other hand, the latter group may be viewed as a special case of the first. For this let H be of the form $H = J \otimes J^*$, with J another finite (co-)commutative Hopf algebra over k. Let $\eta:\ J \otimes J^* \simeq (J \otimes J^*)^*$ be the canonical isomorphism and let $\theta:\ H \to H^*$ be defined by

$$(5)\quad \theta = \eta \circ \omega, \text{ with } \omega:\ J \otimes J^* \to J \otimes J^*,\ \omega(x \otimes \alpha) = x \otimes \varepsilon(\alpha),$$

for $x \in J$, and $\alpha \in J^*$. Then a k-algebra A is a J-dimodule algebra iff A is a θ-dimodule algebra, and $\mathrm{BD}(\theta,\ H) = \mathrm{BD}(J)$, see [6], I, Prop. 5.3 and [19], p. 35.

Lemma 0.1. *The convolution product $\theta * \theta^*$ is equal to η.*

Proof. Let $x\otimes\alpha,\ y\otimes\beta \in J\otimes J^*$. Clearly, $\eta = \eta^*$ because $\langle\eta(x\otimes\alpha),\ y\otimes \beta\rangle = \langle x,\ \beta\rangle\langle\alpha,\ y\rangle = \langle x\otimes\alpha,\ \eta(y\otimes\beta)\rangle$. Now $\langle\eta(x\otimes\alpha),\ \eta^{-1}\omega^*\eta(y\otimes\beta)\rangle = \langle\omega(x\otimes\alpha),\ \eta(y\otimes\beta)\rangle = \varepsilon(\alpha)\varepsilon(y)\langle x,\ \beta\rangle = \langle\eta(x\otimes\alpha),\ \varepsilon(y)\otimes\beta\rangle$, and so $\eta^{-1}\omega^*\eta = \varepsilon\otimes 1$. But $\omega = 1\otimes\varepsilon$, and therefore $\omega * (\eta^{-1}\omega^*\eta) = \mathrm{id}_H$. Thus $\eta = \eta(\omega * \eta^{-1}\omega^*\eta) = \theta * \omega^*\eta = \theta * \theta^*$.

We shall later write "$+$" for the convolution product and $\eta = \theta + \theta^*$ for arbitray $\theta \in \mathrm{Hopf}_k(H,\ H^*)$.

1. The aim of this section is to establish the exact sequence (1). We shall proceed mainly as in [10] and [11], employing results of [20]; π_θ will assign to an H-Galois representative the class of the centralizer of its fixed subalgebra.

Recall that an H-comodule algebra A is said to be an H-Galois extension of a k-algebra Z if $Z \cong A^H = \{a \in A|\ \Sigma a_{(0)} \otimes a_{(1)} = a \otimes 1\}$ and if

$$\gamma_A:\ A \otimes_Z A \to A \otimes H,\ \gamma_A(a \otimes b) = \Sigma a b_{(0)} \otimes b_{(1)},$$

is an isomorphism, see [9], [14], and [20].

Lemma 1.1. *Let A be a θ-Azumaya algebra over k. Then A is equivalent to a θ-Azumaya algebra B that is H-Galois (over B^H).*

Proof. First suppose $A = k$. Choose an H-Galois extension T of k and let $B = T\#H^*$. Then $B \to B \otimes H$, $t \otimes g \mapsto \Sigma(t_{(0)} \otimes g) \otimes t_{(1)}$, makes B a θ-dimodule algebra and B is H-Galois over H^*, cf. [5]. There is a canonical isomorphism $\Phi : T\#H^* \to \mathrm{End}_k T$ defined by $\Phi(t \otimes g)(x) = t(gx)$, and we claim that this is an isomorphism of θ-Azumaya algebras. It suffices to show that Φ is H^*-linear. But the H^*-action on $\mathrm{End}_k T$ is given by (2), and

$$\Phi(gt \otimes g')(x) = (gt)(g'x) = \Sigma g_{(1)}(t(g'\lambda(g_{(2)})x)) = (g \cdot \Phi(t \otimes g'))(x),$$

for g, $g' \in H^*$, and x, $t \in T$. The general case follows now from the next lemma.

Lemma 1.2. *Let A, B, and C be θ-Azumaya algebras over k and suppose that B is H-Galois. Then $A\#B$ and $B\#C$ are also H-Galois.*

Proof. Let $h \in H$ and let $\Sigma v_i \otimes w_i \in B \otimes_{B^H} B$ be the preimage of $1 \otimes h$ under γ_B. Then

$$\gamma_{A\#B}(\Sigma(1 \otimes v_i) \otimes (1 \otimes w_i)) = (1 \otimes 1) \otimes h, \tag{6}$$

so that $\gamma_{A\#B}$ is surjective. But this implies already that $A\#B$ is H-Galois according to [14]. Similarly for $B\#C$.

Let A be a θ-Azumaya algebra over k. We want to show that the centralizer S of $Z = A^H$ is an H^*-Galois extension of k in case A is H-Galois over Z. We first consider a certain subalgebra $\Gamma \subset A \otimes H^*$ which will turn out to be isomorphic with S if A is H-Galois.

Write $A^d = A\#\bar{A}$. We may consider A as a left A^d-module via F, (4), and define a left A^d-module action on $A \otimes H^*$ by

$$(a \otimes b) \cdot (c \otimes g) = \Sigma a(b_{(1)}c)b_{(0)} \otimes b_{(2)}g, \quad a,\ b,\ c \in A,\ g \in H^*,$$

where $hg = \Sigma g_{(1)}\langle g_{(2)},\ h\rangle$ for $h \in H$, $g \in H^*$. Define Γ to consist of all w in $A \otimes H^*$ satisfying $(x \otimes 1) \cdot w = (1 \otimes x) \cdot w$, $x \in A$. For $w = \Sigma b_i \otimes g_i$ this means

$$\Sigma x b_i \otimes g_i = \Sigma(x_{(1)}b_i)x_{(0)} \otimes x_{(2)}g_i, \quad x \in A. \tag{7}$$

We may view $A \otimes H^*$ as an H^*-Galois extension of A with H-action

$$h(a \otimes g) = a \otimes \lambda(h)g, \quad a \in A, \ h \in H, \ g \in H^*.$$

Then Γ is an H-submodule algebra of $A \otimes H^*$.

Lemma 1.3. *Γ is an H^*-Galois extension of k.*

Proof. Let $w \in \Gamma^{H^*} = \Gamma \cap A$, hence $w = b \otimes 1$ for some $b \in A$. By (7), $xb \otimes 1 = \Sigma(x_{(1)}b)x_{(0)} \otimes 1$ for all $x \in A$. But this yields $F'(1 \otimes b) = F'(b \otimes 1)$. Thus $1 \otimes b = b \otimes 1$, and so $b \in k$, cf. [13], p. 31. Next note that the bimodule ${}_{A^d}A_k$ is invertible (in the sense of [1], II. 3.2) since A is faithfully projective over k and $A^d \cong \mathrm{End}_k A$, [1], II. 5.6. Now

$$\Gamma \cong \mathrm{Hom}_{A^d}(A, \ A \otimes H^*) \cong \mathrm{Hom}_{A^d}(A, \ A^d) \otimes_{A^d} (A \otimes H^*),$$

cf. [16], Lemma 1.5, and it follows from Morita theory [1], II, Thm. 4.4, that $A \otimes \Gamma \to A \otimes H^*$, $a \otimes w \mapsto aw$, is an isomorphism. But this map can be used to transform the isomorphism $\gamma_{A \otimes H^*}$ into $1 \otimes \gamma_\Gamma : \ A \otimes \Gamma \otimes \Gamma \to A \otimes \Gamma \otimes H^*$ by a diagram similar to the one in [11], p. 310. It follows that γ_Γ is bijective.

In the following we suppose that A is H-Galois over $Z = A^H$. Let $S = A^Z = \{s \in A | sz = zs, \ z \in Z\}$. Then S is naturally an H^*-submodule algebra of A. Moreover, by [20], S is a (right) H-module satisfying

$$sa = \Sigma a_{(0)}(s \cdot a_{(1)}), \quad s \in S, \ a \in A, \tag{8}$$

or, equivalently, $as = \Sigma(s\lambda(a_{(1)}))a_{(0)}$. Let $h \in H$ and $\Sigma v_i \otimes w_i = \gamma_A^{-1}(1 \otimes h)$. Then

$$s \cdot h = \Sigma v_i w_{i(0)}(s \cdot w_{i(1)}) = \Sigma v_i s w_i, \quad s \in S. \tag{9}$$

This shows that $s \cdot h$ is uniquely determined by (8). Furthermore, we know by [20], Cor. 2.3, that $(gs)h = g(sh)$ for all $g \in H^*$, $s \in S$, and $h \in H$.

We now define a new H-module action on S by

$$s^h = \Sigma(h_{(1)}s)h_{(2)} = \Sigma h_{(1)}(sh_{(2)}), \quad s \in S, \ h \in H. \tag{10}$$

Lemma 1.4. *S consists of all elements $s \in A$ satisfying*

$$\Sigma x_{(0)}(s^{x_{(1)}}) = \Sigma(x_{(1)}s)x_{(0)}, \quad x \in A.$$

Proof. If $s \in A$ satisfies the above equation then $zs = sz$ for $z \in Z$ because $\Sigma z_{(0)} \otimes z_{(1)} = z \otimes 1$. Conversely, for $s \in S$ we obtain from (8): $\Sigma x_{(0)}(s^{x_{(1)}}) = \Sigma x_{(0)}((x_{(1)}s)x_{(2)}) = \Sigma((x_{(1)}s)x_{(2)}\lambda(x_{(3)}))x_{(0)} = \Sigma(x_{(1)}s)x_{(0)}$.

Next observe that (7) implies, when taking $x \in Z$, that $\Gamma \subset S \otimes H^{*\mathrm{in}}$. Hence there is an algebra map

(11) $$\Gamma \to S,\ \Sigma s_i \otimes g_i \mapsto \Sigma s_i \varepsilon(g_i).$$

Proposition 1.5. *Let A be a θ-Azumaya algebra over k and suppose that A is H-Galois over $Z = A^H$. Then the centralizer $S = A^Z$ is an H^*-Galois extension of k with respect to* (10)*; S is isomorphic with Γ via* (11).

Proof. Let $s \in S$ satisfy $s^h = \varepsilon(h)s$ for all $h \in H$. Then lemma 1.4 implies $xs = \Sigma x_{(0)}(s^{x_{(1)}}) = \Sigma(x_{(1)}s)x_{(0)}$ for $x \in A$. Therefore $F'(1 \otimes s) = F'(s \otimes 1)$, and so $s \in k$. Thus $S^{H^*} = k$. By [20], Satz 1.2, it is now enough to show that (11) is H-linear. Let $h \in H$ and let $\Sigma v_i \otimes w_i = \gamma_A^{-1}(1 \otimes h)$. Then

$$\Sigma 1 \otimes h_{(1)} \otimes h_{(2)} = \Sigma v_i w_{i(0)} \otimes w_{i(1)} \otimes w_{i(2)}, \quad \text{in} \quad A \otimes H \otimes H,$$

and therefore (9) yields for $s \in S$

(12) $$s^h = \Sigma(h_{(1)}s)h_{(2)} = \Sigma v_i w_{i(0)}((w_{i(2)}s)w_{i(1)}) = \Sigma v_i(w_{i(1)}s)w_{i(0)}.$$

But this implies for $\Sigma s_j \otimes g_j \in \Gamma$:

$$\Sigma s_j \otimes \lambda(h)g_j = \Sigma v_i w_{i(0)} s_j \otimes \lambda(w_{i(1)})g_j = \Sigma v_i(w_{i(1)}s_j)w_{i(0)} \otimes g_j = \Sigma {s_j}^h \otimes g_j$$

where the second equation holds by (7) with x replaced by $w_{i(0)}$.

Next we want to show that the k-algebra S admits another structure of H^*-Galois extension of k. For this we consider the θ-Azumaya algebra $\bar{A}$ with product "$*$" defined by (3); $\bar{A}$ contains the subalgebra Z^{op}, and we have $Z^{\mathrm{op}} = \bar{A}^H$ since $\bar{A} = A$ as H-comodules. We further assume that A is H-Galois over Z.

Lemma 1.6. *$\bar{A}$ is H-Galois over Z^{op}.*

Proof. Define $\beta : A \otimes H \to A \otimes H$ by $\beta(a \otimes h) = \Sigma a_{(0)} \otimes \theta(a_{(1)}\lambda(h_{(1)})) \cdot h_{(2)}$. Then β is an isomorphism, and it is easy to verify that $\gamma_{\bar{A}}(a \otimes b) = \beta\gamma'_A(b \otimes a)$ where $\gamma'_A : A \otimes_Z A \to A \otimes H$ is defined by $\gamma'_A(b \otimes a) = \Sigma b_{(0)}a \otimes b_{(1)}$. But γ'_A is bijective, [20], and hence so is $\gamma_{\bar{A}}$.

Let $\bar{S} = \bar{A}^{Z^{\mathrm{op}}}$. Then $S = \bar{S}$ as k-modules because $z*a = \Sigma(z_{(1)}a)z_{(0)} = az$, and $a * z = \Sigma(a_{(1)}z)a_{(0)} = \Sigma\langle\theta a_{(1)}, z_{(1)}\rangle z_{(0)}a_{(0)} = \Sigma\varepsilon(a_{(1)})za_{(0)} = za$ for $z \in Z$, $a \in A$. According to prop. 1.5, $\bar{S}$ is an H^*-Galois extension of k, and it is readily verified that the H-action on the algebra $\bar{S}$ is also measuring for the algebra structure of S. Hence, setting

$$ {}^h s = \bar{s}^{\lambda(h)}, \quad s \in S,\ h \in H, \tag{13} $$

gives S another structure of H^*-Galois extension of k, possibly different from that induced by A. The next lemma gives the relation between them.

Lemma 1.7. *For all $h \in H$ and $s \in S$*

$$ {}^h s = \Sigma s_{(0)}^{h_{(1)}}\langle\theta\lambda h_{(2)}, s_{(1)}\rangle\langle\theta s_{(2)}, \lambda h_{(3)}\rangle, \tag{14} $$

(*where* $s \mapsto \Sigma s_{(0)} \otimes s_{(1)}$ *under* $A \to A \otimes H$).

Proof. Let us write $s * h$ for the centralizer action of H on $\bar{S}$ so that (13) reads ${}^h s = \Sigma\lambda(h_{(1)})(s * \lambda(h_{(2)}))$. Consider the isomorphism

$$ \chi : H \to H,\ \chi(h) = \Sigma\theta(\lambda h_{(1)}) \cdot h_{(2)} = \Sigma h_{(1)}\langle\theta\lambda h_{(2)}, h_{(3)}\rangle $$

which has inverse $h \mapsto \Sigma\theta(h_{(1)}) \cdot h_{(2)}$. Fix $h \in H$ and let $\gamma_A(\Sigma v_i \otimes w_i) = 1 \otimes h$. By the proof of lemma 1.6 we have $\gamma_{\bar{A}}(\Sigma w_i \otimes v_i) = 1 \otimes \chi(\lambda h)$, since $\gamma'_A(\Sigma v_i \otimes w_i) = 1 \otimes \lambda(h)$, cf. [20], (1.1). To simplify the notation let us write $\Sigma v \otimes w$ instead of $\Sigma v_i \otimes w_i$. By definition of γ_A, $\Sigma v w_{(0)} \otimes w_{(1)} = 1 \otimes h$, and this implies easily

$$ \Sigma v_{(0)}w_{(0)} \otimes v_{(1)} \otimes w_{(1)} = \Sigma 1 \otimes \lambda(h_{(1)}) \otimes h_{(2)}. \tag{15} $$

Now $s * \chi(\lambda h) = \Sigma w * s * v = \Sigma((w_{(1)}s)w_{(0)}) * v$

$$= \Sigma(s_{(1)}w_{(1)})v((w_{(2)}s_{(0)})w_{(0)})$$
$$= \Sigma v_{(0)}s_{(0)}w_{(0)}\langle\theta(s_{(1)}w_{(1)}),\ v_{(1)}\rangle\langle\theta w_{(2)},\ s_{(2)}\rangle$$
$$= \Sigma v_{(0)}w_{(0)}(s_{(0)}w_{(1)})\langle\theta s_{(1)},\ v_{(1)}\rangle\langle\theta w_{(2)},\ v_{(2)}\rangle\langle\theta w_{(3)},\ s_{(2)}\rangle$$
$$= \Sigma(s_{(0)}h_{(1)})\langle\theta s_{(1)},\ \lambda h_{(2)}\rangle\langle\theta h_{(3)},\ \lambda h_{(4)}\rangle\langle\theta h_{(5)},\ s_{(2)}\rangle.$$

Replacing now h by $\chi^{-1}(h)$ gives $s * \lambda(h) = \Sigma s_{(0)}h_{(1)}\langle\theta s_{(1)},\ \lambda h_{(2)}\rangle\langle\theta h_{(3)},\ s_{(2)}\rangle$, from which the result follows.

When applying (14) to ${}^{h_{(1)}}s_{(0)}$ one obtains

$$s^h = \Sigma^{h_{(1)}}s_{(0)}\langle\theta h_{(2)},\ s_{(1)}\rangle\langle\theta s_{(2)},\ h_{(3)}\rangle, \quad s \in S,\ h \in H.$$

This implies further ${}^h(s^{h'}) = ({}^hs)^{h'}$ for $h,\ h' \in H$, and $s \in S$, and means that S is an H^*-Galois biextension of k in the sense of [5].

In the following we denote by $\mathcal{D}(H^*)$ the category of all H-dimodule algebras that are H^*-Galois biextensions of k such that the dimodule structure commutes with both of the Galois structures. A morphism in $\mathcal{D}(H^*)$ is an isomorphism of k-algebras which respects all the H-(co-)actions involved. Clearly, the above constructed S is an object of $\mathcal{D}(H^*)$ which we denote by $\pi(A)$. Moreover, $S = \pi(A)$ is a θ-dimodule algebra satisfying (14) and

$$\Sigma x_{(0)}(s^{x_{(1)}}) = \Sigma(x_{(1)}s)x_{(0)}, \quad x,\ s \in S, \tag{16}$$

by lemma 1.4. Given $S,\ T \in \mathcal{D}(H^*)$ we can define an object $S \wedge T \in \mathcal{D}(H^*)$ as follows. As an H-dimodule algebra, $S \wedge T \subset S\#T$ is the subalgebra of all $\Sigma s_i \otimes t_i \in S\#T$ satisfying

$$\Sigma s_i{}^h \otimes t_i = \Sigma s_i \otimes {}^ht_i, \quad h \in H. \tag{17}$$

The Galois structures on $S \wedge T$ are defined by

$${}^h(\Sigma s_i \otimes t_i) = \Sigma^h s_i \otimes t_i, \quad \text{and} \quad (\Sigma s_i \otimes t_i)^h = \Sigma s_i \otimes t_i{}^h, \quad h \in H.$$

Let $D(H^*)$ denote the set of isomorphism classes of objects of $\mathcal{D}(H^*)$.

Lemma 1.8. *The correspondence $(S,\ T) \mapsto S \wedge T$ induces a group structure on $D(H^*)$. The classes represented by θ-dimodule algebras satisfying* (14) *and* (16) *form a subgroup $D(\theta,\ H^*)$ of $D(H^*)$. The unit element* *is the class of*

$$I = H^* \text{ with trivial dimodule structure.}$$

The proof of this lemma will be given in section 3.

Lemma 1.9. *Let P be a θ-dimodule which is faithfully projective over k, and suppose that $E = \mathrm{End}_k P$ is H-Galois. Then $\pi(E \otimes B) \cong I$ for any Azumaya algebra B over k.*

Proof. Let us view P and E as H^*-modules. Hence $(g \cdot f)(x) = \Sigma g_{(1)} f(\lambda(g_{(2)})x)$ for $g \in H^*$, $f \in E$, and $x \in P$, (2). Consider

$$\rho : H^* \to \mathrm{End}_k P, \quad \rho_g(x) = gx, \quad g \in H^*,\ x \in P.$$

Write $Z = E^H$ and $S = E^Z$. Then $Z = \mathrm{End}_{H^*}(P)$, and so $\mathrm{Im}(\rho) \subset Z \cap S$ and

$$h\rho_g = \theta(h)\rho_g = \varepsilon(h)\rho_g, \quad h \in H,\ g \in H^*. \tag{18}$$

Concerning the centralizer action, we claim that

$$\rho_g \cdot h = \rho_{hg}, \quad h \in H,\ g \in H^*. \tag{19}$$

First observe that $\Sigma f_{(0)} \circ \rho_{f_{(1)}g} = \rho_g \circ f$ for any $f \in E$; indeed, for $x \in P$

$$\Sigma f_{(0)}(\rho_{f_{(1)}g}(x)) = \Sigma f_{(0)}(\langle f_{(1)}, g_{(1)}\rangle g_{(2)}x) = \Sigma(g_{(1)} \cdot f)(g_{(2)}x) = gf(x).$$

But this implies (19) since the centralizer action is uniquely determined by (8). Now (18) and (19) give $\rho_{hg} = \rho_g \cdot h = \Sigma(h_{(1)}\rho_g) \cdot h_{(2)} = \rho_g{}^h$. Hence $\rho : H^* \to S$ is an isomorphism of right H^*-Galois extensions; ρ is clearly a morphism of θ-dimodule algebras. Finally, we now know $S \subset Z$ so that $\Sigma s_{(0)} \otimes s_{(1)} = s \otimes 1$, and (14) yields ${}^h s = s^h$. Hence $S \cong I$.

Now let B be an Azumaya algebra over k and set $E' = E \otimes B$, $Z' = E'^H$, and $S' = E'^{Z'}$. Then $Z' = Z \otimes B$, and the image of $\rho' : H^* \to E'$, $g \mapsto \rho_g \otimes 1$, is contained in S'. We still have $\rho'(h \cdot g) = \rho'(g) \cdot h$ since $\gamma_{E'}(\Sigma(v_i \otimes 1) \otimes (w_i \otimes 1)) = 1 \otimes h$. It follows as before that ρ' induces an isomorphism $I \simeq S'$.

Theorem 1.10. *There exists an exact sequence of groups*

$$1 \to \mathrm{Br}(k) \to \mathrm{BD}(\theta, H^*) \xrightarrow{\pi_\theta} D(\theta, H^*) \tag{20}$$

where π_θ associates to an H-Galois representative A the isomorphism class of the centralizer $\pi(A) = A^Z$ of the fixed subalgebra $Z = A^H$.

Proof. Let A and B be θ-Azumaya algebras over k which are H-Galois and let $S = \pi(A)$ and $T = \pi(B)$. By lemma 1.2, $C = A\#B$ is also H-Galois and we set $U = \pi(C)$. We first show that the natural map

$$m:\ S\wedge T \to A\#B$$

induces an isomorphism $S\wedge T \simeq U$ in $\mathcal{D}(H^*)$. Let $c = \Sigma s_i\otimes t_i \in S\wedge T$. We claim that $m(c) \in U$, i.e. $m(c)z = zm(c)$ for any $z = \Sigma x_i\otimes y_i$ in $(A\#B)^H$. By abuse of notation, we simply write $z = x\otimes y$ and $c = s\otimes t$. Then $x\otimes y\otimes 1 = \Sigma x_{(0)}\otimes y_{(0)}\otimes x_{(1)}y_{(1)}$, and hence $\Sigma x_{(0)}\otimes y\otimes x_{(1)}\otimes \lambda(x_{(2)}) = \Sigma x_{(0)}\otimes y_{(0)}\otimes x_{(1)}\otimes y_{(1)}$. Now, using (8), for A and $\bar{A}$, and the last two equations, we obtain

$$\begin{aligned}
(s\otimes t)\#(x\otimes y) &= \Sigma s(t_{(1)}x)\otimes t_{(0)}y = \Sigma(t_{(1)}x_{(0)})(sx_{(1)})\otimes t_{(0)}y\\
&= \Sigma(t_{(1)}x_{(0)})(\lambda(x_{(1)})s^{x_{(2)}})\otimes t_{(0)}y = \Sigma(t_{(1)}x_{(0)})(\lambda(x_{(1)})s)\otimes({}^{x_{(2)}}t_{(0)})y\\
&= \Sigma(t_{(1)}x_{(0)})(\lambda(x_{(1)})s)\otimes(\lambda(x_{(2)})t_{(0)} * \lambda(x_{(3)}))y\\
&= \Sigma(t_{(1)}x_{(0)})(y_{(1)}s)\otimes(y_{(2)}t_{(0)} * y_{(3)})y_{(0)}\\
&= \Sigma(t_{(1)}x_{(0)})(y_{(1)}s)\otimes y_{(0)} * (t_{(0)} * y_{(2)}) = \Sigma(t_{(1)}x_{(0)})(y_{(1)}s)\otimes t_{(0)} * y_{(0)}\\
&= \Sigma\langle\theta t_{(1)},\ x_{(1)}\rangle x_{(0)}(y_{(1)}s)\otimes\langle\theta t_{(2)},\ y_{(2)}\rangle y_{(0)}t_{(0)}\\
&= \Sigma\langle\theta t_{(1)},\ x_{(1)}y_{(1)}\rangle x_{(0)}(y_{(2)}s)\otimes y_{(0)}t_{(0)}\\
&= \Sigma\langle\theta t_{(1)},\ 1\rangle x(y_{(1)}s)\otimes y_{(0)}t_{(0)} = \Sigma x(y_{(1)}s)\otimes y_{(0)}t = (x\otimes y)\#(s\otimes t).
\end{aligned}$$

Next we show $m(s\otimes t)^h = m(s\otimes t^h)$ for $h\in H$. Write $\gamma_B{}^{-1}(1\otimes h) = \Sigma v\otimes w$. Then we obtain from (6), (12), (8), and (15):

$$\begin{aligned}
m(s\otimes t)^h &= \Sigma(1\otimes v)\#(w_{(1)}(s\otimes t))\#(1\otimes w_{(0)})\\
&= \Sigma(1\otimes v)\#(w_{(1)}s\otimes w_{(2)}t)\#(1\otimes w_{(0)})\\
&= \Sigma v_{(1)}w_{(1)}s\otimes v_{(0)}(w_{(2)}t)w_{(0)}\\
&= \Sigma v_{(1)}w_{(1)}s\otimes v_{(0)}w_{(0)}((w_{(2)}t)w_{(3)})\\
&= \Sigma\lambda(h_{(1)})h_{(2)}s\otimes(h_{(3)}t)h_{(4)} = m(s\otimes t^h).
\end{aligned}$$

In a similar way one shows ${}^h m(s\otimes t) = m({}^h s\otimes t)$, which is left to the reader. It follows now that $m:\ S\wedge T\to U$ is an isomorphism in $\mathcal{D}(H^*)$.

Next suppose $A\sim B$, i.e. there exist faithfully projective θ-dimodules P and Q such that $A\#E(P)\cong A\#E(Q)$. Note that both sides are H-Galois by lemma 1.2. Let Y be a faithfully projective θ-dimodule such that

$E(Y)$ is H-Galois. Then $\pi(E(Y)) \cong I$ by lemma 1.9, and $\pi(A\#E(P)) \cong \pi(A\#E(P)) \wedge \pi(E(Y)) \cong \pi(A\#E(P))\#E(Y)) \cong \pi(A) \wedge \pi(E(P \otimes Y)) \cong \pi(A)$, cf. [15], Cor. 3.8. Similarly, $\pi(B\#E(Q)) \cong \pi(B)$, and hence $\pi(A) \cong \pi(B)$.

We have now shown that π_θ in (20) is a well-defined group homomorphism, and we know by lemma 1.9 that $\mathrm{Br}(k) \subset \mathrm{Ker}(\pi_\theta)$.

In the following suppose there exists an isomorphism $\rho : I \to A^Z$ in $\mathcal{D}(H^*)$. This means in particular that the H-coaction on A^Z is trivial so that $A^Z \subset Z$ and $\rho(h \cdot g) = \rho(g)^h = \Sigma(h_{(1)}\rho(g))h_{(2)} = \rho(g)h$. But then

$$ga = \Sigma\rho(g_{(1)})a\rho(\lambda g_{(2)}), \quad g \in H^*,\ a \in A. \tag{21}$$

In fact, $ga = \Sigma a_{(0)}\langle g, a_{(1)}\rangle = \Sigma a_{(0)}\langle g_{(1)}, a_{(1)}\rangle\rho(g_{(2)}\lambda g_{(3)}) = \Sigma a_{(0)}\rho((a_{(1)} \cdot g_{(1)})\lambda g_{(2)}) = \Sigma a_{(0)}(\rho(g_{(1)})a_{(1)})\rho(\lambda g_{(2)}) = \Sigma\rho(g_{(1)})a\rho(\lambda g_{(2)})$. Now consider the composite

$$A \otimes A \xrightarrow{\nu} A \otimes A \xrightarrow{F} \mathrm{End}_k A$$

with $\nu(a \otimes b) = \Sigma a\rho(\theta\lambda b_{(1)}) \otimes \rho(\theta b_{(2)})b_{(0)}$. Then (21) with $ga = \theta(b_{(2)})x = b_{(2)}x$ gives

$$F\nu(a \otimes b)(x) = \Sigma a\rho(\theta\lambda b_{(1)})(b_{(2)}x)\rho(\theta b_{(3)})b_{(0)} = axb, \quad x \in A.$$

But ν is bijective with $\nu^{-1}(a \otimes b) = \Sigma a\rho(\theta b_{(1)}) \otimes \rho(\theta\lambda b_{(2)})b_{(0)}$. It follows that A is an Azumaya k-algebra. Furthermore, it is not difficult to see that

$$A\#B \to A \otimes B, \quad a \otimes b \mapsto \Sigma a\rho(\theta b_{(1)}) \otimes b_{(0)},$$

is an isomorphism of H-comodule algebras for any θ-dimodule algebra B. It follows now that there is a well-defined homomorphism $\beta : \mathrm{Ker}(\pi_\theta) \to \mathrm{Br}(k)$, which maps the class of an H-Galois representative A to its class in $\mathrm{Br}(k)$. Clearly, $\beta\iota =$id for the embedding $\iota : \mathrm{Br}(k) \to \mathrm{Ker}(\pi_\theta)$ and it remains to show that β is injective; $\beta(A) = 0$ means there exists a k-algebra isomorphism $\tau : A \simeq \mathrm{End}_k P$ for a faithfully projective k-module P. Then

$$H^* \cong A^Z \to \mathrm{End}_k P, \quad g \mapsto \tau(\rho(g)),$$

makes P an H^*-module and hence $\mathrm{End}_k P$ a θ-dimodule algebra. But τ is then H^*-linear. In fact, we conclude from (21) and (2):

$$\tau(ga) = \tau(\Sigma\rho(g_{(1)})a\rho(\lambda g_{(2)})) = g \cdot \tau(a), \quad a \in A,\ g \in H^*.$$

Hence β is injective and this completes the proof of the theorem.

2. In the next two sections we want to work out the relationship between $D(\theta, H^*)$ and the abelian group $E(H^*)$. Let $\mathcal{E}(H^*)$ denote the category of H^*-Galois extensions of k. Recall that the group law on $E(H^*)$ is given by $[S]\cdot[T] = [S\cdot T]$ for $S,\ T \in \mathcal{E}(H^*)$ where

$$S\cdot T = \{\Sigma s_i \otimes t_i \in S\otimes T \mid \Sigma hs_i \otimes t_i = \Sigma s_i \otimes ht_i,\ h \in H\}$$

The following result from [22] will play an essential role in the rest of the paper.

Lemma 2.1. *Let $S \in \mathcal{E}(H^*)$, and suppose S is an $(H',\ H^*)$-dimodule algebra for another Hopf k-algebra H'.*

(i) *There exists a unique Hopf algebra homomorphism $v_S :\ H' \to H$ such that*

$$h's = v_S(h')s = \Sigma s_{(0)}\langle v_S(h'),\ s_{(1)}\rangle,\quad h' \in H,\ s \in\ S.$$

(ii) *v_S is bijective if and only if S is an H'^*-Galois extension of k.*

(iii) *Let $T \in \mathcal{E}(H^*)$ be another such algebra and suppose there exists an H'-linear isomorphism $S \to T$ in $\mathcal{E}(H^*)$. Then $v_S = v_T$.*

This has been proved in [22] for $H' = H^*$, but the same proof works for arbitrary H'.

In the sequel we shall write $R = \mathrm{Hopf}_k(H,\ H)$, $V = \mathrm{Hopf}_k(H^*,\ H)$, and $\mathrm{Aut}(H)$ will denote the group of Hopf algebra automorphisms of H; R is naturally an associative ring with addition the convolution product,

$$(a+b)(h) = \Sigma a(h_{(1)})b(h_{(2)}),\ \text{ and }\ ab = a\circ b,\quad a,\ b \in R,$$

and V is a left R-module. Note that $\varepsilon = 0$ and $-a = \lambda a$.

Consider now the category $\mathcal{D}'(H^*) = \mathcal{E}(H^*)\times \mathrm{Aut}(H)\times V\times R$ with objects denoted $\mathbf{S} = (S_0,\ u_S,\ v_S,\ a_S)$ and morphisms defined by

$$\mathrm{Hom}(\mathbf{S},\ \mathbf{T}) = \mathrm{Hom}(S_0,\ T_0),\ \text{if } u_S = u_T,\ v_S = v_T,\ \text{and } a_S = a_T,$$

and $\mathrm{Hom}(\mathbf{S},\ \mathbf{T}) = \phi$ otherwise. There is a functor

$$\mathcal{D}'(H^*) \to \mathcal{D}(H^*),\quad (S_0,\ u,\ v,\ a) \mapsto S, \tag{22}$$

where $S = S_0$ as k-algebras, viewed as an object of $\mathcal{D}(H^*)$ by

$$(23) \qquad s^h = hs,\ {}^h s = u(h)s,\ h \cdot s = a(h)s, \text{ and } g \cdot s = v^*(g)s,$$

for $s \in S_0$, $h \in H$, $g \in H^*$; the last two equations define the H-dimodule structure of S with H-coaction $s \mapsto \Sigma s_{(0)} \otimes v(s_{(1)})$.

Lemma 2.2. *The functor* (22) *is an isomorphism of categories.*

Proof. This is an immediate consequence of lemma 2.1.

In the following we shall identify $\mathcal{D}'(H^*)$ with $\mathcal{D}(H^*)$ so that

$$D(H^*) = E(H^*) \times \mathrm{Aut}(H) \times V \times R \quad \text{(as sets)}.$$

The notations $\Sigma s_{(0)} \otimes s_{(1)}$ and hs for $s \in S$ and $h \in H$ will from now on uniquely refer to the structure of S_0. We want to determine the coordinates of the object $S \wedge T$ introduced in section 1. For this we need some preparations.

Let $S \in \mathcal{E}(H^*)$, $\varphi \in V$, and $u \in \mathrm{Aut}(H)$. We define new objects $(S,\ \varphi)$ and S^u of $\mathcal{E}(H^*)$ as follows, cf. [22]. We let $(S,\ \varphi) = S$ as H-modules, and $S^u = S$ as k-algebras; $(S,\ \varphi)$ has multiplication

$$(24) \qquad s \cdot s' = \Sigma s_{(0)} s'_{(0)} \langle \varphi(s_{(1)}),\ s'_{(1)} \rangle, \quad s,\ s' \in S,$$

and S^u has H-action $h \cdot s = u(h)s$, hence H^*-coaction $s \mapsto \Sigma s_{(0)} \otimes u^*(s_{(1)})$. Then $(S,\ u) \mapsto S^u$ makes $E(H^*)$ a right $\mathrm{Aut}(H)$-module. Furthermore,

$$(S,\ \varphi) \cong S \cdot (H^*,\ \varphi),$$

by the proof of [22], Lemma 2, and

$$(25) \qquad i:\ V \to E(H^*),\ \varphi \mapsto [(H^*,\ \varphi)],$$

is a homomorphism of abelian groups.

Remark. Under the conditions of [3], sect. 3, the map i is the zero map as follows from [3], Lemma 3.1.

Lemma 2.3. *Let* $\varphi \in V$ *and* $u \in \mathrm{Aut}(H)$. *Then* $i(\varphi)^u = i(u^{-1}\varphi u^{*-1})$.

Proof. For any $S \in \mathcal{E}(H^*)$ we have $(S, \varphi)^u = (S^u, u^{-1}\varphi u^{*-1})$ as is readily verified. But for $S = H^*$, $u^* : S^u \to S$ is an isomorphism in $\mathcal{E}(H^*)$.

Lemma 2.4. *Let S, $T \in \mathcal{D}(H^*)$. Then $S \wedge T$ defined in section 1 is an object of $\mathcal{D}(H^*)$; it satisfies*

$$(26) \qquad (S \wedge T)_0 = (S_0^{u_T^{-1}}, u_T a_S v_T) \cdot T_0, \quad \textit{in } \mathcal{E}(H^*),$$

$$(27) \quad u_{S\wedge T} = u_T u_S, \; v_{S\wedge T} = v_T + v_S u_T^*, \; \textit{and} \; a_{S\wedge T} = a_T + u_T a_S,$$

where "+" means convolution product.

Proof. It is easy to see that both sides of (26) coincide as H-modules. Let (by abuse of notation) $s \otimes t$, $s' \otimes t' \in S \wedge T$. Then (17) yields $\Sigma s \otimes t_{(0)} \otimes t_{(1)} = \Sigma s_{(0)} \otimes t \otimes u_T^{*-1}(s_{(1)})$, and therefore

$$\begin{aligned}(s \otimes t)\#(s' \otimes t') &= \Sigma s(a_S v_T(t_{(1)})s') \otimes t_{(0)}t' \\ &= \Sigma s_{(0)}(a_S v_T u_T^{*-1}(s_{(1)})s') \otimes tt' \\ &= \Sigma s_{(0)}s'_{(0)}\langle u_T a_S v_T u_T^{*-1}(s_{(1)}), u_T^{*-1}(s'_{(1)})\rangle \otimes tt'.\end{aligned}$$

This shows that both sides of (26) coincide also as k-algebras. The formulae (27) are easier and left to the reader.

That (26) and (27) define a group law on $D(H^*)$ is a special case of a more general result that will be given in section 3. We first need two more lemmas.

Let $S \in \mathcal{E}(H^*)$. Then $S^{H^*} = k$ and the centralizer action (9) is defined on all of S. According to lemma 2.1 it is given by a $\mu_S \in V = \mathrm{Hopf}_k(H^*, H)$, uniquely determined by

$$(28) \qquad sx = \Sigma x_{(0)}(\mu_S(x_{(1)})s), \quad s, x \in S,$$

or, equivalently, $xs = \Sigma(\mu_S(\lambda x_{(1)})s)x_{(0)}$. Clearly, S is commutative iff $\mu_S = \varepsilon$.

Lemma 2.5. *Let S, $T \in \mathcal{E}(H^*)$, $\varphi \in V$, and $u \in \mathrm{Aut}(H)$. Then*

$$\begin{aligned}\mu_{S^*} &= -\mu_S, & \mu_{S\cdot T} &= \mu_S + \mu_T, \\ \mu_{S^u} &= u^{-1}\mu_S u^{*-1}, & \mu_{(S,\varphi)} &= \mu_S + \varphi^* - \varphi,\end{aligned}$$

where "+" means convolution product.

Proof. To show the first equation write $\mu = \mu_S$, and let $x,\ s \in S$. Then $xs = \Sigma s_{(0)}(\mu(s_{(1)})x) = \Sigma s_{(0)}x_{(0)}\langle s_{(1)},\ \mu^*(x_{(1)})\rangle = \Sigma(\mu^*(x_{(1)})s)x_{(0)}$. Therefore, by the uniqueness of the centralizer action, $\mu^* = \mu\lambda$. The other formulae can be proved in the same way.

Lemma 2.6. *Let $S = (S_0,\ u,\ v,\ a) \in \mathcal{D}(H^*)$, and let $\mu = \mu_{S_0}$. The isomorphism class of S belongs to $D(\theta,\ H^*)$ if and only if*

$$a = v^*\theta,\ u = 1 - v^*(\theta + \theta^*),\ \text{and}\ \ \mu = v - v^*\theta v. \tag{29}$$

Proof. Clearly, $a = v^*\theta$ means that S is a θ-dimodule, and it is not difficult to see that the other equations are equivalent to (14) and (16), respectively.

Apparently the product defined by (26) and (27) corresponds to a kind of enlarged crossed product. In order to make this more transparent we consider in the next section a more general situation based on a ring Λ with an involution $^*: \Lambda \to \Lambda$. In the present case Λ is the matrix ring

$$\Lambda = \Lambda(H) = \begin{pmatrix} R & V \\ W & R^* \end{pmatrix} \cong \mathrm{Hopf}_k(H \otimes H^*,\ H \otimes H^*) \tag{30}$$

with $W = \mathrm{Hopf}_k(H,\ H^*)$, and $R^* = \mathrm{Hopf}_k(H^*, H^*)$. It admits the natural involution

$$\Lambda \to \Lambda,\ \begin{pmatrix} a & v \\ w & b \end{pmatrix} \mapsto \begin{pmatrix} b^* & v^* \\ w^* & a^* \end{pmatrix}. \tag{31}$$

3. Let Λ be an associative ring with a non-zero idempotent e and supplied with an involution $\Lambda \to \Lambda$, $a \mapsto a^*$. We set

$$R = e\Lambda e,\ V = e\Lambda e^*,\ \text{and}\ \ W = e^*\Lambda e.$$

Then V is a left R-module and clearly $v^* \in V$ for any $v \in V$. Furthermore, if u is a unit of R, then also

$$vu^* = v^{**}u^* = (uv^*)^* \in V, \quad \text{for } v \in V.$$

In the following we fix a subgroup U of the group of units of R, a right U-module E, and an additive map $i: V \to E$ satisfying

$$i(v)u^{-1} = i(uvu^*),\quad v \in V,\ u \in U. \tag{32}$$

Let $D = E \times U \times V \times R$ with elements written in the forms $s = (x_s, u_s, v_s, a_s)$.

Lemma 3.1. *For $s, t \in D$ define $s \cdot t \in D$ by*

$$s \cdot t = (x_s u_t{}^{-1} + x_t + i(u_t a_s v_t),\ u_t u_s,\ v_t + v_s u_t{}^*,\ a_t + u_t a_s).$$

Then D is a group, with unit element $(0, e, 0, 0)$ and

$$s^{-1} = (i(av)u - xu,\ u^{-1},\ -vu^{*-1},\ -u^{-1}a),\ \textit{for } s = (x, u, v, a).$$

Proof. Let us show associativity for the x-coordinate. Let $r, s, t \in D$. Then

$$\begin{aligned} x_{r(st)} &= x_r(u_t u_s)^{-1} + x_{st} + i(u_t u_s a_r(v_t + v_s u_t{}^*)) \\ &= x_r u_s{}^{-1} u_t{}^{-1} + i(u_t u_s a_r v_t) + i(u_s a_r v_s) u_t{}^{-1} + x_{st} \\ &= (x_r u_s{}^{-1} + i(u_s a_r v_s) + x_s) u_t{}^{-1} + i(u_t(a_s + u_s a_r) v_t) + x_t \\ &= x_{rs} u_t{}^{-1} + i(u_t a_{rs} v_t) + x_t = x_{(rs)t}. \end{aligned}$$

The rest of the proof is left to the reader.

In the following we fix an element $\theta \in W = e^*\Lambda e$ and set $\eta = \theta + \theta^*$. Note that $\theta^* \in W$. Clearly, W is a right R-module and $V \cdot W \subset R$. Suppose we are given an additive map $\mu : E \to V$ satisfying

(33) $$\mu(xu^{-1}) = u\mu(x)u^*, \text{ and } \mu(i(v)) = v^* - v.$$

for all $x \in E$, $u \in U$, and $v \in V$. We shall say that μ is skew if

(34) $$\mu(x)^* = -\mu(x), \quad x \in E.$$

Proposition 3.2. *Let D_θ denote the subset of all $s = (x, u, v, a) \in D$ satisfying*

(35) $$u = e - v^*\eta,\ \mu(x) = v - av, \quad \textit{and} \quad a = v^*\theta,$$

where $\eta = \theta + \theta^*$. Then D_θ is a subgroup of D. For any $s \in D_\theta$

(36) $$\mu(x) = uv + v^*\theta^*v.$$

If μ is skew, then also

$$v^* = -uv,\ u^{-1} = e - v\eta, \quad \textit{and} \quad u^*\eta u = \eta. \tag{37}$$

Proof. We omit the routine proof that D_θ is a subgroup. Let $s \in D_\theta$. Then

$$uv + v^*\theta^*v = v - v^*(\theta + \theta^*)v + v^*\theta^*v = v - v^*\theta v = \mu(x),$$

which shows (36). Now suppose (34). Then $-\mu(x)^* = -v^* + v^*\theta^*v = \mu(x) = uv + v^*\theta^*v$. Thus $uv = -v^*$. Now $v = -u^{-1}v^*$ yields

$$e = u^{-1}u = u^{-1}(e - v^*\eta) = u^{-1} - u^{-1}v^*\eta = u^{-1} + v\eta,$$

that is $u^{-1} = e - v\eta$. Finally, since $u^* = e^* - \eta^*v$ and $\eta^* = \eta$, we have

$$u^*\eta = (e^* - \eta v)\eta = \eta - \eta v\eta = \eta(e - v\eta) = \eta u^{-1}.$$

Proof of Lemma 1.8. Let $E = E(H^*)$, and $(\Lambda,\ ^*)$ be as in (30) and (31). Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $R = e\Lambda e$, $V = e\Lambda e^*$, and $W = e^*\Lambda e$. Define $i : V \to E$ and $\mu : E \to V$ as in section 2. By lemma 2.3 and 2.5, i and μ have properties (32) and (33), (and μ is skew). But $D(H^*) = D$, and $D(\theta,\ H^*) = D_\theta$.

Note that any $s = (x,\ u,\ v,\ a) \in D_\theta$ is uniquely determined by x and v. Let us say that η is $(e,\ e^*)$-invertible if there exists $\eta' \in V$ such that

$$\eta'\eta = e, \quad \text{and} \quad \eta\eta' = e^*. \tag{38}$$

Then η' is unique, and $\eta'^* = \eta'$ since $\eta^* = \eta$. (For $\Lambda = \Lambda(H)$ as above this simply means that $\eta :\ H \to H^*$ is bijective and $\eta' = \eta^{-1}$). If η is $(e,\ e^*)$-invertible, then $s \in D_\theta$ is uniquely determined by $(x,\ u)$ because $v^*\eta = e - u$ yields $v^* = v^*e^* = v^*\eta\eta' = (e - u)\eta'$ and hence $v = \eta'(e^* - u^*)$.

Next we want to prove a result which is similar to [10], Thm. 3.4, and gives some insight into the structure of D_θ in case μ is skew and 2 is a unit of R. Note that if $R = \mathrm{Hopf}_k(H,\ H)$ with $H = k[G]$, G a finite abelian group, then $\mathrm{char}(R) = \exp(G)$.

Let $U_\eta = \{u \in U \mid u^*\eta u = \eta\}$ and assume that μ is skew. Define D_θ^1 by the exact sequence

$$1 \to D_\theta^1 \longrightarrow D_\theta \xrightarrow{\kappa} U_\eta^{\mathrm{op}} \tag{39}$$

with $\kappa(x, u, v) = u$. Let $E_c = \mathrm{Ker}(\mu)$, and $V_\eta = \{v \in V \mid v\eta = 0$, and $v^* = -v\}$. Then, by (37), we have also an exact sequence

$$1 \to E_c \xrightarrow{\tau} D_\theta^1 \xrightarrow{\rho} V_\eta \tag{40}$$

where $\tau(x) = (x, \epsilon, 0)$, and $\rho(x, e, v) = v$.

Proposition 3.3. *Assume that μ is skew and that 2 is a unit of R. Then*

(i) *ρ is surjective,*

(ii) *κ is surjective if η is (e, e^*)-invertible.*

Proof. (i) Let $v \in V_\eta$ and set $m = v - v^*\theta v$. Then $m + m^* = v - v^*\theta v - v - v^*\theta^* v = -v^*\eta v = v\eta v = 0$. Let $x = -i(2^{-1}m)$. Then $\mu(x) = (-2^{-1}m)^* + 2^{-1}m = 2^{-1}(m+m) = m$. Clearly, $e - v^*\eta = e + v\eta = e$, so that $(x, e, v) \in D_\theta^1$.

(ii) Let $u \in U_\eta$. Define $v = (e - u)\eta'$ with η' as in (38), and let m and x be as in (i). It is not difficult to see that $m + m^* = 0$ so that again $\mu(x) = m$. Furthermore,

$$e - v^*\eta = e - \eta'(e - u)^*\eta = e - (\eta' - \eta' u^*)\eta = e - (\eta' - u^{-1}\eta')\eta = u^{-1}.$$

Thus (x, u^{-1}, v) is an element of D_θ.

Suppose in the following that η is (e, e^*)-invertible with inverse η', (38). Then $V \to R$, $v \mapsto v\eta$, and $R \to V$, $a \mapsto a\eta'$, are inverse to each other, and we obtain an involution $R \to R$, $a \mapsto a^{(*)} = \eta' a^* \eta$. Put $\omega = \eta'\theta \in R$. Then

$$\omega^{(*)} = \eta'\theta^*, \quad \text{and} \quad \omega + \omega^{(*)} = 1_R = e.$$

Lemma 3.4. *Let $u \in U_\eta$ such that $\omega u = u\omega^{(*)}$ and $u\omega = \omega^{(*)}u$. Set*

$$x = i(u^{-1}\omega\eta'), \; v = (e - u^{-1})\eta', \; \textit{and} \; a = (e - u)\omega. \tag{41}$$

Then $(x,\ u,\ v,\ a)$ *is an element of* D_θ.

Proof. By assumption, $u^{*-1}\eta = \eta u$, and (33) yields $\mu(x)\eta = (u^{-1}\omega\eta')^*\eta - u^{-1}\omega = \omega^{(*)}u - u^{-1}\omega$. On the other hand

$$\begin{aligned}(e-a)v\eta &= (e-(e-u)\omega)(e-u^{-1}) = e - \omega + u\omega - u^{-1} + \omega u^{-1} - u\omega u^{-1} \\ &= \omega^{(*)} + \omega^{(*)}u - (e-\omega)u^{-1} - \omega^{(*)}uu^{-1} = \omega^{(*)}u - \omega^{(*)}u^{-1} \\ &= \omega^{(*)}u - u^{-1}\omega.\end{aligned}$$

Thus $\mu(x) = v - av$. Also, $v^* = \eta'(e - u^{*-1}) = \eta' - \eta' u^{*-1} = \eta' - u\eta' = (e-u)\eta'$. Therefore $v^*\theta = (e-u)\omega = a$. Finally, $e - v^*\eta = e - (e-u) = u$.

4. Let $(S,\ u,\ v) \in D(\theta,\ H^*)$. In contrary to section 1, the notations $\Sigma s_{(0)} \otimes s_{(1)}$ and hs for $s \in S$ and $h \in H$ will always refer to the right Galois structure of $S = S_0$, hence $hs = s^h$. By (23), the H-dimodule structure on S is then given by $s \mapsto \Sigma s_{(0)} \otimes v(s_{(1)})$ and $h \cdot s = v^*(\theta h)s$. Let $\mu = \mu_S$. In the following we seek to construct a preimage of $(S,\ u,\ v)$ under π_θ.

Consider the subalgebra $K = \{x \in S |\ x^{\sigma(h)} = \varepsilon(h)x,\ h \in H\}$ with

$$\sigma : \ H \to H,\ \sigma(h) = \Sigma h_{(1)} v^*\theta(\lambda h_{(2)}), \tag{42}$$

that is, $\sigma = 1 - v^*\theta$. Then $\mu = \sigma \circ v$ by (29), and (28) implies that K is contained in the centre of S. We assume in the following that K is also contained in the centre of some H-Galois extension T of k and that $\alpha_T : \ T \to T \otimes H$ satisfies

$$\alpha_T(x) = \Sigma x_{(0)} \otimes v(\lambda x_{(1)}), \quad x \in K. \tag{43}$$

We can then define on $S \otimes_K T$ an H-coaction by

$$s \otimes t \mapsto \Sigma(s_{(0)} \otimes t_{(0)}) \otimes v(s_{(1)})\lambda(t_{(1)}), \tag{44}$$

and a product by $(s \otimes t)\#(s' \otimes t') = \Sigma s(\sigma(t_{(1)})s') \otimes t_{(0)}t$. It is easy to see that this makes $S \otimes_K T$ an H-comodule algebra, and hence a θ-dimodule algebra, which we denote by $A = S\#_K T$.

Theorem 4.1. *Suppose that the dimodule algebra* $A = S\#_K T$ *defined above is* θ*-Azumaya over* k. *Then* $\pi_\theta(A) = (S,\ u,\ v)$.

Proof. By definition, we have a homomorphism of θ-dimodule algebras

$$n: \; S \to S\#_K T, \; x \mapsto x \otimes 1.$$

Let $Z = A^H$. Then $n(S) \subset A^Z$. For let $s \otimes t \in Z$, that is, $s \otimes t$ is sent to $s \otimes t \otimes 1$ under (44). Then, since $\mu = \sigma v$, we conclude from (28)

$$\begin{aligned}(s \otimes t)\#(x \otimes 1) &= \Sigma s(\sigma(t_{(1)})x) \otimes t_{(0)} \\ &= \Sigma(\sigma v(\lambda s_{(1)})\sigma(t_{(1)})x)s_{(0)} \otimes t_{(0)} \\ &= \Sigma(\sigma\lambda(v(s_{(1)})\lambda(t_{(1)})x)s_{(0)} \otimes t_{(0)} \\ &= xs \otimes t = (x \otimes 1)\#(s \otimes t).\end{aligned}$$

Next let $h \in H$ and $\gamma_T{}^{-1}(1 \otimes \lambda h) = \Sigma v_i \otimes w_i$. Then $\gamma_A(\Sigma(1 \otimes v_i) \otimes (1 \otimes w_i)) = 1 \otimes h$, which shows in particular that A is H-Galois over Z. Furthermore, by (9),

$$n(x)h = \Sigma(1 \otimes v_i)\#(x \otimes 1)\#(1 \otimes w_i) = \Sigma\sigma(v_{i(1)})x \otimes v_{i(0)}w_i = \sigma(h)x \otimes 1.$$

From this we obtain $n(x)^h = n(x^h)$ since

$$\begin{aligned}n(x)^h &= \Sigma(h_{(1)}n(x))h_{(2)} = \Sigma(v^*\theta(h_{(1)})x \otimes 1)h_{(2)} \\ &= \Sigma v^*\theta(h_{(1)})\sigma(h_{(2)})x \otimes 1 = \Sigma v^*\theta(h_{(1)})v^*\theta(\lambda h_{(2)})h_{(3)}x \otimes 1 \\ &= hx \otimes 1.\end{aligned}$$

We now know that $S \to A^Z$, $x \mapsto x \otimes 1$, is an isomorphism of (right) H^*-Galois extensions of k. Moreover, both admit the same v, hence also the same u since u is determined by v.

In the following we shall give several applications of theorem 4.1. That A is θ-Azumaya will follow in each case from the next lemma.

Let A be an arbitrary (C^*, H)-dimodule algebra over k where C is another finite (co-)commutative Hopf algebra over k. Suppose that A is C-Galois over k, so that the H-coaction on A is defined by an element $w \in \mathrm{Hopf}_k(C, H)$.

Lemma 4.2. *Define* $w': C \to C^*$ *by* $w'(c) = \Sigma\mu_A(c_{(1)})w^*\theta w(c_{(2)})$. *Then A is θ-Azumaya over k if and only if w' is bijective.*

Proof. Consider the isomorphisms $\Phi, \Phi': A \otimes C^* \to \mathrm{End}_k A$ defined by $\Phi(a \otimes \beta)(x) = a(\beta x)$, and $\Phi'(a \otimes \beta)(x) = (\beta x)a$, cf. [20]. Then for

$a,\ b,\ x \in A$

$$\begin{aligned} F(a \otimes b)(x) &= \Sigma a(\theta w(b_{(1)}) \cdot x) b_{(0)} = \Sigma a(w^* \theta w(b_{(1)}) x) b_{(0)} \\ &= \Sigma a b_{(0)} (\mu_A(b_{(2)}) w^* \theta w(b_{(1)}) x) = \Sigma \Phi(a b_{(0)} \otimes w'(b_{(1)}))(x). \end{aligned}$$

This means $F = \Phi \circ (1 \otimes w') \circ \gamma_A$. Similarly, $F' = \Phi' \circ (1 \otimes w'^*) \circ {\gamma_A}'$ with ${\gamma_A}'(a \otimes b) = \Sigma a_{(0)} b \otimes a_{(1)}$. Thus F and F' are bijective if and only if w' is bijective.

Corollary 4.3. *Let $(S,\ u,\ v) \in D(\theta,\ H^*)$ and suppose that σ defined in* (42) *is an isomorphism. Then $(S,\ u,\ v)$ is contained in the image of π_θ.*

Proof. By assumption, S is H^*-Galois with respect to σ, and $K = k$. We can choose $T = H$ with $\alpha_T(h) = \Sigma h_{(1)} \otimes h_{(2)}$. Then $A = S \#_\sigma H$ is naturally an $H^* \otimes H$-Galois extension and (44) is obtained by changing the coaction by $w:\ H^* \otimes H \to H,\ w(g \otimes h) = v(g)\lambda(h)$. Then $w^*:\ H^* \to H \otimes H^*$ is defined by $w^*(g) = \Sigma v^*(g_{(1)}) \otimes \lambda(g_{(2)})$. Moreover, it is easily checked that $\mu_A:\ H^* \otimes H \to H \otimes H^*$ is given by $\mu_A(g \otimes h) = \Sigma \mu_S(g_{(1)}) \sigma(\lambda h) \otimes \sigma^*(g_{(2)})$. Let us view μ_A and $w^* \theta w$ as elements of $\Lambda(H)$, (30). Then, since $\sigma = 1 - v^*\theta$ and $\mu_S = v - v^*\theta v$,

$$w' = \mu_A + w^*\theta w = \begin{pmatrix} -\sigma & \mu_S \\ 0 & \sigma^* \end{pmatrix} + \begin{pmatrix} -v^*\theta & v^*\theta v \\ \theta & -\theta v \end{pmatrix} = \begin{pmatrix} -1 & v \\ \theta & \sigma^* - \theta v \end{pmatrix}.$$

But this is invertible with inverse $\begin{pmatrix} v\sigma^{*-1}\theta - 1 & v\sigma^{*-1} \\ \sigma^{*-1}\theta & \sigma^{*-1} \end{pmatrix}$. Hence A is a θ-Azumaya algebra over k and the result follows from theorem 4.1.

Let us consider the case $\theta = \varepsilon$. The group $\mathrm{BD}(\varepsilon,\ H)$ is commutative since the smash product coincides with the tensor product. Furthermore, $\mathrm{Br}(k) \to \mathrm{BD}(\varepsilon,\ H)$ admits a natural splitting since ε-Azumaya implies Azumaya.

Corollary 4.4.(Beattie [2].) *There is a split exact sequence*

$$0 \to \mathrm{Br}(k) \longrightarrow \mathrm{BD}(\varepsilon,\ H) \xrightarrow{\pi_\varepsilon} E(H^*) \to 0.$$

Proof. $D(\varepsilon, H^*)$ consists of all elements $(S, 1, \mu_S, \varepsilon)$ with $S \in E(H^*)$ and therefore $D(\varepsilon, H^*) \cong E(H^*)$. For each such element we have $\sigma = id_H$ so that cor. 4.3 yields the surjectivity of π_ε.

Next we consider the subgroup $\mathrm{BD}^s(\theta, H) \subset \mathrm{BD}(\theta, H)$ of algebras split by a faithfully flat extension of k, [6]. Each such algebra is Azumaya over k as follows from [15], Prop. 3.7, and [1], p. 105, (6).

Let $E_c(H^*) \subset E(H^*)$ denote the subgroup of commutative Galois extensions. It is embedded into $D(\theta, H^*)$ by $S \mapsto (S, 1, \varepsilon)$, see (40).

Corollary 4.5. *There is an exact sequence*

$$0 \to \mathrm{Br}(k) \longrightarrow \mathrm{BD}^s(\theta, H) \xrightarrow{\pi_\theta} E_c(H^*) \to 0. \tag{45}$$

Assume that the convolution product $\eta = \theta + \theta^* : H \to H^*$ is bijective. Then

$$0 \to \mathrm{BD}^s(\theta, H) \longrightarrow \mathrm{BD}(\theta, H) \xrightarrow{\psi} \mathrm{Aut}(H)^{\mathrm{op}} \tag{46}$$

is exact where $\psi = \kappa \circ \pi_\theta$ with $\kappa : D(\theta, H^*) \to \mathrm{Aut}(H)^{\mathrm{op}}$, $(S, u) \mapsto u$, cf. (39).

Proof. Let $[A] \in \mathrm{BD}(\theta, H)$ and $\pi_\theta(A) = (S, u, v)$. First suppose $v = \varepsilon$. Then, by (29), S is commutative (and $u = 1$), After base extension by S we obtain $\pi_{\theta\otimes 1}(A \otimes S) = 0$ since $S \otimes S \cong S \otimes H^*$. Hence $[A \otimes S] \in \mathrm{Br}(S)$ and A is split by a faithfully flat extension $k \subset S \subset K$. Conversely, suppose A is split by some faithfully flat extension $k \subset K$. Then $\pi_{\theta\otimes 1}(A \otimes K) = 0$ so that $v \otimes 1 : H^* \otimes K \to H \otimes K$ is equal to $\varepsilon \otimes 1$. But then $v = \varepsilon$ so that S is again commutative and $u = 1$. Furthermore, cor.4.3 implies that any $S \in E_c(H^*)$ is in $\mathrm{Im}(\pi_\theta)$ because in this case $\sigma = \mathrm{id}_H$. It follows that (45) is exact. Finally note that $u = 1$ yields $v = \varepsilon$ if η is bijective, and this shows that (46) is also exact.

A more complete description of $\mathrm{BD}^s(\theta, H)$ as a central group extension of $E_c(H^*)$ by $\mathrm{Br}(k)$ has been given by different methods in Caenepeel [6, I]; see also [4] and [7] for the group ring case. The corresponding 2-cocycle is derived from the pairing $E_c(H^*) \times E_c(H) \to \mathrm{Br}(k)$, $[S, T] \mapsto [S\#T]$, originally considered in [12], (cf. [21]).

Let $H = J \otimes J^*$ and $\theta = \eta \circ \omega$ be as in (5). In [6, II], Caenepeel proved that there exists a homomorphism $\beta : \mathrm{BD}(J) \to \mathrm{Aut}(H)$ with kernel $\mathrm{BD}^s(J)$; moreover, each element $u \in \mathrm{Im}(\beta)$ satisfies

$$q \circ u = q, \quad \text{for } q: J \otimes J^* \to k, \; x \otimes \alpha \mapsto \langle x, \alpha \rangle. \tag{47}$$

The construction of β relies on a dimodule version of the Rosenberg-Zelinsky sequence and there seems to be no evident proof that β coincides (up to sign) with ψ of (46). However we can prove:

Proposition 4.6. *Let H and θ be as above, and let $(S, u, v) \in D(\theta, H^*)$. Then u satisfies* (47).

Proof. Recall that $\omega = 1 \otimes \varepsilon$, and so for any $x \otimes \alpha \in J \otimes J^*$

$$\Sigma\langle x_{(1)} \otimes \alpha_{(1)}, \omega(\lambda x_{(2)} \otimes \lambda\alpha_{(2)})\rangle = \langle \lambda x, \alpha\rangle = \langle x, \lambda\alpha\rangle = q(x \otimes \lambda\alpha).$$

Furthermore, (28) implies $\Sigma\langle g_{(1)}, \mu_S(g_{(2)})\rangle = \varepsilon(g)$ for any $g \in H^*$, and we know by (29) that $v = v^*\theta v - \mu_S$. From this we obtain, since $\theta = \eta\omega$,

$$\begin{aligned}\Sigma\langle g_{(1)}, v(\lambda g_{(2)})\rangle = \Sigma\langle g_{(1)}, v^*\theta v(\lambda g_{(2)})\rangle &= \Sigma\langle v(g_{(1)}), \omega\lambda(v(g_{(2)}))\rangle \\ &= q((1 \otimes \lambda)v(g)).\end{aligned}$$

By (37), $u^{-1} = 1 - v\eta$ and therefore

$$\begin{aligned} qu^{-1}(x \otimes \alpha) &= q(\Sigma(x_{(1)} \otimes \alpha_{(1)})v\eta(\lambda x_{(2)} \otimes \lambda\alpha_{(2)})) \\ &= \Sigma q(x_{(1)} \otimes \alpha_{(1)})q(v\eta(\lambda x_{(2)} \otimes \lambda\alpha_{(2)})) \\ &\qquad \langle x_{(3)} \otimes \alpha_{(3)}, v\eta(\lambda x_{(4)} \otimes \lambda\alpha_{(4)})\rangle \\ &= \Sigma q(x_{(1)} \otimes \alpha_{(1)})q(v\eta(x_{(2)} \otimes \alpha_{(2)})) \\ &\qquad q((1 \otimes \lambda)(v\eta(x_{(3)} \otimes \alpha_{(3)})). \end{aligned}$$

But we always have $\Sigma\langle y_{(1)}, \beta_{(1)}\rangle\langle y_{(2)}, \lambda\beta_{(2)}\rangle = \varepsilon(y)\varepsilon(\beta)$ so that the last term is simply $q(x \otimes \alpha)$ as was to be shown.

In the following we want to apply theorem 4.1 for an element (S, u, v) of the form (41). Let $H = J \otimes J^*$ and $\theta = \omega \circ \eta$ be as in (5), and suppose there exists a Hopf algebra isomorphism $\varphi: J \to J^*$. Consider

$$u: J \otimes J^* \to J \otimes J^*, \; x \otimes \alpha \mapsto \varphi^{-1}(\alpha) \otimes \varphi^*(x). \tag{48}$$

Then $u^{-1}(x \otimes \alpha) = \varphi^{*-1}(\alpha) \otimes \varphi(x)$, and it is easily verified that $u^*\eta = \eta u^{-1}$, that is, $u \in U_\eta$. Also, $\omega u(x \otimes \alpha) = \varphi^{-1}(\alpha) \otimes \varepsilon(x) = u(\varepsilon(x) \otimes \alpha)$, so that

$\omega u = u(\eta^{-1}\omega^*\eta) = u\omega^{(*)}$, cf. the proof of lemma 0.1. Equally, $u\omega = \omega^{(*)}u$. Hence we know by lemma 3.4 that the following element (S, u, v) belongs to $D(\theta, H^*)$.

Proposition 4.7. *Define $S \in \mathcal{E}(H^*)$ by $S = (H^*, u^{-1}\omega\eta^{-1})$, and let v be as in (41). Then (S, u, v) is contained in the image of π_θ.*

Proof. We have $\sigma = 1 - v^*\theta = 1-(1-u)\omega$, and hence for $x\otimes\alpha \in J\otimes J^*$

$$\sigma(x\otimes\alpha) = \Sigma(x_{(1)}\otimes\alpha_{(1)})(\lambda x_{(2)}\otimes\varepsilon(\alpha_{(2)}))u(x_{(3)}\otimes\varepsilon(\alpha_{(3)})) \tag{49}$$
$$= 1\otimes\varphi^*(x)\alpha.$$

Let us identify $S = (J\otimes J^*, u^{-1}\omega)$ via η. According to (24) the product on S is then defined by

$$(x\otimes\alpha)\cdot(y\otimes\beta) = \Sigma x_{(1)}y_{(1)}\otimes\alpha_{(1)}\beta_{(1)}\langle u^{-1}(x_{(2)}\otimes\varepsilon(\alpha_{(2)}),\ y_{(2)}\otimes\beta_{(2)}\rangle$$
$$= \Sigma x_{(1)}y_{(1)}\otimes\alpha\beta\langle\varphi(x_{(2)}),\ y_{(2)}\rangle.$$

Thus $S = J^\varphi\otimes J^*$ for $J^\varphi = (J, \varphi)$. Now (49) implies $K = k\otimes J^* \subset J^\varphi\otimes J^*$. Let $T = J\otimes J^*$. Then T is an H-Galois extension of k if we define $\alpha_T : T \to T\otimes H$ by

$$\alpha_T(x\otimes\alpha) = \Sigma(x_{(1)}\otimes\alpha_{(1)})\otimes(\varphi^{*-1}(\alpha_{(2)})\otimes\varphi(x_{(2)})\lambda(\alpha_{(3)})).$$

Let us show that (43) holds. We have $v\eta = 1-u^{-1}$ so that for $\alpha\in J^*$

$$v\eta(1\otimes\alpha) = \Sigma(1\otimes\alpha_{(1)})(\varphi^{*-1}(\lambda\alpha_{(2)})\otimes 1) = \Sigma\varphi^{*-1}(\lambda\alpha_{(2)})\otimes\alpha_{(1)}.$$

This implies (43) because

$$\alpha_T(1\otimes\alpha) = \Sigma(1\otimes\alpha_{(1)})\otimes(\varphi^{*-1}(\alpha_{(2)})\otimes\lambda\alpha_{(3)})$$
$$= \Sigma(1\otimes\alpha_{(1)})\otimes v\eta(1\otimes\lambda\alpha_{(2)}).$$

It remains to show that $A = S\#_K T$ is θ-Azumaya over k. There is an isomorphism of k-algebras

$$B = (J^\varphi\#J^*)\otimes J^* \to A,\ (x\otimes\beta)\otimes\alpha \mapsto (x\otimes\alpha)\otimes(\varphi^{-1}(\beta)\otimes 1),$$

which is also H-colinear if we define $\alpha_B : B\to B\otimes H$ by

$$\alpha_B(x\otimes\beta\otimes\alpha) = \Sigma(x_{(1)}\otimes\beta_{(1)}\otimes\alpha_{(1)})\otimes v\eta(x_{(2)}\otimes\alpha_{(2)})\lambda(\beta_{(2)}).$$

But B is naturally an $H \otimes J^*$-Galois extension of k, and α_B is obtained by changing the Galois coaction by $w: H \otimes J^* \to H$,

$$w(x \otimes \beta \otimes \alpha) = v\eta(x \otimes \alpha)\lambda(\beta) = \Sigma x_{(1)}\varphi^{*-1}(\lambda\alpha_{(1)}) \otimes \alpha_{(2)}\varphi(\lambda x_{(2)})\lambda(\beta).$$

Then $w^*: J \otimes J^* \to J^* \otimes J \otimes J$ is given by $x \otimes \alpha \mapsto \Sigma\varphi^*(\lambda x_{(1)})\alpha_{(1)} \otimes \lambda x_{(2)} \otimes x_{(3)}\varphi^{-1}(\lambda\alpha_{(2)})$. Furthermore, it is easy to check, using (28), that $\mu_B: H \otimes J^* \to J^* \otimes J \otimes J$ satisfies

$$\mu_B(x \otimes \beta \otimes \alpha) = \Sigma\varphi^*(x_{(1)})\varphi(\lambda x_{(2)})\lambda(\beta) \otimes x_{(3)} \otimes \varepsilon(\alpha).$$

It follows that $w': H \otimes J^* \to J^* \otimes J \otimes J$ of lemma 4.2 is given by

$$w'(x \otimes \beta \otimes \alpha) = \Sigma\varphi(\lambda x_{(1)})\lambda(\alpha_{(1)})\lambda(\beta) \otimes \varphi^{*-1}(\alpha_{(2)}) \otimes x_{(2)}\varphi^{*-1}(\lambda\alpha_{(3)}).$$

But this is bijective since $x \otimes \beta \otimes \alpha \mapsto \Sigma\varphi(\lambda x_{(1)})\lambda(\alpha_{(1)})\lambda(\beta) \otimes \alpha_{(2)} \otimes x_{(2)}$ is evidently bijective, thereby completing the proof.

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(*Recieved March 30, 1992*)