Math. J. Okayama Univ. 35(1993), 29-34

STRUCTURE OF p-SOLVABLE GROUPS WITH THREE p-REGULAR CLASSES II

YASUSHI NINOMIYA

J. B. Olsson and his student Madsen kindly pointed out that the proof of [2, Lemma 3.1] is incorrect and gave counterexamples to the lemma. The counterexamples show that there are missing groups in the list of [2, Theorem B]. I thank them for their care and for bringing this error to my attention. In this note, we give all the missing groups by proving Theorem below.

We preserve the notation of [2]. In particular, recall that given a finite group G, $r_{p'}(G)$ and $\pi(G)$ denote, respectively, the number of p-regular classes in G and the set of primes diving the order of G.

Let G be a finite p-nilpotent group with $O_p(G) = \{1\}$ and assume $r_{p'}(G) = 3$. Then $|\pi(V)| \leq 2$, where $V = O_{p'}(G)$. Let $|\pi(V)| = 2$, and set $\pi(V) = \{q, r\}$. Let Q and R be Sylow q- and r-subgroups of V respectively. By our assumption, V is a Frobenius group, and so we may assume that R is the Frobenius kernel of V. Then $r_{p'}(G/R) = 2$. In the proof of [2, Lemma 3.1], we applied Theorem A to the group G/R. For this G/R has to satisfy the condition that $O_p(G/R) = \{1\}$. However, G/R does not always satisfy the condition. The purpose of this note is to give all the isomorphism classes of finite p-nilpotent groups G with $O_p(G) = \{1\}$ which satisfy $r_{p'}(G) = 3$ and $|\pi(V)| = 2$.

Theorem. *Let G be a finite p-nilpotent group with $O_p(G) = \{1\}$. Suppose $r_{p'}(G) = 3$. If $|\pi(O_{p'}(G))| = 2$ then one of the following holds:*

(1) $p \neq 2$ *and* $G \simeq \mathbb{Z}_r \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{p^n})$, *where* $r = 2p^n + 1$ is a prime.

(2) $p \neq 2, 3$ *and* $G \simeq E_{3^l} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{p^n})$, *where* $3^l = 2p^n + 1$.

(3) $p = 2$ *and* $G \simeq E_{5^2} \rtimes H$, *where* $H = \langle w, a\rangle$; $w^3 = a^8 = 1$, $a^{-1}wa = w^{-1}$.

(4) $p = 2$ *and* $G \simeq E_{5^2} \rtimes H$, *where* $H = \langle w, a, b\rangle$; $w^3 = a^8 = b^2 = 1$, $a^{-1}wa = w$, $b^{-1}wb = w^{-1}$, $b^{-1}ab = a^5$.

(5) $p = 2$ *and* $G \simeq E_{3^4} \rtimes H$, *where* $H = \langle w, a, b\rangle$; $w^5 = a^8 = 1$, $b^4 = a^4$, $a^{-1}wa = w$, $b^{-1}wb = w^2$, $b^{-1}ab = a^3$.

(6) *$p = 2$ and $G \simeq E_{3^4} \rtimes H$, where $H = \langle w, a, b\rangle$; $w^5 = a^{16} = b^4 = 1$, $a^{-1}wa = w$, $b^{-1}wb = w^2$, $b^{-1}ab = a^{11}$.*

Although we used Lemma 3.1 in the proof of [2, Proposition 3.3] and in [2, Sections 4 and 5], we can now assume, in virtue of Theorem, that $V = O_{p'}(G)$ is a q-group for some prime $q \neq p$, so that the failure of the lemma affects nothing. Nevertheless, we need to fill a gap in the proof of Proposition 3.3. In the proof, we applied Theorem A to the group G/V_0. For this it is necessary to show that $O_p(G/V_0) = \{1\}$. Madsen simplified my original proof and here we give the proof due to him: Let $O_p(G/V_0) = SV_0/V_0$ where S is a p-group. Since

$$[O_p(G/V_0),\ O_{p'}(G/V_0)] = \{1\},$$

we get $[S, V] \subset V_0$, and so S acts trivially on V/V_0. Thus noting that $V_0 = \Phi(V)$, we have $S \subset C_G(V)$ by [1, Corollary 5.1.4]. On the other hand, G is q-solvable and $O_{q'}(G) = O_p(G) = \{1\}$, $O_q(G) = V$. Hence $C_G(V) \subset V$ by [1, Theorem 6.3.2]. Therefore $S \subset S \cap V = \{1\}$. This proves that $O_p(G/V_0) = \{1\}$.

Thus we see that [2, Theorem B], which we will use in the proof of our theorem, holds except when G is p-nilpotent and $|\pi(O_{p'}(G))| = 2$. To prove the theorem we need a number theoretical lemma.

Lemma. *Let $q = 2^x + 1$ be a Fermat prime. If a prime number r satisfies the relation $r^l - 1 = 2^m q$ for some $l \geq 2$ and $m \geq 1$ then one of the following holds:*

(1) $r = 3,\ l = 4,\ q = 5$;
(2) $r = 5,\ l = 2,\ q = 3$;
(3) $r = 7,\ l = 2,\ q = 3$.

Proof. Since $2^m q = (r-1)(r^{l-1} + \cdots + 1)$ and q is prime, q is a divisor of $r - 1$ or $r^{l-1} + \cdots + 1$. We distinguish two cases:

Case 1. $q|(r^{l-1} + \cdots + 1)$. We show that (1) or (2) holds. Write $r - 1 = 2^a$, $r^{l-1} + \cdots + 1 = 2^b q$, where $a + b = m$. Then $b \neq 0$, for otherwise $r^{l-1} + \cdots + r = q - 1 = 2^x$, which is impossible. Hence l is even. We first show that if $a = 1$ then (1) holds. In this case, as $r = 3$, $l > 2$. If $2 \parallel l$ then because

$$3^l - 1 = (3^2 - 1)(3^{l-2} + \cdots + 3^2 + 1)$$

and $3^{l-2}+\cdots+3^2+1$ is odd, we have $3^{l-2}+\cdots+3^2+1=q$, and so $3^{l-2}+\cdots+3^2=q-1=2^x$, which is impossible. Hence $4|l$. Then, as $(3^4-1)|(3^l-1)$, we have $q=5$. Set $l=4l'$. Then $5\cdot 2^m=3^l-1=(3^4-1)(3^{4(l'-1)}+\cdots+3^4+1)$. This shows that $3^{4(l'-1)}+\cdots+3^4+1$ is a power of 2, and we have $l'=1$. Indeed, if $l'\neq 1$ then l' is even and $(3^8-1)|(3^l-1)$, and so $5\cdot 41|(3^l-1)$, which is not the case. Thus (1) holds. We next show that if $a>1$ then (2) holds. Since $2(2^{a-1}+1)=(r+1)|2^b q$, we have $q=2^{a-1}+1$. Since q and $r=2^a+1$ are both Fermat primes, $a-1$ and a are both powers of 2. This forces a to be 2, and so $q=3$ and $r=5$. The equality

$$2^m q=5^l-1=(5^2-1)(5^{l-2}+\cdots+5^2+1)$$

implies that $5^{l-2}+\cdots+5^2+1$ is a power of 2. Suppose $l\neq 2$. Then $l/2$ must be even. Then $(5^4-1)|(5^l-1)$, and so $3\cdot 13|(5^l-1)$, which is not the case. We therefore have $l=2$, and hence (2) holds.

Case 2. $q|(r-1)$. We show that (3) holds. Write $r-1=2^a q$, $r^{l-1}+\cdots+1=2^b$, where $a+b=m$. Since l must be even, $(r+1)|(r^{l-1}+\cdots+1)=2^b$. Hence $r+1=2^{b'}$ for some $b'\leq b$. Then $2^a q=r-1=2(2^{b'-1}-1)$. Thus we have $q=2^{b'-1}-1$. Therefore, as q is a Fermat prime, we have $b'=3$ and consequently $q=3$ and $r=7$. From the equality

$$7^l-1=(7^2-1)(7^{l-2}+\cdots+7^2+1),$$

it follows that $7^{l-2}+\cdots+7^2+1$ is a power of 2. This forces l to be 2, for otherwise $l/2$ is even and $(7^4-1)|(7^l-1)$, and so $3\cdot 5^2|(7^l-1)$, which is not the case. Hence (3) holds.

Proof of Theorem. Set $V=O_{p'}(G)$ and $\pi(V)=\{q,r\}$ and let Q and R be Sylow q- and r-subgroups of V respectively. Because V is a Frobenius group, we may assume that R is the Frobenius kernel of V. Then we have $Q\simeq \mathbb{Z}_q$. If $O_p(G/R)=\{1\}$, then we can apply [2, Theorem A] to G/R, and we reach a contradiction by using an argument in the proof of [2, Lemma 3.1]. We therefore have $O_p(G/R)\neq\{1\}$. Now let H be a complement of R in G and let P be a Sylow p-subgroup of H. We first assume that $O_p(H)=P$. Then as $r_{p'}(H)=2$, we have $|Q|=2$. Since H acts transitively on $R^\#$, $R^\#$ is a union of two orbits under the action of P. Therefore (7) and (8) of [2, Theorem B] apply to the group $R\rtimes P$. Thus (1) or (2) holds in this case. We next assume that $O_p(H)\neq P$ and set $O_p(H) = S$. Since $r_{p'}(H/S) = 2$ and Q is cyclic, by [2, Theorem A(d)], $p = 2$ and q is a Fermat prime. Set $|P| = 2^n$ and $|R| = r^l$. Since H acts transitively on $R^{\#}$, we have $r^l - 1 = 2^m q$, $1 \leq m \leq n$. We note that H is a nonabelian group contained isomorphically in Aut R. Hence $l \geq 2$. Therefore by Lemma we have the following possibilities:

$$(r,\ l,\ q) = (3,\ 4,\ 5) \text{ or } (5,\ 2,\ 3) \text{ or } (7,\ 2,\ 3).$$

We show that for the first case (5) or (6) holds; for the second case (3) or (4) holds; and the third case is impossible. Now let $(r,\ l,\ q) = (5,\ 2,\ 3)$. Since $r_{2'}(H/S) = 2$ and $|Q| = 3$, we have $|P : S| = 2$. Further, by $|R^{\#}| = 3 \cdot 8$, we have $|P| \geq 8$, and so $|S| \geq 4$. We may regard H as a subgroup of GL(2, 5). Then because $[Q,\ S] = \{1\}$, S centralizes a cyclic subgroup of GL(2, 5) of order 3, which is a Sylow 3-subgroup of GL(2, 5). Now choose an element $u = \begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix}$ of GL(2, 5). Because u is of order 3, we may assume that S centralizes u. We can see that $C_{\mathrm{GL}(2,\ 5)}(u) = \langle u,\ a\rangle$, where $a = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, which implies that $|S| \leq 8$ because a is of order 8. Hence $|S| = 4$ or 8. We show that if $|S| = 4$ then (3) holds. In this case, because $|P| = 8$, P acts semiregularly on $R^{\#}$, and $P \simeq \mathbb{Z}_8$ or Q_8. Because $a^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is a central element of GL(2, 5), we see that S is a central subgroup of H. If $P \simeq Q_8$, each element of order 4 lying in P is not a central element. This shows that $P \simeq \mathbb{Z}_8$, and hence (3) holds in this case. Indeed, $w = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ and a satisfy the relation described in (3), and the semidirect product of the 2-dimensional vector space over GF(5) by $H = \langle w,\ a\rangle$ is a group of type (3). Suppose next $|S| = 8$. A Sylow 2-subgroup of $N_{\mathrm{GL}(2,5)}(\langle u\rangle)$ is given by $\langle a,\ b\rangle$, where $b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and its order is 16. Therefore, because P is of order 16 and normalizes $\langle u\rangle$, we have $P \simeq \langle a,\ b\rangle$. This shows that (4) holds. We next consider the case $(r,\ l,\ q) = (3,\ 4,\ 5)$. Since $r_{2'}(H/S) = 2$ and $|Q| = 5$, we have $P/S \simeq \mathbb{Z}_4$. Further, by $|R^{\#}| = 5 \cdot 16$, $|P| \geq 16$ and $|S| \geq 4$. By regarding H as a subgroup of GL(4, 3), we see that S centralizes a cyclic subgroup of GL(4, 3) of order 5, which is a Sylow 5-subgroup of GL(4, 3).

An element

$$u = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

of GL(4, 3) is of order 5. Hence we may assume that S centralizes u. Now the elements

$$s = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

are of order 16 and 4 respectively and

$$C_{\mathrm{GL}(4,\,3)}(u) = \langle u,\ s\rangle, N_{\mathrm{GL}(4,3)}(\langle u\rangle) = \langle u,\ s,\ t\rangle,$$

where the action of t on $\langle u,\ s\rangle$ is given by

$$t^{-1}ut = u^2, t^{-1}st = s^{11}.$$

Thus obviously $4 \leq |S| \leq 16$. We first show that the case $|S| = 4$ is impossible. Suppose $|S| = 4$. Then $|P| = 16$ and so P acts semiregularly on $R^{\#}$. Hence $P \simeq \mathbb{Z}_{16}$ or Q_{16}. Because P has a cyclic group of order 4 as a factor group, it is not isomorphic to Q_{16}. On the other hand, if $P \simeq \mathbb{Z}_{16}$ then $P = \langle st^2\rangle$ because $P \neq \langle s\rangle$ and the subgroups of $\langle s, t\rangle$ of order 16 are $\langle s\rangle$ and $\langle st^2\rangle$. Hence, because $(st^2)^2 = s^{10}$ and s^{10} centralizes u, the maximal subgroup of P centralizes u, which contradicts the fact that $S = C_P(u)$. Thus we have $|S| = 8$ or 16. We now show that if $|S| = 8$ then (5) holds. Since $|P| = 32$, by regarding P as a subgroup of $\langle s, t\rangle, P = \langle s^2,\ t\rangle$ or $\langle s^2,\ st\rangle$. For the former case, we can check that $R^{\#}$ is a union of two orbits under the actionof $H = \langle u,\ s^2, t\rangle$. This is not the case. On the other hand, for the latter case, $H = \langle u,\ s^2,\ st\rangle$ acts transitively on $R^{\#}$. Hence, setting $w = u$, $a = s^2$, $b = st$, we see that (5) holds. Assume $|S| = 16$. Then P is a Sylow 2-subgroup $\langle s,\ t\rangle$ of $N_{\mathrm{GL}(4,\,3)}(\langle u\rangle)$. Because $H = \langle u,\ s,\ t\rangle$ contains $\langle u,\ s^2,\ st\rangle$, which acts transitively on $R^{\#}$, it acts transitively on $R^{\#}$. Thus (6) holds. In final, we show that the case $(r,\ l,\ q) = (7,\ 2,\ 3)$ is impossible. Suppose by way of contradiction that this case occurs. Since $r_{2'}(H/S) = 2$, $|Q| = 3$ and $|R^{\#}| = 3 \cdot 16$, we have $|P/S| = 2$ and $|P| \geq 16$. We may regard H as a subgroup of GL(2, 7). A Sylow 3-subgroup of GL(2, 7) is given by $\langle u, v\rangle$, where $u = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $v = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$. Because u is a central element, we may assume that S centraliezes v. So $|S|$ is a divisor of $|C_{\mathrm{GL}(2,\,7)}(v)|$, but it is impossible, for $|S| \geq 8$ and $|C_{\mathrm{GL}(2,\,7)}(v)| = 4 \cdot 9$. This shows that the case $(r, l, q) = (7, 2, 3)$ does not occur, and the theorem is proved.

In conclusion, by adding six types of groups given in Theorem to the list of [2, Theorem B], we obtain all the finite p-solvable groups G with $O_p(G) = \{1\}$ which have exactly three p-regular classes.

REFERENCES

[1] D. GORENSTEIN: Finite Groups, Harper & Row, New York, 1968.
[2] Y. NINOMIYA: Structure of p-solvable groups with three p-regular classes, Canad. J. Math. **43**(1991), 559–579.

DEPARTMENT OF MATHEMATICS
FACULTY OF LIBERAL ARTS
SHINSHU UNIVERSITY
MATSUMOTO 390, JAPAN

(*Received September 24, 1992*)