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EXTENTIONS OF ORDERINGS OF HIGHER LEVEL ON RINGS

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1. Introduction. In the 1930's Artin, Schreier and Baer[1,2,3] each studied orders on a field. Recently Becker broadened this to orderings of higher level on a field [7] (with further work in [9] by Becker, Harman, and Rosenberg). Orders were also generalized by Coste and Coste-Roy to orderings of level 1 on a commutative ring with unit [14] (see also Becker [6], and Lam [20]).

In a previous paper [4] we defined orderings of level n on a commutative ring, modeling the presentation on that of [6]. As defined, orderings of level n on a commutative ring A are in one to correspondence with sets $(\wp, \chi)$, where $\wp$ is a prime ideal of A and χ is a signature from the quotient field of $A/\wp$ into the $2n^{\text{th}}$ roots of unity. Thus we identified an ordering of level n on a commutative ring with a signature, in [8, 20] orderings of higher level are given a different definition and are identified with the kernels of such signatures. There also exists a presentation [18,19] based on a definition of a signature on a ring and the generalization of the infinite primes of Harrison [17].

We defined the real spectrum of level n of A, denoted R_n-spec$A$, to be the set of all orderings of level n on A. A topology on R_n-spec$A$ was defined, and it was shown that R_n-spec acts a contravariant functor from the category of commutative rings with unit into the category of topological spaces. For other work in this area see also Berr [11] and Kanzaki [18,19]. In future papers we hope to further investigate real algebraic geometry of higher level. In this paper we will investigate how orderings may be extended or restricted from one ring to another. We start by restating some definitions and conclusions of [4].

Giving a ring A, let $\dot{A} = A \setminus \{0\}$. If $\wp$ is a prime ideal of A, let $k(\wp)$ be the quotient field of $A/\wp$. Lastly, let $\mu(2n)$ denote the $2n^{\text{th}}$ roots of unity $\mu(2n)$.

Definition 1.1. An ordering α of level n on a ring A is an ordered collection of subsets of A: $\alpha_1, \dots, \alpha_{2n}$, such that

i) $A = \alpha_1 \cup \cdots \cup \alpha_{2n}$

ii) $\alpha_i \cap \alpha_j = \wp_\alpha$ a prime ideal, and denoting $\alpha_i^* = \alpha_i \setminus \wp_\alpha$

iii) $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$

iv) $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_k^*$ where $k = i + j$ if $i + j \leq 2n$, and $k = i + j - 2n$ if $i + j > 2n$.

Remark 1.2 The arrangement (or indexing) is considered part of the definition. This yields a bijection between orderings of level n and pairs $(\wp, \chi)$, where $\wp$ is Spec$A$ and χ is a signature of level n on $\dot{k}(\wp)$. The bijection is given by $\alpha \to (\wp_\alpha, \chi_\alpha)$ where $\chi_\alpha(\alpha_i + \wp_\alpha) = \zeta^i$ and $\zeta = e^{\pi i/n}$.

Definition 1.3. The set of all orderings of level n on a ring A is called the real spectrum of level n of A and denoted by R_n-spec$A$.

Definition and Remarks 1.4. If α is an ordering of level n whose associated signature χ_α maps $\dot{k}(\wp_\alpha)$ onto $\mu(2n)$, then α and χ_α are both said to have exact level n. The set of all orderings of exact level n is denoted by $R_{\underline{n}}$-spec$A$.

If α is an ordering of level n on A associated with a signature χ mapping $\dot{k}(\wp_\alpha)$ into $\mu(2n)$, then $\alpha_j = \{a \in A \mid (a + \wp_\alpha) \in \chi^{-1}((e^{\pi i/n})^j) \cup \{0\}\}$. If χ maps into $\mu(2m) \subsetneqq \mu(2n)$, so $n/m \in Z$, then there is an ordering β of level m associated to χ. This implies $\beta_j = \alpha_{j \cdot n/m}$, where $\alpha_j = \beta_{jm/n}$ if n/m divides j and $\alpha_j = \wp_\beta$ if n/m does not divide j. Therefore every ordering α of level n associated with a signature χ which maps onto $\mu(2m) \not\subseteq \mu(2n)$ is associated to an ordering β of level m. We shall not distinguish between orderings obtained from the same signature, and will indicate the level by a left subscript where needed. Thus, for β as above, $\beta = {}_m\alpha$. That is, when α is written as an ordering of level m it equals β.

Definition 1.5. We write $\alpha \subseteq \beta$ if $\alpha_i \subseteq \beta_i$ for all i, and we say β is a *specialization* of α. In this situation, α is called a *generalization* of β. Let α and β be orderings of levels m and n respectively, where $s = \text{lcm}(m, n)$. Then we write $\alpha \subseteq \beta$ if ${}_s\alpha \subseteq {}_s\beta$. Note that $\alpha \subseteq \beta$ if and only if ${}_t\alpha \subseteq {}_t\beta$ for all t divisible by s, and that $\alpha \subseteq \beta$ implies the level of β is less than or equal to the level of α.

Definition and Remark 1.6. In order to topologize R_n-spec$A$

we view $f \in A$ as a function from R_n-spec into $\Pi k(\wp_\alpha)$ by setting $f(\alpha) = f + \wp_\alpha \in k(\wp_\alpha)$. Given χ_α and ζ as in Remark 1.2, we define $D(f,t) = \{\alpha \in R_n\text{-spec}A \mid \chi_\alpha(f(\alpha)) = \zeta^t\} = \{\alpha \mid f \in \alpha_t^*\}$ where $1 \leq t \leq 2n$. The sets $D(f,t)$ make up a subbasis for a topology of R_n-spec$A$. By definition its basis is given by the sets $\cap_{i=1}^r D(f_i,t_i)$. This topology is called the Coste-Roy (C-R) topology on R_n-spec$A$. When considering the topology of R_n-spec$A$ this is the topology to which we are referring. There is a second topology called the Tychonoff topology. A set is called constructible if it can be obtained from the sets $D(f,t)$ by a finite sequence of taking unions, intersections, and complements. The constructible subsets of R_n-spec$A$ form a basis of the Tychonoff topology. The Tychonoff topology is finer than the C-R topology. In the C-R topology R_n-spec$A$ is compact, and in the Tychonoff topology it is compact and Hausdorff.

2. Extensions of orderings. Let K be a field, V a valuation ring of K and I the maximal ideal of V. A character χ on $\dot{K}$ is compatible with V if $(1+I) \subseteq \ker\chi$. Each character χ, compatible with V, yields a character $\bar{\chi}$ of V/I called the pushdown of χ (with respect to V). Furthermore, if χ is a signature then $\bar{\chi}$ is a signature [9]. Therefore if α is an ordering of level n on K and $1+I \subseteq \alpha_{2n}^* = \ker\chi_\alpha$ we may define the pushdown of α on V/I. It is written as $\bar{\alpha}$, where $\bar{\alpha}_i$ is defined to be the image of $\alpha_i \cap V$ under the canonical map $i^*: V \to V/I$.

Definition and Remark 2.1 Given F a field and $\alpha \in R_n$-spec$F$, let $A(\alpha_{2n}^*) = \{a \in F \mid \exists_{t\in N} \text{ with } t \pm a \in \alpha_{2n}^*\}$. If $A(\chi)$ is defined as in Becker, Harman and Rosenberg [9, page 60], and if χ_α is the signature associated to α by Remark 1.2, then $A(\alpha_{2n}^*) = A(\chi_\alpha)$. Similarly, if $P = \alpha_{2n}$, and $A(P)$ is defined as by Becker [7, page 14], then $A(P) = A(\alpha_{2n}^*)$. The maximal ideal of $A(\alpha_{2n}^*)$ is $I(\alpha_{2n}^*) = \{a \in F \mid \forall_{n\in N}\ 1/n \pm a \in \alpha_{2n}^*\}$, and this agrees with the definition of $I(P)$ and $I(\chi_\alpha)$ found in the literature [7,9, respectively].

Becker has shown [6, Theorem 3.4] that $A(\alpha_{2n}^*) = A(\chi_\alpha)$ is the smallest valuation ring compatible with χ_α, and that the pushdown of χ_α to $A(\alpha_{2n}^*)/I(\alpha_{2n}^*)$ is the signature of an archimedean ordering of level 1 [9, Theorem 2.7(iii)].

Definition 2.2. Let $A \supset B$ be rings, with $\alpha \in R_n$-spec$A$ and $\beta \in$

R_m-spec$B$. If $l = \text{lcm}(m,n)$, and ${}_l\alpha$ and ${}_l\beta$ are defined as in Remark 1.4. Then we say α (or ${}_l\alpha$) is an extension of β (or ${}_l\beta$), or α extends β, if ${}_l\alpha_i \cap B = {}_l\beta_i$, for $i = 1, \ldots, 2n$. If α extends β, and if the exact level of α equals the exact level of β, then α is called a faithful extension of β.

Thorem 2.3. *Let $A \supset B$ be rings with $\alpha \in R_{\underline{n}}$-spec$A$ and $\beta \in R_{\underline{m}}$-spec$B$. If α extends β then m divides n, so that in particular $n \geq m$.*

Proof. Let $l = \text{lcm}(m,n)$ and assume $l \neq n$. By Remark 1.4, we know ${}_l\alpha_i = \wp_\alpha$ if i is not divisible by l/n. This implies ${}_l\beta_i = \wp_\beta$ when i is not divisible by l/n. Therefore $\text{Im}(\chi_\beta) \subseteq \mu(2n)$, but the image of χ_β is $\mu(2m)$, so m divides n.

We will now prove a generalized version of Brumfiels place extension theorem [13, page 152, 7.7.4]. This requires that we extend the idea of convex sets from the setting of orderings of level 1, [6, pages 26 and 27], to that of orderings of level n. It also requires that we define an archimedean property similar to that of Becker [5, pages 20–21].

Definition 2.4. Let $A \supset B$ be rings, and let $\alpha \in R_n$-spec$A$. If I is a subset of B we say I is convex in B with respect to α, if given c, d, and $c - d$ in $(\alpha_{2n}^* \cap B)$ then $c \in I$ implies $d \in I$.

Definition 2.5. Let $K \supset F$ be fields, and $\alpha \in R_n$-spec$K$. We say K is archimedean over F with respect to α if for all $a \in K$ there exists $b \in (\alpha_{2n}^* \cap F)$ such that $b \pm a \in \alpha_{2n}^*$.

Theorem 2.6. *Let L be a field of characteristic zero, and $\alpha \in R_n$-spec$L$. If R is a local subring of L whose maximal ideal M is convex in R with respect to α, and $U(R)$ is the units of R then*

i) *$Z^+ \subset (U(R) \cap \alpha_{2n}^*)$, where Z^+ is the positive integers.*

ii) *$\hat{R} = \{a \in L \mid \exists r \in (U(R) \cap \alpha_{2n}^*)$ such that $r \pm a \in \alpha_{2n}^*\}$ is a valuation ring compatible with χ_α and has maximal ideal $\hat{M} = \{a \in L \mid \forall r \in (U(R) \cap \alpha_{2n}^*)$ we have $r \pm a \in \alpha_{2n}^*\}$.*

iii) *R is dominated by $\hat{R}$ (i.e. $R \subset \hat{R}$ and $\hat{M} \cap R = M$).*

iv) *Since χ_α is compatible with $\hat{R}$ we may push down χ_α to $\bar{\chi}_\alpha$ a signature of $\hat{R}/\hat{M}$ [9]. The residue field $\hat{R}/\hat{M}$ is archimedean over R/M with respect to the ordering on $\hat{R}/\hat{M}$ induced by $\bar{\chi}_\alpha$, (where archimedean* *is defined as in Definition 2.5 since $\bar{\chi}_\alpha$ is not necessarily of level 1).*

Proof. i) This is clear since $Z^+ \subset (R \cap \alpha_{2n}^*)$ and M is convex in R.

ii) Clearly 1 and 0 are in $\hat{R}$, and if $a \in \hat{R}$ then $-a \in \hat{R}$. Let $a, b \in \hat{R}$ then there exists r_a and r_b in $(U(R) \cap \alpha_{2n}^*)$ such that $r_a \pm a$ and $r_b \pm b$ lie in α_{2n}^*. This implies $r_a + r_b \pm (a+b)$ and $r_a + r_b + 1 \pm (a+b)$ are in α_{2n}^*. Since both $r_a + r_b$ and $r_a + r_b + 1$ lie in α_{2n}^*, and at least one of them lies in $U(R)$, there exists an $r \in (U(R) \cap \alpha_{2n}^*)$ such that $r \pm (a+b) \in \alpha_{2n}^*$. Therefore $\hat{R}$ is additively closed. Note also that $(r_a r_b + ab) = (1/2)\{(r_a + a)(r_b + b) + (r_a - a)(r_b - b)\}$ is in α_{2n}^*, and that a similar identity implies $(r_a r_b - ab) \in \alpha_{2n}^*$. So $(r_a r_b \pm ab) \in \alpha_{2n}^*$ and $\hat{R}$ is multiplicatively closed. Therefore $\hat{R}$ is a ring. By definition $\hat{R}$ contains $A(\chi_\alpha)$, but $A(\chi_\alpha)$ is a valuation ring compatible with χ_α. Therefore, by [9], $\hat{R}$ is also a valuation ring compatible with χ_α.

Let $a, b \in \hat{M}$ and $r \in \hat{R}$. To show $\hat{M}$ is an ideal, we note that if $t \in (U(R) \cap \alpha_{2n}^*)$ then $t \pm (a-b) = (t/2 \pm a) + (t/2 \pm b)$ is in α_{2n}^*. So $(a-b)$ is in $\hat{M}$. It remains to show that $ra \in \hat{M}$. By definition, $(t \pm a) \in \alpha_{2n}^*$ for all t in $(U(R) \cap \alpha_{2n}^*)$, and there exists $s_r \in (U(R) \cap \alpha_{2n}^*)$ such that $(s_r \pm r)$ is in α_{2n}^*. As above $[s_r t \pm ra] \in \alpha_{2n}^*$. Since any s in $(U(R) \cap \alpha_{2n}^*)$ can be written as $s_r t$ for some $t \in (U(R) \cap \alpha_{2n}^*)$, we see that $(s \pm ra)$ lies in α_{2n}^* for all s in $(U(R) \cap \alpha_{2n}^*)$. Therefore $ra \in \hat{M}$ and $\hat{M}$ is an ideal.

If I is the maximal ideal of $\hat{R}$ then $I \supset \hat{M}$, and since $\hat{R}$ is compatible with χ we have $1 + I \subset \ker \chi_\alpha = \alpha_{2n}^*$. Let $a \in I$ and $r \in (U(R) \cap \alpha_{2n}^*)$, then $r \pm a = r(1 \pm r^{-1}a) \in (\alpha_{2n}^*)(1+I) \subset \alpha_{2n}^*$. Thus $a \in I \Rightarrow a \in \hat{M}$. Hence $\hat{M}$ is the maximal ideal.

iii) By Hardy and Wright [16, page 235, using $d = 2n$]

$$(2n)!X = \sum_{h=0}^{2n-1} (-1)^{2n-1-h} \binom{2n-1}{h} [(X+h)^{2n} - h^{2n}].$$

for $X \in R$. Since $(2n)! \in (U(R) \cap \alpha_{2n}^*)$, this implies $R \subseteq (\alpha_{2n}^* \cap R) - (\alpha_{2n}^* \cap R)$. Thus $R \subseteq \hat{R}$ if $(R \cap \alpha_{2n}^*) \subseteq \hat{R}$. By definition $(U(R) \cap \alpha_{2n}^*) \subseteq \hat{R}$. So assume $m \in (M \cap \alpha_{2n}^*)$, then $(m+1) \in (U(R) \cap \alpha_{2n}^*) \subseteq \hat{R}$. Since $1 \in \hat{R}$ this implies $m \in \hat{R}$. Therefore $R \subseteq \hat{R}$.

Since $R \subseteq \hat{R}$ we have $M \supseteq (\hat{M} \cap R)$, it remains to show $M \subseteq \hat{M} \cap R$. Let $a \in M$ and suppose $a \notin \hat{M}$ (so $a \neq 0$); then $a^{2n} \in (M \cap \alpha_{2n}^*)$ but $a^{2n} \notin \hat{M}$. Let $b = a^{2n}$. Then $1/b \in \hat{R}$, which implies there exists $r \in (U(R) \cap \alpha_{2n}^*)$ such that $(r \pm 1/b) \in \alpha_{2n}^*$. But $b \in \alpha_{2n}^*$ and $r \in (U(R) \cap \alpha_{2n}^*)$, so $(b \pm 1/r) \in \alpha_{2n}^*$. Thus by the convexity of M we have $1/r \in M$. This is impossible since $r \in U(R)$, therefore $(\hat{M} \cap R) = M$.

iv) Since $\hat{R}$ is a valuation ring of L compatible with χ_α, we know that $\bar{\chi}_\alpha$ is a signature of $\hat{R}/\hat{M}$. So $\bar{\chi}_\alpha$ induces an ordering on $\hat{R}/\hat{M}$ via $\bar{\chi}_\alpha(\bar{\epsilon}) = \chi(\epsilon)$. Let $\bar{x} \in \hat{R}/\hat{M}$, if $x \in \hat{R}$ then by definition of $\hat{R}$ there exists $y \in (U(R) \cap \ker\chi_\alpha)$ such that $(y \pm x) \in \ker\chi_\alpha$, but then $(\bar{y} \pm \bar{x})$ lies in $\ker\bar{\chi}_\alpha$, with $\bar{y} \in (R/M \cap \ker\bar{\chi}_\alpha)$. Therefore $\hat{R}/\hat{M}$ is archimedean over R/M.

The following theorem follows directly from [7, Satz 3.1], even with our more restrictive definition of a faithful extension.

Theorem 2.7. *Let L/K be a finite extension of fields and $\alpha \in R_n$-spec$K$. Let k be the residue class field of $A(\alpha_{2n}^*)$, and $\bar{\alpha}$ the pushdown of α. Let $\hat{A}$ be a valuation ring of L such that $\hat{A} \cap K = A(\alpha_{2n}^*)$, and let l be the residue class field of $\hat{A}$. If $[v(\hat{A}): v(A(\alpha_{2n}^*))]$ is relatively prime to n, and $\bar{\alpha}$ has a faithful extension to l; then α has a faithful extension to L.*

Definition 2.8. Given a field F with $\alpha \in R_n$-spec$F$, we define K to be a real closure of level n of (F, α), if K is a maximal algebraic extension of F admitting a faithful extension of α. If $\hat{\alpha}$ is that extension we write $(K, \hat{\alpha})$ is a real closure of (F, α). A field L is a real closed of level n if for some $\beta \in R_n$-spec$L$, (L, β) is the real closure of (L, β).

Remark 2.9. By Becker [7, Satze 3.6 and 3.7], if (L, β) is real closed of level n, and α is any ordering in R_n-spec$L$, then $\alpha_{2n} = \beta_{2n}$. By [9, Proposition 3.21] χ_α and χ_β have the same exact level and $\chi_\alpha = (\chi_\beta)^r$ where $(r, n) = 1$. Therefore there are exactly $\varphi(n)$ orderings of exact level n, and if $\alpha \in R_n$-spec$L$ then $\alpha_i = \beta_k$ where $k \equiv ij \bmod 2n$ and $(j, n) = 1$. Thus if (L, β) is real closed of level n for some β, then L is real closed of level n with respect to all of its orderings of level n. Since our definition of faithful extension is more restrictive than in Becker [7], our definition of real closure is different. In our case $(K, \hat{\alpha})$ is a real closure of (F, α) implies $\hat{\alpha}_i \cap F = \alpha_i$; in Becker's notation it only implies $\hat{\alpha}_{2n} \cap F = \alpha_{2n}$. Thus in Becker's notation there are $\varphi(n)$ choices for the ordering on K that make K a real closure of (F, α). However the definitions of when L

is real closed of level n agree, since in our case (L, β) real closed of level n implies it is real closed for all orderings of level n on L.

Definition 2.10. Let A be a commutative ring and $\alpha \in R_{\underline{n}}\text{-spec}A$. We will denote by $\tilde{\alpha}$ the induced ordering on $k(\wp_\alpha) = qf(A/\wp_\alpha)$. Furthermore we shall denote by $k(\alpha)$ an arbitrary real closure, of level n, of $(k(\wp_\alpha), \tilde{\alpha})$.

Definition 2.11. Let A and B be commutative rings with orderings α and β respectively, both of level n. Then any ring homomorphism $\lambda: A \to B$ is called order preserving if $\lambda(\alpha_i) \subseteq \beta_i$ for all i.

With this definition the following lemma is clear.

Lemma 2.12. *If α and β are in $R_{\underline{n}}\text{-spec}A$, then $\alpha \subseteq \beta$ if and only if $\wp_\alpha \subseteq \wp_\beta$ and the canonical homomorphism $\mu_{\alpha,\beta}: A/\wp_\alpha \twoheadrightarrow A/\wp_\beta \subseteq k(\wp_\beta)$ defined by $\mu_{\alpha,\beta}(a + \wp_\alpha) = (a + \wp_\beta)$ is order preserving.*

In order to prove our next thorem we need the following lemma.

Lemma 2.13 *Let $(K, \hat{\alpha})$ and (k, β) be two real closed fields of level n. If there exists a place λ from K onto $k \cup \infty$ with valuation ring V and $\alpha_i = (\hat{\alpha}_i \cap V)$, then there exists an r with $(r, n) = 1$ such that $\lambda(\alpha_i) = \beta_j$ for some $j \equiv ir \bmod 2n$.*

Proof. Without loss of generality we may assume λ is the canonical place associated to V and k is its residue class field. By Becker, Harman and Rosenberg [9, Theorem 1.12 and Prop. 2.5], there exists an ordering of exact level n on K, called it γ, such that $\lambda(\gamma_i \cap V) = \beta_i$. And by Remark 2.9, we see $\hat{\alpha}_i = \gamma_j$ where $j \equiv ir \bmod 2n$ and $(n, r) = 1$. Therefore $\lambda(\alpha_i) = \lambda(\hat{\alpha}_i \cap V) = \lambda(\gamma_j \cap V) = \beta_j$ where $j \equiv ir \bmod 2n$ and $(n, r) = 1$.

Theorem 2.14. *For $\alpha, \beta \in R_{\underline{n}}\text{-spec}A$, $\alpha \subseteq \beta$ implies $\wp_\alpha \subseteq \wp_\beta$ and the map $\mu_{\alpha,\beta}: A/\wp \to A/\wp_\beta$ extends to a place $\hat{\mu}: k(\alpha) \to K \cup \infty$ where K is a real closed (of level n) extension of $(k(\wp_\beta), \tilde{\beta})$ and $k(\alpha)$ is an arbitrary real closure of $(k(\wp_\alpha), \tilde{\alpha})$. Conversely, if $\wp_\alpha \subseteq \wp_\beta$ and the map $\mu_{\alpha,\beta}: A/\wp_\alpha \to A/\wp_\beta$ extends to a place $\hat{\mu}: k(\alpha) \to K \cup \infty$ where K is a real closed of level n extension of $(k(\wp_\beta), \tilde{\beta})$ then $\alpha_i \subseteq \beta_j$ where $j \equiv ir \bmod 2n$ for some r such that $(r, n) = 1$.*

Proof. The second statement can be proved using Lemma 2.13. To prove the first statement we assume $\alpha \subseteq \beta$. Then $\mu_{\alpha,\beta}\colon A/\wp_\alpha \to A/\wp_\beta$ is order preserving and $\ker\mu_{\alpha,\beta} = \wp_\beta/\wp_\alpha$. Set $A_{\alpha,\beta} = (A/\wp_\alpha)_{\ker\mu_{\alpha,\beta}}$. The ring $A_{\alpha,\beta}$ is a local subring of $k(\wp_\alpha)$ and $\mu_{\alpha,\beta}$ extends uniquely to a homomorphism $\mu\colon A_{\alpha,\beta} \to k(\wp_\beta)$ via $\mu(r/s) = \mu_{\alpha,\beta}(r)/\mu_{\alpha,\beta}(s)$ where s is not in the kernel of $\mu_{\alpha,\beta}$. Since $\mu_{\alpha,\beta}$ is order preserving clearly μ is as well.

To apply Theorem 2.6 we need to show that the maximal ideal of $A_{\alpha,\beta}$, which is $(\ker\mu_{\alpha,\beta})_{\ker\mu_{\alpha,\beta}} = \ker\mu$, is convex on $A_{\alpha,\beta}$ with respect to $\tilde{\alpha}$. Let $c \in (\ker\mu \cap \tilde{\alpha}^*_{2n})$ with d and $c-d$ in $(\tilde{\alpha}^*_{2n} \cap A_{\alpha,\beta})$, by Definition 2.4 we need to show that $d \in \ker\mu$. Let $e = (c-d) \in (\tilde{\alpha}^*_{2n} \cap A_{\alpha,\beta})$, then $\mu(e) = \mu(c-d) = \mu(c) - \mu(d) = -\mu(d)$. Since μ is order preserving both $-\mu(d) = \mu(e)$ and $\mu(d)$ are in $\tilde{\beta}_{2n}$. This implies $\mu(d) = 0$, so $d \in \ker\mu$, and the maximal ideal of $A_{\alpha,\beta}$ is convex in $A_{\alpha,\beta}$.

Applying Theorem 2.6 we may extend μ to a place $\hat{\mu}$ as in the following diagram.

$$\begin{array}{ccc}
k(\alpha) & & \\
| & & \\
\hat{A}_{\alpha,\beta} & \xrightarrow{\hat{\mu}} & K' \\
\cup & & \cup \\
A_{\alpha,\beta} & \xrightarrow{\mu} & k(\wp_\beta)
\end{array}$$

Here $\hat{A}_{\alpha,\beta}$ is a valuation ring compatible with the ordering associated with $k(\alpha)$, and we may assume K' is the residue field of $k(\alpha)$ under $\hat{\mu}$.

The only statement left to prove in the first direction is that K' is a real closed (of level n) extension of $(k(\wp_\beta), \tilde{\beta})$. Using the notation $\hat{\tilde{\alpha}}$ for the faithful extension of $\tilde{\alpha}$ to $k(\alpha)$ we see $(\hat{\tilde{\alpha}} \cap \hat{A}_{\alpha,\beta})$ gives an ordering of exact level n on $A_{\alpha,\beta}$. Now $\hat{\mu} = \varphi \cdot i^*$, where φ is an isomorphism and i^* is the canonical pushdown map onto the residue class field. Therefore, since the level of the pushdown of any ordering is less than or equal to the level of the original ordering, if $\hat{\tilde{\beta}}_i = \hat{\mu}(\hat{\tilde{\alpha}}_i \cap \hat{A}_{\alpha,\beta})$, then $\hat{\tilde{\beta}}$ is an ordering of level less than or equal to n on K'. On the other hand $\hat{\tilde{\beta}}$ extends $\tilde{\beta}$, so by Theorem 2.3, the exact level of $\hat{\tilde{\beta}}$ is greater than or equal to n, the exact level of $\tilde{\beta}$. Therefore $\hat{\tilde{\beta}}$ is an ordering or exact level n.

If K' is not real closed of level n, then there is a finite extension N of K' which has an ordering that is a faithful extension of $\hat{\tilde{\beta}}$. Let $m = [N\colon K']$. By Endler [15, Theorem 27.1, page 206], there exists a finite separable extension L of $k(\alpha)$ of degree m, with valuation ring B over $\hat{A}_{\alpha,\beta}$ and associated valuation w lving over the valuation v of $\hat{A}_{\alpha,\beta}$, whose value group is the value group of v, and whose residue class field is N. Now by Theorem 2.7 there is a faithful extension of $\hat{\alpha}$ from $k(\alpha)$ to L. But this contradicts the fact that $k(\alpha)$ is real closed of level n. Therefore K is real closed of level n.

3. Real closures of higher level. We first note some definitions and results of [4]. Let A and B be commutative rings and $\varphi: A \to B$ be a unitary ring homomorphism. If $\beta \in R_n$-spec$B$, and α is the indexed collection of subsets of A where $\alpha_i = \varphi^{-1}(\beta_i)$, then α lies in R_n-spec$A$. The map φ_*: R_n-spec$B \to R_n$-spec$A$ defined by $\varphi_*(\beta) = \alpha$ where $\alpha_i = \varphi^{-1}(\beta_i)$ is continuous in the C-R topology of the n^{th} level real spectra, and in the Tychonoff topology. Note $\beta \subseteq \delta$ if and only if $\varphi_*(\beta) \subseteq \varphi_*(\delta)$. In addition R_n-spec is a contravariant functor from the category of commutative rings with unit into the category of topological spaces, thus $(\varphi\psi)_* = \psi_*\varphi_*$.

For clarity in the proof of the next theorem we here restate a theorem of Becker using the definitions and notations of this paper.

Theorem 3.1 [10, Theorem 3.8]. *If β is an ordering of level n on a field L then the following statements are equvalent:*

i) *(L, β) is real closed.*

ii) *$A(\beta^*_{2n})$ is henselian, and its residue field is real closed of level* 1. *If v is the valuation associated with $A(\beta^*_{2n})$, then $v(\beta^*_{2n}) = nv(L)$ and $pv(L) = v(L)$ for every rational prime p not dividing n.*

Theorem 3.2. *Let $(k, \tilde{\tilde{\alpha}})$ be a real closed extension of (F, α); as defined in Definition 2.8. Let $(R, \tilde{\alpha})$ be the algebraic closure of (F, α) in $(K, \tilde{\tilde{\alpha}})$, then $(R, \tilde{\alpha})$ is also real closed.*

Proof. Since $(K, \tilde{\tilde{\alpha}})$ is real closed, all the conditions of statement Theorem 3.1(ii) hold for $A(\tilde{\tilde{\alpha}}^*_{2n})$. We must show that they hold for $A(\tilde{\alpha}^*_{2n})$. Note that by definitoin, $A(\tilde{\alpha}^*_{2n}) = A(\tilde{\tilde{\alpha}}^*_{2n}) \cap R$ and $A(\alpha^*_{2n}) = A(\tilde{\alpha}^*_{2n}) \cap F$. For ease of notation we let $\tilde{\tilde{A}} = A(\tilde{\tilde{\alpha}}^*_{2n})$, $\tilde{A} = A(\tilde{\alpha}^*_{2n})$ and $A = A(\alpha^*_{2n})$. The maximal ideals were defined in Definition 2.1 and are denoted $\tilde{\tilde{I}} = I(\tilde{\tilde{\alpha}}^*_{2n})$, $\tilde{I} = I(\tilde{\alpha}^*_{2n})$, and $I = I(\alpha^*_{2n})$. We shall denote the respective residue class fields by $\tilde{\tilde{k}}$, $\tilde{k}$ and k. If $\tilde{\tilde{v}}$, $\tilde{v}$ and v are the respective valuations, then $\tilde{\tilde{v}}(R) = \tilde{v}(R)$ and $\tilde{\tilde{v}}(F) = \tilde{v}(F) = v(F)$, so we may write all the valuations as v.

Let $f(x) \in \tilde{A}[x]$ be a monic polynomial, and let $b_0 \in \tilde{A} \subseteq \tilde{\tilde{A}}$ such that $\bar{b}_0$ is a simple root of $\bar{f}(x)$ in $\tilde{k}$. Since $\tilde{\tilde{A}}$ is henselian there exists a root b of $f(x)$ in $\tilde{\tilde{A}}$ such that $b \equiv b_0 \bmod \tilde{\tilde{I}}$. This b also lies in R, since R is algebraically closed in K, so $b \in (R \cap \tilde{\tilde{A}}) = \tilde{A}$. Now since $(\tilde{\tilde{I}} \cap \tilde{A}) = \tilde{I}$, this implies $b \equiv b_0 \bmod \tilde{I}$. Therefore $\tilde{A}$ is a henselian valuating ring. It is easy to show that $\tilde{k}$ is algebraically closed in $\tilde{\tilde{k}}$, a real closed field of level 1, and hence $\tilde{k}$ is a real closed of level 1.

It remains to show $v(\tilde{\alpha}^*_{2n}) = nv(R)$ and $pv(R) = v(R)$ for every rational prime p not dividing n. To show $pv(R) = v(R)$ we note $pv(R) = v(R^p) = v(K^p \cap R)$ which Brown [12, Prop. 2.5] has shown is $v(K^p) \cap v(R)$. If p does not divide n, then since K is real closed $v(K^p) = v(K)$. Therefore, $pv(R) = (v(K) \cap v(R)) = v(R)$, if p does not divide n.

To show $v(\tilde{\alpha}^*_{2n}) = nv(R)$, we start by showing $v(\tilde{\tilde{\alpha}}^*_{2n} \cap R) = (v(\tilde{\tilde{\alpha}}^*_{2n}) \cap v(R))$. Obviously, $v(\tilde{\tilde{\alpha}}^*_{2n} \cap R)$ is contained in $v(\tilde{\tilde{\alpha}}^*_{2n}) \cap v(R)$. To obtain the other inclusion, let $a \in K$ such that $nv(a)$ is an arbitrary element of $(nv(K) \cap v(R)) = (v(\tilde{\tilde{\alpha}}^*_{2n}) \cap v(R))$. Then $v(a^n) = v(b)$ for some b in R, where a^n/b is a unit of $\tilde{\tilde{A}}$. Consider the pushdown $\overline{a^n/b}$ of a^n/b. Since $\tilde{\tilde{\alpha}}$ is an ordering of level 1 either $\overline{a^n/b}$ or $\overline{-a^n/b}$ is in $\tilde{\tilde{\alpha}}_{2n}$. Without loss of generality we may assume $\overline{a^n/b} \in \tilde{\tilde{\alpha}}_{2n}$, since $v(b) = v(-b)$. Because the residue class field $\tilde{\tilde{k}}$ is real closed of level 1, the polynomial $\overline{x^n - a^n/b}$ in $\tilde{\tilde{k}}[x]$ has a root in $\tilde{\tilde{k}}$. Since $\tilde{\tilde{A}}$ is henselian, K contains a root of the polynomial $x^n - a^n/b$. Therefore $a^n/b = d^n$ for some $d \in K$. This implies $(a/d)^n = b \in R$, so $nv(a) = v(b) = v((a/d)^n) \in v(K^n \cap R)$. But since $K^n \subseteq (\tilde{\tilde{\alpha}}^*_{2n} \cup -\tilde{\tilde{\alpha}}^*_{2n})$, we also have $(a/d)^n \in \pm\tilde{\tilde{\alpha}}^*_{2n}$. Therefore since $v(-(a/d)^n) = v((a/d)^n)$ we see that $nv(a) = v(b) = v((a/d)^n) \in v(\alpha^*_{2n} \cap R)$.

We have shown $v(\tilde{\tilde{\alpha}}^*_{2n} \cap R) = (v(\tilde{\tilde{\alpha}}^*_{2n}) \cap v(R))$. Therefore, $v(\tilde{\alpha}^*_{2n}) = v(\tilde{\tilde{\alpha}}^*_{2n} \cap R) = v(\tilde{\tilde{\alpha}}^*_{2n}) \cap v(R) = nv(K) \cap v(R) = v(K^n) \cap v(R) = v(R^n) = nv(R)$.

Lemma 3.3. *Let K and L be fields, and let φ be an isomorphism from K onto L, where α and β are orderings of exact level n of K and L, respectively. Assume further that $\varphi^*(\beta) = \alpha$, then given any real closure $k(\alpha)$ of (K, α), φ can be extended to an isomorphism $\bar{\varphi}$ from $k(\alpha)$ onto some real closure of (L, β). (We shall show later that $\bar{\varphi}$ is unique in the* *sense that for a particular* $k(\alpha)$ *and* $\bar{L}$*, if this isomorphism exists, it is unique.)*

Proof. Since $k(\alpha)$ is an algebraic extension there exists a normal closure, F, of $k(\alpha)$ over K. This F is the splitting field over K of some set S of polynomials in $K[x]$. Let S' be the corresponding set of polynomials in $L[x]$, and let $\bar{\bar{L}}$ be the splitting field of S' over L. Then φ extends to an isomorphism $\bar{\varphi}: F \to \bar{\bar{L}}$ and we have

$$\begin{array}{ccc} F & \overset{\bar{\varphi}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \bar{\bar{L}} \\ | & & | \\ k(\alpha) & & | \\ | & & | \\ K & \overset{\varphi}{\rightarrowtail\!\!\!\twoheadrightarrow} & L \end{array}$$

Let $\bar{L} = \bar{\varphi}(k(\alpha))$, and let $\tilde{\beta}_i = \bar{\varphi}(\tilde{\alpha}_i)$ where $\tilde{\alpha}_i$ is the extension of α_i to $k(\alpha)$. Clearly the $\tilde{\beta}_i$ form and ordering. Since $\bar{\varphi}^{-1}(\tilde{\beta}_i \cap L) = \bar{\varphi}^{-1}(\tilde{\beta}_i) \cap \bar{\varphi}^{-1}(L) = \tilde{\alpha}_i \cap K = \alpha_i$ we see that $\tilde{\beta}_i \cap L = \beta_i$, therefore $\bar{L}$ has an ordering of level n that extends β. By Theorem 2.3 it has exact level n. Therefore $\tilde{\beta}$ is a faithful extension of β.

We now have

$$\begin{array}{ccc} F & \overset{\bar{\varphi}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \bar{\bar{L}} \\ | & & | \\ k(\alpha) & \overset{\bar{\varphi}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \bar{L} \\ | & & | \\ k & \overset{\varphi}{\rightarrowtail\!\!\!\twoheadrightarrow} & L \end{array}$$

where $\bar{L}$ is an algebraic extension of L with an ordering of higher level faithfully extending the ordering of L. If $\bar{L}$ is not real closed of level n we may repeat the above argument on $\bar{\varphi}^1: \bar{L} \to k(\alpha)$ to get an isomorphism from a real closure of $\bar{L}$ to an algebraic extension of $k(\alpha)$ having a faithful extension of α. Since $k(\alpha)$ is real closed of level n, this extension must be $k(\alpha)$ itself. Hence $\bar{L}$ is a real closure of (L, β).

The following corollary is clear.

Corollary 3.4. *If* (K, α) *is a real closed field of level* n *and* L *is isomorphic to* K*, then* L *is real closed of level* n *for some (and hence by Remark 2.9 all) orders of level* n*.*

Lemma 3.5. *Let $(\bar{L}, \tilde{\beta})$ be a real closure of (L, β) where β (and thus $\tilde{\beta}$) has exact level n. If φ is an automorphism of $\bar{L}$ fixing L, then it is the identity. That is,* $\mathrm{Aut}(\bar{L}/L) = 1$.

Proof. Assume n is odd, then by Becker [7, Satz 3.6], $\bar{L}$ has a single ordering of level 1, namely $\tilde{\alpha}$ where $\tilde{\alpha}_2 = \bar{L}^2$ and $\tilde{\alpha}_1 = -\bar{L}^2$. Since $\varphi(\tilde{\alpha}_2) = \varphi(\bar{L}^2) = \bar{L}^2 = \tilde{\alpha}_2$, by the theory of ordinary real closed fields we see that φ extends to a unique automorphism of the real closure of level 1 of $\bar{L}$ [1], here denoted by $(K, \tilde{\tilde{\alpha}})$. In this case $\tilde{\tilde{\alpha}}_2 = K^2$ and $\tilde{\tilde{\alpha}}_1 = -K^2$. In addition, we have

$$\begin{array}{ccc} (K, \tilde{\tilde{\alpha}}) & \xrightarrow{\bar{\varphi}} & (K, \tilde{\tilde{\alpha}}) \\ | & & | \\ (\bar{L}, \tilde{\alpha}) & \xrightarrow{\varphi} & (\bar{L}, \tilde{\alpha}) \\ | & & | \\ L & \xrightarrow{\mathrm{id}} & L \end{array}$$

Since K is a real closure of level 1 of L we know that $\mathrm{Aut}(K/L) = 1$. Therefore $\bar{\varphi}$ is the identity and this implies φ is the identity.

Next we assume n is even, then by Becker [7, Satz 3.7] $\bar{L}$ has exactly two orderings of level 1; $\gamma_2 = \bar{L}^2 \cup \tau\bar{L}^2$ and $\delta_2 = \bar{L}^2 \cup -\tau\bar{L}^2$ where $\tau \in \bar{L}$ such that $\tau \notin \bar{L}^2 \cup -\bar{L}^2$. Either $\varphi(\gamma_2) = \gamma_2$ and $\varphi(\delta_2) = \delta_2$ or $\varphi(\gamma_2) = \delta_2$ and $\varphi(\delta_2) = \gamma_2$. Note that $\tilde{\beta}_i \cap (\bar{L}^2 \cup -\bar{L}^2) = \emptyset$ for every odd i, but that $(\tilde{\beta}_i \cap L) \neq 0$ since $\tilde{\beta}$ is a faithful extension of β. Therefore we may pick τ in L but not in $(\bar{L}^2 \cup -\bar{L}^2)$ such that $\gamma_2 = \bar{L}^2 \cup \tau\bar{L}^2$ and $\delta_2 = \bar{L}^2 \cup -\tau\bar{L}^2$. We now see $\gamma_2 \cap L = L^2 \cup \tau L^2 \neq L^2 \cup -\tau L^2 = \delta_2 \cap L$. Thus $\gamma_2 \cap L$ and $\delta_2 \cap L$ are distinct orderings of level 1 on L. Furthermore, $\varphi(\delta_2 \cap L) = \delta_2 \cap L$ and $\varphi(\gamma_2 \cap L) = \gamma_2 \cap L$ since φ is the identity on L. Thus $\varphi(\gamma_2) = \gamma_2$ and $\varphi(\delta_2) = \delta_2$. Therefore any automorphism $\varphi: (\bar{L}, \tilde{\beta}) \rightarrowtail\!\!\!\!\rightarrow (\bar{L}, \tilde{\beta})$ sends each of the two orderings of level 1 of $\bar{L}$ to itself. Now, by the theory of ordinary real closed fields [1], we see that φ extends to an automorphism of a real closure of $(\bar{L}, \gamma)$, here denoted by $(K, \tilde{\alpha})$. Thus we have

$$\begin{array}{ccc} (K, \tilde{\gamma}) & \xrightarrow{\bar{\varphi}} & (K, \tilde{\gamma}) \\ | & & | \\ (\bar{L}, \gamma) & \xrightarrow{\varphi} & (\bar{L}, \gamma) \\ | & & | \\ L & \xrightarrow{\mathrm{id}} & L. \end{array}$$

Since $\text{Aut}(K/L) = 1$, the isomorphism φ is the identity. Therefore if $(\bar{L}, \tilde{\beta})$ is a real closure of level n of (L, β) then $\text{Aut}(\bar{L}/L) = 1$.

Lemma 3.5 allows us to add uniqueness to the statement of Lemma 3.3. We restate the lemma here as Theorem 3.6.

Theorem 3.6. *Let $\varphi: K \to L$ be an isomorphism from K onto L where α and β are orderings of exact level n of K and L respectively, such that $\varphi_*(\beta) = \alpha$. Then given any real closure $k(\alpha)$ of (K, α), φ can be extended to an isomorphism $\bar{\varphi}$ from $k(\alpha)$ onto some real closure, $\bar{L}$, of (L, β). The extension is unique in the sense that, for a particular $k(\alpha)$ and $\bar{L}$, if this is isomorphism exists it is unique.*

Proposition 3.7. *Let $A \xrightarrow{\varphi} B \xrightarrow{\Psi} C$ be ring homomorphisms and α, β and γ orderings of A, B and C respectively.*

i) *If $\varphi_*(\beta) = \alpha$ then for each $k(\beta)$, a real closure of $(k(\wp_\beta), \tilde{\beta})$, there exists a monomorphism from some real closure of $k(\wp_\alpha)$ into $k(\beta)$. This monomorphism is the unique monomorphism between these fields extending φ.*

ii) *If $\varphi_*(\beta) = \alpha$ and $\Psi_*(\gamma) = \beta$, and if $\tilde{\Psi}$ is an extension of Ψ mapping a real closure $k(\beta)$, of $(k(\wp_\beta), \tilde{\beta})$, into a real closure $k(\gamma)$, of $(k(\wp_\gamma), \tilde{\gamma})$, and $\tilde{\varphi}$ is an extension of φ mapping a real closure $k(\alpha)$, of $(k(\wp_\alpha), \tilde{\alpha})$, into $k(\beta)$, then there is an extension $\tilde{\theta}$ of $\Psi \cdot \varphi$ mapping $k(\alpha)$ into $k(\gamma)$ such that $\tilde{\Psi} \cdot \tilde{\varphi} = \tilde{\theta}$.*

Proof. i) By [4] the map φ induces an order preserving monomorphism $\bar{\varphi}: (k(\wp_\alpha), \tilde{\alpha}) \to (k(\wp_\beta), \tilde{\beta})$. Threrefore, we have $k(\wp_\alpha) \overset{\bar{\varphi}}{\rightarrowtail} M \subseteq k(\wp_\beta)$. Let $k(\beta)$ be a real closure of $(k(\wp_\beta), \tilde{\beta})$ and $\bar{M}$ be the algebraic closure of M in $k(\beta)$. Then by Lemma 3.2, the field $\bar{M}$ is real closed of level n. By Lemma 3.3, there is an extension, $\bar{\bar{\varphi}}^{-1}$ of $\bar{\varphi}^{-1}$, mapping $\bar{M}$ isomorphically onto some real closure of level n of $(k(\wp_\alpha), \tilde{\alpha})$, denote it by $k(\alpha)$. Thus $(\bar{\bar{\varphi}}^{-1})^{-1} = \bar{\bar{\varphi}}$ maps $k(\alpha)$ isomorphically onto $\bar{M} \subseteq k(\beta)$. If $\sigma: k(\alpha) \to k(\beta)$ is another such monomorphism then we have

$$\begin{array}{ccc} k(\alpha) & \overset{\bar{\bar{\varphi}}}{\rightarrowtail} & \bar{M} \subseteq k(\beta) \\ | & & \\ k(\wp_\alpha) & \overset{\bar{\varphi}}{\rightarrowtail} & M \subseteq k(\wp_\beta) \end{array} \quad \text{and} \quad \begin{array}{ccc} k(\alpha) & \overset{\sigma}{\rightarrowtail} & \bar{N} \subseteq k(\beta) \\ | & & \\ k(\wp_\alpha) & \overset{\bar{\varphi}}{\rightarrowtail} & M \subseteq k(\wp_\beta) \end{array}$$

where $\bar{N}$ is a real closed of level n by Corollary 3.4. In fact $\bar{N}$ is a real closure of M, since $k(\alpha)$ is algebraic over $k(\wp_\alpha)$ implies that $\bar{N}$ is algebraic over M. If $\bar{M} \neq \bar{N}$ then we consider $\overline{MN}$, the composite of $\bar{M}$ and $\bar{N}$ in $k(\beta)$. Then $\overline{MN}$ is algebraic over $\bar{M}$, and since $\overline{MN} \subseteq k(\beta)$ we see that the field $\overline{MN}$ has an ordering faithfully extending the ordering of M. But $\bar{M}$ is real closed, therefore $\overline{MN} = \bar{M}$. Similarly $\bar{N} = \overline{MN}$, so $\bar{M} = \bar{N}$. Therefore, by the uniqueness in Theorem 3.6, we have $\sigma = \tilde{\bar{\varphi}}$.

ii) We have the diagrams

$$\begin{array}{ccccc} k(\alpha) & \xrightarrow{\tilde{\varphi}} & k(\beta) & \xrightarrow{\tilde{\Psi}} & k(\gamma) \\ | & & | & & | \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\Psi} & C \end{array} \qquad \begin{array}{ccc} & & k(\gamma) \\ & & | \\ A & \xrightarrow{\Psi\cdot\varphi} & C \end{array}$$

Since $\tilde{\Psi}\cdot\tilde{\varphi}$ is an extension of $\Psi\cdot\varphi$ mapping $k(\alpha)$ monomorphically into $k(\gamma)$, such extensions exist. Let $\tilde{\theta}$ be any extension of $\Psi\cdot\varphi$ mapping $k(\alpha)$ monomorphically into $k(\gamma)$. By the uniqueness of such extensions in part (i) we have $\tilde{\theta} = \tilde{\Psi}\cdot\tilde{\varphi}$.

Theorem 3.8. *Let $\alpha \in R_n$-spec$A$, and $\varphi: A \to B$ a ring homomorphism, so that B may be viewed as an A algebra. Then we have the commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{i} & B \otimes_A k(\wp_\alpha) \\ \varphi| & & |j \\ A & \xrightarrow{\pi_\alpha} & k(\wp_\alpha) \end{array}$$

where $k(\wp_\alpha) = qf(A/\wp_\alpha)$, π_α is the canonical map $a \to a + \wp_\alpha$ and i and j are the canonical maps given by $i(b) = b \otimes 1$ and $j(c) = 1 \otimes c$. Let $\tilde{\alpha}$ be the ordering on $k(\wp_\alpha)$ induced by α, then the map i_* induces a homeomorphism between $j_*^{-1}(\tilde{\alpha})$ and $\varphi_*^{-1}(\alpha)$.

Proof. We first assume $j_*^{-1}(\tilde{\alpha})$ is not empty. Since $i\varphi = j\pi_\alpha$ we know that $\varphi_* i_* = \pi_{\alpha *} j_*$. This implies $\varphi_* i_*(j_*^{-1}(\tilde{\alpha})) = \pi_{\alpha *} j_*(j_*^{-1}(\tilde{\alpha})) = \pi_{\alpha *}(\tilde{\alpha}) = \alpha$, therefore $i_*(j_*^{-1}(\tilde{\alpha})) \subseteq \varphi_*^{-1}(\alpha)$. To show i_* maps $j_*^{-1}(\tilde{\alpha})$ onto $\varphi_*^{-1}(\tilde{\alpha})$ we first show that $\varphi_*^{-1}(\alpha) \subseteq$ image i_*. Let $\beta \in R_n$-spec$B$ with $\varphi_*(\beta) = \alpha$. By [4] there exists an order preserving map $\tilde{\varphi}: k(\wp_\alpha) \to k(\wp_\beta)$ such that $\tilde{\varphi}\pi_\alpha = \pi_\beta \varphi$. Therefore, we may define $\pi: B \otimes_A k(\wp_\alpha) \to k(\wp_\beta)$ via $\pi(b \otimes c) = \pi_\beta(b)\tilde{\varphi}(c)$, since $\pi_\beta(b)\tilde{\varphi}(c)$ is bilinear and balanced over A. By definition $\pi i = \pi_\beta$, thus if $\tilde{\beta}$ is the unique ordering on $k(\wp_\beta)$ extending β then $i_* \pi_*(\tilde{\beta}) = \pi_{\beta *}(\tilde{\beta}) = \beta$. Therefore $\beta \in$ image i_*. Let $\gamma \in$

R_n-spec$(B \otimes_A k(\wp_\alpha))$ such that $\beta = i_*(\gamma)$. Then $\alpha = \varphi_*(\beta) = \varphi_* i_*(\gamma) = \pi_{\alpha*} j_*(\gamma)$. Thus, $j_*(\gamma) = \tilde{\alpha}$, so that $\gamma \in j_*^{-1}(\tilde{\alpha})$. Hence, $\beta \in \varphi_*^{-1}(\alpha)$ implies $\beta \in i_*(j_*^{-1}(\tilde{\alpha}))$ and so $i_*(j_*^{-1}(\tilde{\alpha})) = \varphi_*^{-1}(\alpha)$

To show i_* is one-to-one on $j_*^{-1}(\tilde{\alpha})$, we first note that since $k(\wp_\alpha) = qf(A/\wp_\alpha)$, every element of $(B \otimes_A k(\wp_\alpha))$ can be written as $b \otimes 1/\bar{a}$ where $\bar{a} \in A/\wp_\alpha$ and $b \in B$. Therefore, to show two orderings on $(B \otimes_A k(\wp_\alpha))$ are the same it is enough to examine elements of the form $b \otimes a$. Let $\beta = i_*(\gamma) = i_*(\delta)$ with γ and δ in $j_*^{-1}(\tilde{\alpha})$. Then $1 \otimes c \in \gamma_k$ if and only if $1 \otimes c \in \delta_k$. Similarly $b \otimes 1 \in \gamma_j$ if and only if $b \otimes 1 \in \delta_j$. Since $b \otimes c = (b \otimes 1)(1 \otimes c)$ we see that $b \otimes c \in \delta_l$ if and only if $b \otimes c \in \gamma_l$. Therefore $\gamma = \delta$ and i_* is one-to-one on $j_*^{-1}(\tilde{\alpha})$.

We have shown $i_*: j_*^{-1}(\tilde{\alpha}) \to \varphi_*^{-1}(\tilde{\alpha})$ is a bijection. It remains to show i_* is homeomorphic on $j_*^{-1}(\tilde{\alpha})$. By [4, Prop. 4.4] the map i_* is continuous, so we need only show i_* is an open map. Note that $\varphi_*^{-1}(\alpha)$ is a Tychonoff closed subset of the compact Tychonoff space R_n-spec$B$, and $j_*^{-1}(\tilde{\alpha})$ is a Tychonoff closed subset of the compact Tychonoff space R_n-spec$(B \otimes_A k(\wp_\alpha))$. Therefore $i_*: j_*^{-1}(\tilde{\alpha}) \to \varphi_*^{-1}(\alpha)$ is a homeomorphism in the Tychonoff topology, since it is continuous and both spaces are compact. Now since $\delta \subset \gamma$ if and only if $i_*(\delta) \subset i_*(\gamma)$ we see by [4, Prop. 3.6] that i_* is an open map. Hence i_* is a homeomorphism from $j_*^{-1}(\tilde{\alpha})$ to $\varphi_*^{-1}(\alpha)$.

We now consider the case $j_*^{-1}(\tilde{\alpha}) = \emptyset$. Suppose $\varphi_*(\beta) = \alpha$ so that $\varphi_*^{-1}(\alpha) \neq \emptyset$. We shall construct an ordering γ on $B \otimes_A k(\wp_\alpha)$ such that $j_*(\gamma) = \tilde{\alpha}$. Define $b \otimes 1 \in \gamma_i$ if and only if $b \in \beta_i$, and define $1 \otimes c \in \gamma_j$ if and only if $c \in \tilde{\alpha}_j$. Now define $b \otimes c \in \gamma_k$ if and only if $(b \otimes 1)) \in \gamma_i$ and $(1 \otimes c) \in \gamma_j$, with i and j such that $k = i + j$ if $i + j \leq 2n$ and $k = i + j - 2n$ if $i + j > 2n$. As above, every element of $(B \otimes_A k(\wp_\alpha))$ may be written as $b \otimes c$, so we examine only $b \otimes c$. If $(b \otimes c) = (d \otimes e)$ then $(b \otimes c) = (b \otimes ef) = (b\varphi(f) \otimes e)$ for some $f \in k(\wp_\alpha)$, therefore it is easily shown that two representations of the same element lie in the same γ_i. It remains to verify that the γ_i form an ordering.

From the definition it is clear that $B \otimes_A k(\wp_\alpha) = \gamma_1 \cup \cdots \cup \gamma_{2n}$. If $(a \otimes b)$ lies in $(\gamma_i \cap \gamma_j)$ then either $a \in (\beta_r \cap \beta_s) = \wp_\beta$ for some r and s or $b \in (\tilde{\alpha}_m \cap \tilde{\alpha}_n) = 0$ for some m and n. Therefore $a \otimes b \in (\gamma_i \cap \gamma_j)$ if and only if $a \otimes b \in \wp_\beta \otimes_A k(\wp_\alpha)$. This is a prime ideal, since $\wp_\beta$ is prime and $k(\wp_\alpha)$ has no zero divisors.

To show $\gamma_i^* + \gamma_i^* \subseteq \gamma_i^*$, let $a \otimes b$ and $c \otimes d$ be in γ_i^*, then a and c do not lie in $\wp_\beta$, and neither b nor d are zero. As before, $a \otimes b = e \otimes 1/\bar{f}$ and $c \otimes d = g \otimes 1/\bar{f}$ where $e \otimes 1/\bar{f}$ and $g \otimes 1/\bar{f}$ lie in γ_i^*. Therefore $(a \otimes b) + (c \otimes d) = (d \otimes 1/\bar{f}) + (g \otimes 1/\bar{f}) = ((e+g) \otimes 1/\bar{f}) \in \varphi_i^*$ since e, g and $e+g$ are all in the same β_j^*.

Lastly, it is easy to see $\gamma_i^* \cdot \gamma_j^* \subseteq \gamma_k^*$ where $k = i+j$ if $i+j \leq 2n$ and $k = i+j-2n$ if $i+j > 2n$. Therefore γ is an ordering on $(B \otimes_A k(\wp_\alpha))$ with $j_*(\gamma) = \tilde{\alpha}$. But this contradicts $j_*^{-1}(\tilde{\alpha}) = \emptyset$, so $j_*^{-1}(\tilde{\alpha}) = \emptyset$ implies $\varphi_*^{-1}(\alpha) = \emptyset$.

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