Math. J. Okayama Univ. 35(1993), 41-43

ON AN AACDMZ QUESTION

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Let D be a (commutative) integral domain with quotient field K. Let $F(D)$ denote the set of nonzero fractional ideals of D and let $f(D)$ be the subset of finitely generated members of $F(D)$. For each $A \in F(D)$, we set $D:_K A = A^{-1}$ and $(A^{-1})^{-1} = A_v$. The function on $F(D)$ defined by $A \mapsto A_v$ is called the v-operation on D. If for each $A \in f(D)$, there exists a $B \in F(D)$ with $(AB)_v = D$, then D is called a v-domain. If there is a set of prime ideals $\{P_i \mid i \in I\}$ of D such that $D = \bigcap_{i \in I} D_{P_i}$ and each D_{P_i} is a valuation domain, then D is called an essential domain. [1] investigated characterizations of v-domains and related properties. Among other Theorems it proved the following,

Theorem 1 ([1, Theorem 7]).
(1) *If D is an essential domain, then*

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_1, \cdots, A_n \in f(D)$.
(2) *If D is integrally closed and*

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_1, \cdots, A_n \in f(D)$, then D is a v-domain.

Relating with Theorem 1 it posed the following,

Question ([1, p.7]). Does any v-domain D satisfy

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_i \in f(D)$?

The aim of this paper is to give an affirmative answer to the question. We will prove the following,

Theorem 2. *Let D be a v-domain. Then we have*

$$(A_1 \cap \cdots \cap A_n)_v = (A_1)_v \cap \cdots \cap (A_n)_v$$

for all $A_i \in f(D)$.

First we recall the definition and some properties of the Kronecker function ring of D with respect to the v-operation. Let $D[X]$ be the polynomial ring of an indeterminate X over D. For each $f \in K[X]$, we denote the fractional ideal of D generated by the coefficients of f by $c(f)$.

Lemma 3 (cf. [2,(32.7)]). *Let D be a v-domain. Set*

$$D^v = \{0\} \cup \{f/g \mid f,g \in D[X] - \{0\} \quad \text{and} \quad c(f)_v \subset c(g)_v\}.$$

Then,

(1) *D^v is a domain with quotient field $K(X)$.*

(2) *If A is a nonzero finitely generated ideal of D, then $AD^v \cap K = A_v$.*

D^v is called the Kronecker function ring of D with respect to the v-operation.

Lemma 4. *Let D be a v-domain. Let $a \in K - \{0\}$ and $C \in F(D)$. If $aA_v \subset B_v$ and $BA^{-1} \subset C$ are satisfied for some $A \in f(D)$ and some $B \in F(D)$, then $a \in C_v$.*

Proof. We note that $(AA^{-1})_v = D$, since D is a v-domain. Then we have

$$a \in a(AA^{-1})_v = (aA_vA^{-1})_v \subset (B_vA^{-1})_v = (BA^{-1})_v \subset C_v.$$

Proof of Theorem 2. Let D be a v-domain with quotient field K. Let D^v be the Kronecker function ring of D with respect to the v-operation. Let $A_1, \cdots, A_n \in f(D)$. Choose elements $a_{i1}, \cdots, a_{ik(i)}$ of $K - \{0\}$ such that $A_i = (a_{i1}, \cdots, a_{ik(i)})D$ for $1 \leq i \leq n$. We set

$$f_i = a_{i1}X + a_{i2}X^2 + \cdots + a_{ik(i)}X^{k(i)}$$

for $1 \leq i \leq n$. Since, for each j, $a_{ij}/f_i \in D^v$, we have $A_iD^v = f_iD^v$ for $1 \leq i \leq n$. Set $h_i = f_1 \cdots f_{i-1}f_{i+1} \cdots f_n$, and let $\mathrm{d}(i)$ denote the degree of h_i for $1 \leq i \leq n$. We set

$$h_1 + h_2X^{\mathrm{d}(1)} + h_3X^{\mathrm{d}(1)+\mathrm{d}(2)} + \cdots + h_nX^{\mathrm{d}(1)+\cdots+\mathrm{d}(n-1)} = g.$$

Since, for each j,$h_j/g \in D^v$, it immediately follows that $(h_1, \cdots, h_n)D^v = gD^v$, and so

$$(1/f_1, \cdots, 1/f_n)D^v = (g/(f_1 \cdots f_n))D^v.$$

By taking the inverses, we see that

$$f_1 D^v \cap \cdots \cap f_n D^v = ((f_1 \cdots f_n)/g) D^v.$$

Now let $0 \neq a \in (A_1)_v \cap \cdots \cap (A_n)_v$. Tten we have

$$a \in f_1 D^v \cap \cdots \cap f_n D^v = ((f_1 \cdots f_n)/g) D^v.$$

It follows $ag/(f_1 \cdots f_n) \in D^v$. Hence we have $ac(g)_v \subset c(f_1 \cdots f_n)_v$. On the other hand, we have

$$c(f_1, \cdots, f_n)\, c(g)^{-1} \subset A_1 \cap \cdots \cap A_n,$$

since for each i,

$$\begin{aligned} c(f_1, \cdots, f_n)\, c(g)^{-1} &= c(f_i h_i)(c(h_1) + \cdots + c(h_n))^{-1} \\ &\subset c(f_i h_i)\, c(h_i)^{-1} \subset c(f_i) \\ &= A_i. \end{aligned}$$

Then Lemma 4 can be applied to obtain $a \in (A_1 \cap \cdots \cap A_n)_v$. Thus

$$(A_1)_v \cap \cdots \cap (A_n)_v \subset (A_1 \cap \cdots \cap A_n)_v.$$

Since the reverse containment is obvious, the proof is now complete.

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(*Received August 17, 1992*)