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HADAMARD TOURNAMENTS OF ORDER 23

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1. Introduction The purpose of the present paper is to classify Hadamard tournaments of order 23. There exist precisely twenty four inequivalent Hadamard tournaments of order 23 (§4).

Now it is well known that Hadamard tournaments are directly related with skew Hadamard matrices [4]. First we clarify the relation between Hadamard matrices and Hadamard 2-designs introducing the concept of Hadamard designs (§2). An equivalence class of Hadamard designs corresponds to an equivalence class of Hadamard matrices. Once if we know the order of the automorphism group of an Hadamard design D of size $2n$, then we know the number of inequivalent Hadamard 2-designs involved in H. For instance, if the automorphism group of D has order two, then there exist precisely $n(n-1)$ inequivalent Hadamard 2-designs involved in D. Next to clarify the relation between skew Hadamard matrices and Hadamard tournaments we take an approach by skew labelings of blocks (§3). There we propose two questions on skew labelings which are practically important and are verified affirmatively for order 23. At any rate our classification principle relies on [3].

2. Hadamard designs, Hadamard 3-designs and Hadamard 2-designs. We begin with the definition of an Hadamard design.

Definition. Let n be a positive multiple of four and $P=\{1, 2, \cdots, n, 1^\circ, 2^\circ, \cdots, n^\circ\}$ a $2n$-set. Elements of P are called points. Further let $B=\{a_1, a_2, \cdots, a_n, a_1{}^\circ, a_2{}^\circ, \cdots, a_n{}^\circ\}$ be a family of n-subsets of P with $a_i{}^\circ = P - a_i$, $1 \leqq i \leqq n$. Elements of B are called blocks. Now $D=(P, B)$ is called an Hadamard design if the following conditions are satisfied:

(1) Each point is contained precisely in n blocks. Namely D is a 1-design.

(2) Each pair of points except $\{i, i^\circ\}$, $1 \leqq i \leqq n$, is contained precisely in $n/2$ blocks. $\{i, i^\circ\}$, $1 \leqq i \leqq n$ is contained in no blocks.

(3) Each trio of points not containing $\{i, i^\circ\}$, $1 \leqq i \leqq n$, is contained precisely in $n/4$ blocks.

(4) Each pair of blocks except $\{a_i,\ a_i{}^{\circ}\}, 1 \leqq i \leqq n$, meets precisely in $n/2$ points.

(5) Each trio of blocks not containing $\{a_i,\ a_i{}^{\circ}\}$, $1 \leqq i \leqq n$, meets precisely in $n/4$ points.

A permutation σ on P is an automorphism of D if $B\sigma = B$. The set of all automorphisms of D forms a group, the automorphism group $\mathrm{Aut}D$ of D.

Proposition 1. *If σ is an automorphism of D, Then $i^{\circ}\sigma = (i\sigma)^{\circ}$.*

Proof. Otherwise, there exists a block containing $i\sigma$ and $i^{\circ}\sigma$. Then there exists a block containing i and i°.

Proposition 2. *$\omega = (1,\ 1^{\circ})(2, 2^{\circ})\cdots(n,\ n^{\circ})$ belongs to the center of* $\mathrm{Aut}D$.

Proof. Let σ be any automorphism of D. Then $i\omega\sigma\omega = i^{\circ}\sigma\omega = (i\sigma)^{\circ}\omega = i\sigma$.

Let a be a fixed block of D. Then put $D(a) = (P(a),\ B(a))$, where $P(a) = a$ and $B(a) = \{b \cap a;\ b \in B,\ b \neq a,\ a^{\circ}\}$.

Proposition 3. *If $b \cap a = c \cap a$, where $b,\ c \in B$ and $b,\ c \neq a$, then $b = c$.*

Proof. If $b \neq c$, then $b \cap a = c \cap a = b \cap c \cap a$. We have that $|b \cap a| = n/2$ and $|b \cap c \cap a| = n/4$ by definition of D.

Proposition 4. *$D(a)$ is a 3-design. Actually $D(a)$ is called an Hadamard 3-design.*

Proof. Take any three distinct points i, j and k of $P(a)$. Then there exist precisely $n/4$ blocks of B which contain i, j and k including a. Thus there exist presisely $(n/4) - 1$ blocks of $D(a)$ containing i, j and k.

We also notice that there exist precisely $(n/2) - 1$ blocks of $D(a)$ containing i and j, and there exist precisely $n-1$ blocks of $D(a)$ containing i. Furthermore, the size of blocks of $D(a)$ equals $n/2$, and the intersection of any two distinct blocks of $D(a)$ not of the form $b \cap a$, $b^{\circ} \cap a$ meets in $n/4$ points.

Proposition 5. *Let a and b be two blocks of D. Further assume that $D(a)$ and $D(b)$ are isomorphic, and that σ is an isomorphism from $D(a)$ to $D(b)$. Then σ can be extended to an automorphism of D.*

Proof. We have that $a\sigma = b$. For any $i \in a$ it holds that $i\sigma = j \in b$. Thus by defining $i^\circ\sigma = j^\circ$, σ can be regarded as a permutation on P. We show that $B\sigma = B$. Let c be a block of D and k a point of c. If k belongs to $a \cap c$, then $k\sigma = \ell$ belongs to $b \cap d$, where d is a block of D. If k belongs to $a^\circ \cap c$, then k° belongs to $a \cap c^\circ$. In this case $\ell^\circ = k^\circ\sigma$ belongs to $b \cap d^\circ$ which implies that ℓ belongs $b^\circ \cap d$. Thus we obtain that $c\sigma = d$.

Proposition 6. *The automorphism group of $D(a)$ can be regarded as the stabilizer of a in* Aut$D$.

Proof. This is immediate from Proposition 5.

We say that $D(a)$ is involved in D. Then the proof of Proposition 5 can be modified to show that two Hadamard designs involving equivalent Hadamard 3-designs are equivalent.

Proposition 7. *Let i be a point of $D(a)$ and put $P(a,\ i) = a - \{i\}$ and $B(a,\ i) = \{b \cap a - \{i\};\ i \in b \in B(a)\}$. Then $D(a,\ i)=(P(a,\ i),\ B(a,\ i))$ is a symmetric 2-$(n-1, (n/2)-1,\ (n/4)-1)$ design which is called an Hadamard 2-design.*

Proof. It is straightforward.

Proposition 8. *If $D(a,\ i)$ and $D(b,\ j)$ are equivalent, then $D(a)$ and $D(b)$ are equivalent.*

Proof. Let σ be an isomorphism from $D(a,\ i)$ to $D(b,\ j)$. Then $(a - \{i\})\sigma = b - \{j\}$, and for any block $c \neq a$ of $D(a)$ containing i there exists some block $d \neq b$ of $D(b)$ containing j such that $(c \cap a - \{i\})\sigma = d \cap b - \{j\}$. Now we define $i\sigma = j$. Then we have that $(c \cap a)\sigma = d \cap b$, which implies that $(c^\circ \cap a)\sigma = d^\circ \cap b$. So σ maps every block of $D(a)$ to a block of $D(b)$.

We say that $D(a,\ i)$ is involved in $D(a)$ and also in D. Then the proofs of Propositions 5 and 8 can be modified to show that two Hadamard designs involving equivalent Hadamard 2-designs are equivalent.

Proposition 9. *The automorphism group of $D(a, i)$ can be regarded as the stabilizer of i in the automorphism group of $D(a)$.*

Proof. This is immediate from Proposition 8.

3. Skew Hadamard designs and Hadamard tournaments. We introduce the following definition.

Definition. Let D an Hadamard design. Then a labeling of blocks of D with the following properties is called skew:

(1) For every i the block $a(i)$ contains i°,

and

(2) For $i \neq j$ $a(i)$ contains j if and only if $a(j)$ contains i°.

If D allows a skew labeling, then D is also called skew.

Proposition 10. *Let D be a skew Hadamard design. Then for every i $D(a(i), i^\circ)$ can be regarded as an Hadamard tournament. If $a(i)$ and $a(j)$ are inequivalent, then $D(a(i), i^\circ)$ and $D(a(j), j^\circ)$ are inequivalent.*

Proof. We regard $D(a(i), i^\circ)$ as a digraph as follows: For simplicity of notation we normalize $a(i)$ so that $a(i) - \{i^\circ\} = \{1, 2, \cdots, i-1, i+1, \cdots, n\}$. The set of vertices is $a(i) - \{i^\circ\}$, and the out-neighborhood of the vertex j is $a(i) \cap a(j) - \{i^\circ\}$. Now since D is skew, for two distinct vertices j and k $a(i) \cap a(j) - \{i^\circ\}$ contains k if and only if $a(i) \cap a(k) - \{i^\circ\}$ does not contain j. So $D(a(i), i^\circ)$ is a tournament. Further we have that $|a(i) \cap a(j) \cap a(k) - \{i^\circ\}| = (n/4) - 1$. So it is easy to see that $D(a(i), i^\circ)$ is an Hadamard tournament. The second assertion is obvious by Proposition 8.

Conversely let $D(a, i)$ be an Hadamard 2-design. Then a labeling of blocks of $D(a, i)$ with the following properties is called skew: For simplicity of notation we normalize a so that $a - \{i\} = \{1, 2, \cdots, i-1, i+1, \cdots, n\}$. Then

(1) For every point j of $D(a, i)$ $a \cap a(j) - \{i\}$ does not contain j,

and

(2) For two distinct points j and k of $D(a, i)$ $a \cap a(j) - \{i\}$ contains k if and only if $a \cap a(k) - \{i\}$ does not contain j.

If $D(a, i)$ allows a skew labeling, then $D(a, i)$ is also called skew.

Then it is easy to see that a skew Hadamard 2-design can be regarded as an Hadamard tournament and vice versa. Further by labeling a as $a(i^{\circ})$ it is easy to see that if an Hadamard design D involves a skew Hadamard 2-design, then D is skew.

Finally we summarize our situation to list inequivalent Hadamard tournaments; Let n be given. Then

(i) List all skew Hadamard designs of size n.

(ii) In each skew Hadamard design list all inequivalent Hadamard 3-designs involved.

(iii) In each such Hadamard 3-design list all skew Hadamard 2-designs.

However, we mention the following two facts which we have checked through for $n = 24$. We wonder these facts may be true in general.

Fact 1. If Hadamard 2-designs $D(a,\ i)$ and $D(a,\ j)$ allow skew labelings, then i and j belong to the same orbit of the automorphism group of the Hadamard 3-design $D(a)$.

Fact 2. If an Hadamard 2-design $D(a,\ i)$ allows two skew labelings, the resulting two tournaments are equivalent.

Thus we reach our goal inspecting all inequivalent Hadamard 3-designs of size 24. There exist twenty four Hadamard tournaments of order 23. The list is given in the next section.

4. The list of Hadamard tournaments of order 23. We list the out-neighborhood of each vertex together with the automorphism group. The labels of Hadamard 3-designs and Hadamard matrices are those of [1].

(1) $D10$ at 11 ($H8$).
1-2,3,5,9,10,14,17,19,20,21,23
2-3,5,6,7,11,13,15,16,17,21,23
3-4,5,6,8,9,12,16,17,18,20,21
4-1,2,6,9,11,13,16,18,19,20,23
5-4,7,9,10,11,13,14,15,17,18,20
6-1,5,8,10,11,12,13,14,16,17,19
7-1,3,4,6,10,14,15,16,18,19,21
8-1,2,4,5,7,12,15,17,18,19,23
9-2,6,7,8,10,12,14,15,16,20,23
10-2,3,4,8,11,12,13,14,18,21,23
11-1,3,7,8,9,12,13,15,19,20,21
12-1,2,4,5,7,13,14,16,20,21,22
13-1,3,7,8,9,14,16,17,18,22,23
14-2,3,4,8,11,15,16,17,19,20,22
15-1,3,4,6,10,12,13,17,20,22,23
16-1,5,8,10,11,15,18,20,21,22,23
17-4,7,9,10,11,12,16,19,21,22,23
18-1,2,6,9,11,12,14,15,17,21,22
19-2,3,5,9,10,12,13,15,16,18,22
20-2,6,7,8,10,13,17,18,19,21,22
21-4,5,6,8,9,13,14,15,19,22,23
22-1,2,3,4,5,6,7,8,9,10,11
23-3,5,6,7,11,12,14,18,19,20,22.

Aut(1) *is of order* 55 *and it is generated by*

σ =(1,6,7,9,5)
(2,11,13,8,4)
(12,18,23,13,21)
(15,20,17,19,16)

and

τ =(1,8,10,7,11,6,9,5,2,4,3)
(12,14,15,13,16,20,17,23,18,
21,19).

(2) *D*28 at 4 (*H*16)

1-2,5,6,8,10,12,13,16,19,21,23
2-3,4,7,8,10,13,14,15,19,21,22
3-1,5,7,8,9,12,14,16,18,21,22
4-1,3,6,9,10,12,13,15,18,22,23
5-2,4,6,7,9,14,15,16,18,19,23
6-2,3,8,9,11,13,14,17,18,21,23
7-1,4,6,8,11,12,15,17,18,19,21
8-4,5,9,10,11,15,16,17,21,22,23
9-1,2,7,10,11,12,14,17,19,22,23
10-3,5,6,7,11,13,16,17,18,19,22
11-1,2,3,4,5,12,13,14,15,16,17
12-2,5,6,8,10,14,15,17,18,20,22
13-3,5,7,8,9,12,15,17,19,20,23
14-1,4,7,8,10,13,16,17,18,20,23
15-1,3,6,9,10,14,16,17,19,20,21
16-2,4,6,7,9,12,13,17,20,21,22
17-1,2,3,4,5,18,19,20,21,22,23
18-1,2,8,9,11,13,15,16,19,20,22
19-3,4,6,8,11,12,14,16,20,22,23
20-1,2,3,4,5,6,7,8,9,10,11
21-4,5,9,10,11,12,13,14,18,19,20
22-1,5,6,7,11,13,14,15,20,21,23
23-2,3,7,10,11,12,15,16,18,20,21.

Aut(2) *is trivial.*

(3) *D*29 at 3 (*H*16)

1-2,4,8,9,10,12,14,16,20,22,23
2-4,5,6,7,10,12,15,17,19,20,23
3-1,2,6,7,8,15,16,17,21,22,23
4-3,5,7,8,9,12,14,15,19,21,22
5-1,3,6,9,10,14,16,17,19,20,21
6-1,4,7,9,11,14,15,17,18,20,22
7-1,5,8,10,11,14,15,16,18,19,23
8-2,5,6,9,11,12,16,17,18,19,22
9-2,3,7,10,11,12,15,16,18,20,21
10-3,4,6,8,11,12,14,17,18,21,23
11-1,2,3,4,5,18,19,20,21,22,23
12-3,5,6,7,11,13,14,16,20,22,23
13-1,2,3,4,5,6,7,8,9,10,11
14-2,3,8,9,11,13,15,17,19,20,23
15-1,5,8,10,11,12,13,17,20,21,22
16-2,4,6,10,11,13,14,15,19,21,22
17-1,4,7,9,11,12,13,16,19,21,23
18-1,2,3,4,5,12,13,14,15,16,17
19-1,3,6,9,10,12,13,15,18,22,23
20-3,4,7,8,10,13,16,17,18,19,22
21-1,2,6,7,8,12,13,14,18,19,20
22-2,5,7,9,10,13,14,17,18,21,23
23-4,5,6,8,9,13,15,16,18,20,21.

Aut(3) *is cyclic of order* 5. *It is generated by*

σ =(2,8,10,9,4)
(3,7,5,6,11)
(14,20,22,23,16)
(15,19,17,18,21).

(4) *D*38 at 11 (*H*21)

1-3,4,5,8,9,13,15,16,18,20,22
2-1,3,6,8,10,12,13,17,18,21,22
3-7,8,9,10,11,12,13,14,18,19,20
4-2,3,6,9,11,12,15,16,18,19,21
5-2,3,4,7,10,14,16,17,18,19,22
6-1,3,5,7,11,14,15,17,18,20,21
7-1,2,4,8,11,13,16,17,19,20,21

8-4,5,6,10,11,12,13,14,15,16,17
9-2,5,6,7,8,13,14,15,19,21,22
10-1,4,6,7,9,12,14,16,20,21,22
11-1,2,5,9,10,12,15,17,19,20,22
12-1,5,6,7,9,13,16,17,18,19,23
13-4,5,6,10,11,18,19,20,21,22,23
14-1,2,4,7,11,12,13,15,18,22,23
15-2,3,5,7,10,12,13,16,20,21,23
16-2,3,6,9,11,13,14,17,20,22,23
17-1,3,4,9,10,13,14,15,19,21,23
18-7,8,9,10,11,15,16,17,21,22,23
19-1,2,6,8,10,14,15,16,18,20,23
20-2,4,5,8,9,12,14,17,18,21,23
21-1,3,5,8,11,12,14,16,19,22,23
22-3,4,6,7,8,12,15,17,19,20,23
23-1,2,3,4,5,6,7,8,9,10,11.
Aut(4) *is trivial.*

(5) *D*50 at 10 (*H*21)
1-2,3,4,8,10,12,14,17,18,21,22
2-4,5,6,10,11,15,16,17,21,22,23
3-2,4,6,7,9,13,15,17,18,20,21
4-7,8,9,10,11,13,14,17,20,22,23
5-1,3,4,9,11,12,16,17,18,20,23
6-1,4,5,7,8,12,13,14,15,16,17
7-1,2,5,9,11,13,14,16,18,21,22
8-2,3,5,7,10,12,13,16,20,21,23
9-1,2,6,8,11,12,14,15,20,21,23
10-3,5,6,7,9,12,14,15,18,22,23
11-1,3,6,8,10,13,15,16,18,20,22
12-2,3,4,7,11,14,15,16,19,20,22
13-1,2,5,9,10,12,15,17,19,20,22
14-2,3,5,8,11,13,15,17,18,19,23
15-1,4,5,7,8,18,19,20,21,22,23
16-1,3,4,9,10,13,14,15,19,21,23
17-7,8,9,10,11,12,15,16,18,19,21
18-2,4,6,8,9,12,13,16,19,22,23
19-1,2,3,4,5,6,7,8,9,10,11
20-1,2,6,7,10,14,16,17,18,19,23
21-4,5,6,10,11,12,13,14,18,19,20
22-3,5,6,8,9,14,16,17,19,20,21
23-1,3,6,7,11,12,13,17,19,21,22.
Aut(5) *is trivial.*

(6) *D*47 at 1 (*H*26)
1-2,5,6,9,11,12,14,16,19,22,23
2-3,4,6,7,9,12,13,14,15,16,17
3-1,5,7,9,10,12,13,15,18,22,23
4-1,3,6,10,11,13,16,17,18,19,23
5-2,4,7,10,11,14,15,17,18,19,22
6-3,5,7,8,11,12,15,17,19,20,23
7-1,4,8,9,11,13,15,16,19,20,22
8-1,2,3,4,5,13,14,17,20,22,23
9-4,5,6,8,10,12,16,17,18,20,22
10-1,2,6,7,8,12,13,14,18,19,20
11-2,3,8,9,10,14,15,16,18,20,23
12-4,5,7,8,11,13,14,16,18,21,23
13-1,5,6,9,11,14,15,17,18,20,21
14-3,4,6,7,9,18,19,20,21,22,23
15-1,4,8,9,10,12,14,17,19,21,23
16-3,5,6,8,10,13,14,15,19,21,22
17-1,3,7,10,11,12,14,16,20,21,22
18-1,2,6,7,8,15,16,17,21,22,23
19-2,3,8,9,11,12,13,17,18,21,22
20-1,2,3,4,5,12,15,16,18,19,21
21-1,2,3,4,5,6,7,8,9,10,11
22-2,4,6,10,11,12,13,15,20,21,23
23-2,5,7,9,10,13,16,17,19,20,21.
Aut(6) *is trivial.*

(7) *D*67 at 21 (*H*31)
1-13,14,15,16,17,18,19,20,21,22,23
2-1,3,6,7,10,12,13,16,18,20,22
3-1,4,5,7,8,10,14,15,18,22,23

4-1,2,5,8,11,12,13,17,18,20,23
5-1,2,6,7,10,11,14,17,18,19,21
6-1,3,4,8,11,12,15,16,18,19,21
7-1,4,6,8,9,12,13,14,17,21,22
8-1,2,5,9,10,12,15,16,17,19,22
9-1,2,3,4,5,6,19,20,21,22,23
10-1,4,6,7,9,11,15,16,17,20,23
11-1,2,3,7,8,9,13,14,15,19,20
12-1,3,5,9,10,11,13,14,16,21,23
13-3,5,6,8,9,10,15,17,18,20,21
14-2,4,6,8,9,10,13,16,18,19,23
15-2,4,5,7,9,12,14,16,18,20,21
16-3,4,5,7,9,11,13,17,18,19,22
17-2,3,6,9,11,12,14,15,18,22,23
18-7,8,9,10,11,12,19,20,21,22,23
19-2,3,4,7,10,12,13,15,17,21,23
20-3,5,6,7,8,12,14,16,17,19,23
21-2,3,4,8,10,11,14,16,17,20,22
22-4,5,6,10,11,12,13,14,15,19,20
23-2,5,6,7,8,11,13,15,16,21,22.
Aut(7) *is trivial.*

(8) $D80$ at 15 ($H38$)
1-2,3,5,7,10,11,14,17,19,20,23
2-3,4,5,6,7,8,13,14,15,16,17
3-7,8,9,10,13,14,16,18,19,20,22
4-1,3,6,7,9,11,14,15,18,22,23
5-3,4,10,12,15,16,17,18,19,22,23
6-1,3,5,8,11,12,13,17,18,20,22
7-5,6,11,12,14,15,16,19,20,21,22
8-1,4,5,7,10,12,13,14,21,22,23
9-1,2,5,6,7,8,10,12,15,18,19
10-2,4,6,7,11,12,13,16,18,20,23
11-2,3,5,8,9,12,14,16,18,21,23
12-1,2,3,4,13,14,15,18,19,20,21
13-1,4,5,7,9,11,16,17,18,19,21
14-5,6,9,10,13,15,17,18,20,21,23
15-1,3,6,8,10,11,13,16,19,21,23
16-1,4,6,8,9,12,14,17,19,20,23
17-3,4,7,8,9,10,11,12,15,20,21
18-1,2,7,8,15,16,17,20,21,22,23
19-2,4,6,8,10,11,14,17,18,21,22
20-2,4,5,8,9,11,13,15,19,22,23
21-1,2,3,4,5,6,9,10,16,20,22
22-1,2,9,10,11,12,13,14,15,16,17
23-2,3,6,7,9,12,13,17,19,21,22.
Aut(8) *is trivial.*

(9) $D81$ at 3 ($H38$)
1-2,4,7,9,11,13,16,17,19,22,23
2-3,4,5,10,11,12,13,15,17,20,22
3-1,5,7,9,10,12,15,16,19,20,23
4-3,6,7,8,11,15,16,18,19,20,22
5-1,4,6,8,9,10,11,12,18,22,23
6-1,2,3,12,15,16,17,18,21,22,23
7-2,5,6,9,10,17,18,19,20,21,22
8-1,2,3,6,7,10,11,14,17,20,23
9-2,4,6,8,10,12,14,15,16,17,19
10-1,4,6,13,14,15,19,20,21,22,23
11-3,6,7,9,10,12,13,14,16,21,22
12-1,4,7,8,10,13,16,17,18,20,21
13-3,4,5,6,7,8,9,15,17,21,23
14-1,2,3,4,5,6,7,12,13,18,19
15-1,5,7,8,11,12,14,17,19,21,22
16-2,5,7,8,10,13,14,15,18,22,23
17-3,4,5,10,11,14,16,18,19,21,23
18-1,2,3,8,9,10,11,13,15,19,21
19-2,5,6,8,11,12,13,16,20,21,23
20-1,5,6,9,11,13,14,15,16,17,18
21-1,2,3,4,5,8,9,14,16,20,22
22-3,8,9,12,13,14,17,18,19,20,23
23-2,4,7,9,11,12,14,15,18,20,21.
Aut(9) *is trivial.*

(10) $D94$ at 10 ($H43$)
1-4,5,6,10,11,15,16,17,21,22,23
2-1,4,5,7,8,12,13,15,18,21,22
3-1,2,5,9,10,14,15,17,18,19,21
4-3,5,6,8,9,12,14,15,19,22,23
5-7,8,9,10,11,12,13,14,15,16,17
6-2,3,5,7,11,13,15,16,18,19,23
7-1,3,4,8,11,13,14,17,19,21,23
8-1,3,6,9,10,12,13,16,18,21,23
9-1,2,6,7,11,12,14,16,19,21,22
10-2,4,6,7,9,13,14,17,18,22,23
11-2,3,4,8,10,12,16,17,18,19,22
12-1,3,6,7,10,13,15,17,19,20,22
13-1,3,4,9,11,14,15,16,18,20,22
14-1,2,6,8,11,12,15,17,18,20,23
15-7,8,9,10,11,18,19,20,21,22,23
16-2,3,4,7,10,12,14,15,20,21,23
17-2,4,6,8,9,13,15,16,19,20,21
18-1,4,5,7,9,12,16,17,19,20,23
19-1,2,5,8,10,13,14,16,20,22,23
20-1,2,3,4,5,6,7,8,9,10,11
21-4,5,6,10,11,12,13,14,18,19,20
22-3,5,6,7,8,14,16,17,18,20,21
23-2,3,5,9,11,12,13,17,20,21,22.
Aut(10) *is trivial.*

(11) $D95$ at 2 ($H43$)
1-4,5,6,7,10,12,13,16,20,22,23
2-1,3,6,10,11,13,15,17,19,20,22
3-1,5,7,9,11,12,15,17,18,20,23
4-2,3,7,9,10,12,16,17,18,19,22
5-2,4,6,9,11,13,15,16,18,19,23
6-3,4,7,8,11,12,14,15,19,22,23
7-2,5,8,10,11,14,16,17,19,20,23
8-1,2,3,4,5,12,13,14,15,16,17
9-1,2,6,7,8,12,13,14,18,19,20
10-3,5,6,8,9,13,14,17,18,22,23
11-1,4,8,9,10,14,15,16,18,20,22
12-2,5,7,10,11,13,14,15,18,21,22
13-3,4,6,7,11,14,16,17,18,20,21
14-1,2,3,4,5,18,19,20,21,22,23
15-1,4,7,9,10,13,14,17,19,21,23
16-2,3,6,9,10,12,14,15,20,21,23
17-1,5,6,9,11,12,14,16,19,21,22
18-1,2,6,7,8,15,16,17,21,22,23
19-1,3,8,10,11,12,13,16,18,21,23
20-4,5,6,8,10,12,15,17,18,19,21
21-1,2,3,4,5,6,7,8,9,10,11
22-3,5,7,8,9,13,15,16,19,20,21
23-2,4,8,9,11,12,13,17,20,21,22.
Aut(11) *is trivial.*

(12) $D98$ at 4 ($H44$)
1-2,5,6,7,11,12,13,16,20,22,23
2-3,4,7,8,11,12,14,16,18,21,23
3-1,5,7,8,10,12,14,17,20,21,22
4-1,3,6,10,11,13,16,17,18,20,21
5-2,4,6,8,10,13,14,17,18,22,23
6-2,3,7,9,10,12,15,17,18,20,23
7-4,5,9,10,11,12,13,14,15,16,17
8-1,4,6,7,9,12,13,15,18,21,22
9-1,2,3,4,5,15,16,17,21,22,23
10-1,2,8,9,11,14,15,16,18,20,22
11-3,5,6,8,9,13,14,15,20,21,23
12-4,5,9,10,11,18,19,20,21,22,23
13-2,3,6,9,10,12,14,16,19,21,22
14-1,4,6,8,9,12,16,17,19,20,23
15-1,2,3,4,5,12,13,14,18,19,20
16-3,5,6,8,11,12,15,17,18,19,22
17-1,2,8,10,11,12,13,15,19,21,23
18-1,3,7,9,11,13,14,17,19,22,23
19-1,2,3,4,5,6,7,8,9,10,11
20-2,5,7,8,9,13,16,17,18,19,21
21-1,5,6,7,10,14,15,16,18,19,23

22-2,4,6,7,11,14,15,17,19,20,21
23-3,4,7,8,10,13,15,16,19,20,22.
Aut(12) *is trivial.*

(13) *D*104 at 3 (*H*44)
1-2,5,7,10,11,13,14,15,18,21,22
2-3,4,7,8,10,12,14,17,20,21,22
3-1,5,7,9,10,12,15,17,18,20,23
4-1,3,8,9,11,13,14,15,20,21,23
5-2,4,8,9,11,12,13,17,18,22,23
6-1,2,3,4,5,12,13,14,15,16,17
7-4,5,6,8,11,12,15,16,18,20,21
8-1,3,6,10,11,13,16,17,18,20,22
9-1,2,6,7,8,15,16,17,21,22,23
10-4,5,6,7,9,13,14,16,20,22,23
11-2,3,6,9,10,12,14,16,18,21,23
12-1,4,8,9,10,14,15,16,18,19,22
13-2,3,7,9,11,12,15,16,19,20,22
14-3,5,7,8,9,13,16,17,18,19,21
15-2,5,8,10,11,14,16,17,19,20,23
16-1,2,3,4,5,18,19,20,21,22,23
17-1,4,7,10,11,12,13,16,19,21,23
18-2,4,6,9,10,13,15,17,19,20,21
19-1,2,3,4,5,6,7,8,9,10,11
20-1,5,6,9,11,12,14,17,19,21,22
21-3,5,6,8,10,12,13,15,19,22,23
22-3,4,6,7,11,14,15,17,18,19,23
23-1,2,6,7,8,12,13,14,18,19,20.
Aut(13) *is trivial.*

(14) *D*102 at 7 (*H*47)
1-7,8,9,10,11,15,16,17,21,22,23
2-1,4,5,8,9,12,16,17,19,20,23
3-1,2,5,10,11,12,14,15,20,22,23
4-1,3,6,9,11,12,14,17,18,21,23
5-1,4,6,7,10,14,15,16,18,19,23
6-1,2,3,7,8,18,19,20,21,22,23
7-2,3,4,9,10,12,14,16,19,21,22
8-3,4,5,7,11,14,15,17,19,20,21
9-3,5,6,8,11,12,15,16,18,19,22
10-2,4,6,8,9,14,15,17,18,20,22
11-2,5,6,7,10,12,16,17,18,20,21
12-1,5,6,8,10,13,14,17,19,21,22
13-1,2,3,4,5,6,7,8,9,10,11
14-1,2,6,9,11,13,15,16,19,20,21
15-2,4,6,7,11,12,13,17,19,22,23
16-3,4,6,8,10,12,13,15,20,21,23
17-3,5,6,7,9,13,14,16,20,22,23
18-1,2,3,7,8,12,13,14,15,16,17
19-1,3,4,10,11,13,16,17,18,20,22
20-1,4,5,7,9,12,13,15,18,21,22
21-2,3,5,9,10,13,15,17,18,19,23
22-2,4,5,8,11,13,14,16,18,21,23
23-7,8,9,10,11,12,13,14,18,19,20.
Aut(14) *is trivial.*

(15) *D*106 at 13 (*H*48)
1-2,5,6,7,8,11,13,16,20,21,23
2-3,4,6,8,9,10,14,15,20,21,23
3-1,5,6,8,9,11,14,17,19,21,22
4-1,3,6,8,10,12,13,16,19,22,23
5-2,4,6,7,8,12,15,17,19,20,22
6-7,8,9,10,11,12,13,14,15,16,17
7-2,3,4,8,10,11,16,17,18,19,21
8-13,14,15,16,17,18,19,20,21,22,23
9-1,4,5,7,8,10,13,15,18,21,22
10-1,3,5,8,11,12,15,17,18,20,23
11-2,4,5,8,9,12,14,16,18,22,23
12-1,2,3,7,8,9,13,14,18,19,20
13-2,3,5,7,10,11,14,15,19,22,23
14-1,4,5,7,9,10,16,17,19,20,23
15-1,3,4,7,11,12,14,16,20,21,22
16-2,3,5,9,10,12,13,17,20,21,22
17-1,2,4,9,11,12,13,15,19,21,23

18-1,2,3,4,5,6,13,14,15,16,17
19-1,2,6,9,10,11,15,16,18,20,22
20-3,4,6,7,9,11,13,17,18,22,23
21-4,5,6,10,11,12,13,14,18,19,20
22-1,2,6,7,10,12,14,17,18,21,23
23-3,5,6,7,9,12,15,16,18,19,21.
Aut(15) *is trivial.*

(16) *D*108 at 16 (*H*48)
1-2,5,6,8,9,10,14,16,19,20,22
2-3,5,6,7,8,10,15,17,18,20,23
3-1,4,6,7,8,11,15,16,19,20,21
4-1,2,6,9,10,11,13,17,18,20,21
5-3,4,6,7,9,11,13,14,20,22,23
6-13,14,15,16,17,18,19,20,21,22,23
7-1,4,6,8,9,12,14,17,18,19,23
8-4,5,6,10,11,12,16,17,21,22,23
9-2,3,6,8,11,12,14,15,18,21,22
10-3,5,6,7,9,12,13,16,18,19,21
11-1,2,6,7,10,12,13,15,19,22,23
12-1,2,3,4,5,6,13,14,15,16,17
13-1,2,3,7,8,9,16,17,21,22,23
14-2,3,4,8,10,11,13,16,18,19,23
15-1,4,5,7,8,10,13,14,18,21,22
16-2,4,5,7,9,11,15,17,18,19,22
17-1,3,5,9,10,11,14,15,19,21,23
18-1,3,5,8,11,12,13,17,19,20,22
19-2,4,5,8,9,12,13,15,20,21,23
20-7,8,9,10,11,12,13,14,15,16,17
21-1,2,5,7,11,12,14,16,18,20,23
22-2,3,4,7,10,12,14,17,19,20,21
23-1,3,4,9,10,12,15,16,18,20,22.
Aut(16) *is trivial.*

(17) *D*111 at 11 (*H*48)
1-7,8,9,10,11,18,19,20,21,22,23
2-1,3,5,8,10,14,16,17,18,20,21
3-1,4,6,7,8,12,13,16,18,21,23
4-1,2,5,9,11,13,14,16,19,21,23
5-1,3,6,9,10,12,13,17,19,21,22
6-1,2,4,7,11,12,14,17,20,21,22
7-2,4,5,8,9,12,15,17,18,19,22
8-4,5,6,10,11,12,13,14,18,19,20
9-2,3,6,8,11,13,14,17,18,22,23
10-3,4,6,7,9,14,16,17,19,20,23
11-2,3,5,7,10,12,13,16,20,22,23
12-1,2,4,9,10,13,15,17,18,20,23
13-1,2,6,7,10,14,15,16,18,19,22
14-1,3,5,7,11,12,15,17,18,19,23
15-1,2,3,4,5,6,7,8,9,10,11
16-1,5,6,8,9,12,14,15,20,22,23
17-1,3,4,8,11,13,15,16,19,20,22
18-4,5,6,10,11,15,16,17,21,22,23
19-2,3,6,9,11,12,15,16,18,20,21
20-3,4,5,7,9,13,14,15,18,21,22
21-7,8,9,10,11,12,13,14,15,16,17
22-2,3,4,8,10,12,14,15,19,21,23
23-2,5,6,7,8,13,15,17,19,20,21.
Aut(17) *is trivial.*

(18) *D*118 at 10 (*H*52)
1-5,6,9,10,11,14,15,16,17,22,23
2-1,4,5,8,10,13,15,16,19,20,22
3-1,2,9,10,11,12,13,14,16,18,20
4-1,3,6,8,10,12,16,17,18,19,23
5-3,4,7,8,9,12,14,16,17,20,22
6-2,3,5,7,11,12,13,16,19,22,23
7-1,2,3,4,9,14,15,18,19,22,23
8-1,3,6,7,11,13,15,17,18,20,22
9-2,4,6,8,11,13,14,17,19,20,23
10-5,6,7,8,9,12,13,14,15,18,19
11-2,4,5,7,10,12,15,17,18,20,23
12-1,2,7,8,9,13,15,16,17,21,23
13-1,4,5,7,11,14,16,17,18,19,21

14-2,4,6,8,11,12,15,16,18,21,22
15-3,4,5,6,9,13,16,18,20,21,23
16-7,8,9,10,11,18,19,20,21,22,23
17-2,3,6,7,10,14,15,16,19,20,21
18-1,2,5,6,9,12,17,19,20,21,22
19-1,3,5,8,11,12,14,15,20,21,23
20-1,4,6,7,10,12,13,14,21,22,23
21-1,2,3,4,5,6,7,8,9,10,11
22-3,4,9,10,11,12,13,15,17,19,21
23-2,3,5,8,10,13,14,17,18,21,22.
Aut(18) *is cyclic of order* 5.
It is generated by
σ =(1,7,10,3,5)
(2,8,11,4,6)
(12,16,22,15,18)
(13,20,17,23,19).

(19) D119 at 17 (H52)
1-5,6,7,8,15,16,17,18,19,20,21
2-1,4,5,7,9,12,14,16,18,21,23
3-1,2,7,8,13,14,15,19,21,22,23
4-1,3,6,7,9,12,13,17,20,21,22
5-3,4,11,12,13,15,17,18,19,21,23
6-2,3,5,8,9,11,13,14,18,20,21
7-5,6,9,10,13,14,17,18,19,22,23
8-2,4,5,7,10,12,13,15,18,20,22
9-1,3,5,8,10,12,13,16,19,20,23
10-1,2,3,4,5,6,13,14,15,16,17
11-1,2,3,4,7,8,9,10,17,18,19
12-1,3,6,7,10,11,14,15,18,20,23
13-1,2,11,12,14,16,17,18,19,20,22
14-1,4,5,8,9,11,15,17,20,22,23
15-2,4,6,7,9,11,13,16,19,20,23
16-3,4,5,6,7,8,11,12,14,19,22
17-2,3,6,8,9,12,15,16,18,22,23
18-3,4,9,10,14,15,16,19,20,21,22
19-2,4,6,8,10,12,14,17,20,21,23
20-2,3,5,7,10,11,16,17,21,22,23
21-7,8,9,10,11,12,13,14,15,16,17
22-1,2,5,6,9,10,11,12,15,19,21
23-1,4,6,8,10,11,13,16,18,21,22.
Aut(19) *is trivial.*

(20) D122 at 1 (H54)
1-2,4,7,9,10,12,15,16,19,20,23
2-3,5,7,9,10,13,15,17,18,20,22
3-1,4,5,12,13,16,17,18,19,22,23
4-2,5,7,8,11,12,15,16,18,21,22
5-1,8,9,10,11,18,19,20,21,22,23
6-1,2,3,4,5,8,9,14,16,20,22
7-3,5,6,8,10,12,14,15,19,22,23
8-1,2,3,12,13,14,15,18,19,20,21
9-3,4,7,8,10,14,16,17,18,19,21
10-3,4,6,8,11,13,15,16,18,20,23
11-1,2,3,6,7,8,9,12,17,18,23
12-2,5,6,9,10,13,14,16,18,21,23
13-1,4,5,6,7,9,11,14,15,18,19
14-1,2,3,4,5,10,11,15,17,21,23
15-3,5,6,9,11,12,16,17,19,20,21
16-2,5,7,8,11,13,14,17,19,20,23
17-1,4,5,6,7,8,10,12,13,20,21
18-1,6,7,14,15,16,17,20,21,22,23
19-2,4,6,10,11,12,14,17,18,20,22
20-3,4,7,9,11,12,13,14,21,22,23
21-1,2,3,6,7,10,11,13,16,19,22
22-1,8,9,10,11,12,13,14,15,16,17
23-2,4,6,8,9,13,15,17,19,21,22.
Aut(20) *is trivial.*

(21) D126 at 2 (H56)
1-2,5,6,8,11,13,15,16,19,20,23
2-3,4,6,8,11,12,15,17,18,20,22
3-1,4,5,12,13,16,17,18,19,22,23
4-1,8,9,10,11,12,13,14,15,16,17

5-2,4,6,9,10,12,14,17,19,20,23
6-3,4,7,8,10,13,16,17,19,20,21
7-1,2,3,4,5,10,11,15,17,21,23
8-3,5,7,10,11,12,13,14,20,22,23
9-1,2,3,6,7,8,10,12,16,18,23
10-1,2,3,12,13,14,15,18,19,20,21
11-3,5,6,9,10,12,15,16,19,21,22
12-1,6,7,14,15,16,17,20,21,22,23
13-2,5,7,9,11,12,16,17,18,20,21
14-1,2,3,6,7,9,11,13,17,19,22
15-3,5,6,8,9,13,14,17,18,21,23
16-2,5,7,8,10,14,15,17,18,19,22
17-1,8,9,10,11,18,19,20,21,22,23
18-1,4,5,6,7,8,11,12,14,19,21
19-2,4,7,8,9,12,13,15,21,22,23
20-3,4,7,9,11,14,15,16,18,19,23
21-1,2,3,4,5,8,9,14,16,20,22
22-1,4,5,6,7,9,10,13,15,18,20
23-2,4,6,10,11,13,14,16,18,21,22.
Aut(21) *is trivial.*

(22) *D*127 at 3 (*H*57)
1-3,10,11,12,13,14,16,18,19,20,23
2-1,4,6,9,11,12,13,15,20,22,23
3-2,4,6,9,10,14,15,16,18,19,22
4-1,5,6,8,11,13,14,16,18,21,22
5-1,2,3,8,9,10,11,13,15,19,21
6-1,5,7,10,11,12,15,17,18,19,22
7-1,2,3,4,5,12,13,14,15,16,17
8-1,2,3,6,7,9,11,14,17,18,23
9-1,4,6,7,10,13,16,17,19,21,23
10-2,4,7,8,11,13,14,17,19,20,22
11-3,7,9,13,15,16,17,18,20,21,22
12-3,4,5,8,9,10,11,16,17,22,23
13-3,6,8,12,14,15,17,19,21,22,23
14-2,5,6,9,11,12,16,17,19,20,21
15-1,4,8,9,10,12,14,17,18,20,21
16-2,5,6,8,10,13,15,17,18,20,23
17-1,2,3,4,5,18,19,20,21,22,23
18-2,5,7,9,10,12,13,14,21,22,23
19-2,4,7,8,11,12,15,16,18,21,23
20-3,4,5,6,7,8,9,12,13,18,19
21-1,2,3,6,7,8,10,12,16,20,22
22-1,5,7,8,9,14,15,16,19,20,23
23-3,4,5,6,7,10,11,14,15,20,21.
Aut(22) *is trivial.*

(23) *D*128 at 19 (*H*58)
1-3,4,5,6,9,10,13,14,19,20,21
2-1,3,6,8,9,11,14,15,18,21,23
3-5,6,9,10,11,12,15,16,17,18,19
4-2,3,6,8,9,12,13,16,17,20,23
5-2,4,6,7,9,12,13,15,18,21,22
6-7,8,9,10,11,12,19,20,21,22,23
7-1,2,3,4,9,10,17,18,19,22,23
8-1,3,5,7,9,11,13,15,17,20,22
9-13,14,15,16,17,18,19,20,21,22,23
10-2,4,5,8,9,11,14,16,17,21,22
11-1,4,5,7,9,12,14,16,18,20,23
12-1,2,7,8,9,10,13,14,15,16,19
13-2,3,6,7,10,11,14,16,18,20,22
14-3,4,5,6,7,8,15,16,19,22,23
15-1,4,6,7,10,11,13,16,17,21,23
16-1,2,5,6,7,8,17,18,19,20,21
17-1,2,5,6,11,12,13,14,19,22,23
18-1,4,6,8,10,12,14,15,17,20,22
19-2,4,5,8,10,11,13,15,18,20,23
20-2,3,5,7,10,12,14,15,17,21,23
21-3,4,7,8,11,12,13,14,17,18,19
22-1,2,3,4,11,12,15,16,19,20,21
23-1,3,5,8,10,12,13,16,18,21,22
Aut(23) *is of order* 55 *and it is generated by*
$\sigma = (1,3,6,12,7)$

(2,4,5,11,8)
(13,18,20,15,23)
(14,17,21,16,22)

and

τ =(1,3,12,14,6,9,16,7,22,21,17)
(2,4,11,13,5,10,15,8,23,20,18).

(24) *D*130 at 1 (*H*60)
1-2,3,4,5,7,9,10,13,14,17,19
2-3,4,5,6,8,10,11,14,15,18,20
3-4,5,6,7,9,11,12,15,16,19,21
4-5,6,7,8,10,12,13,16,17,20,22
5-6,7,8,9,11,13,14,17,18,21,23
6-1,7,8,9,10,12,14,15,18,19,22
7-2,8,9,10,11,13,15,16,19,20,23
8-1,3,9,10,11,12,14,16,17,20,21
9-2,4,10,11,12,13,15,17,18,21,22
10-3,5,11,12,13,14,16,18,19,22,23
11-1,4,6,12,13,14,15,17,19,20,23
12-1,2,5,7,13,14,15,16,18,20,21
13-2,3,6,8,14,15,16,17,19,21,22
14-3,4,7,9,15,16,17,18,20,22,23
15-1,4,5,8,10,16,17,18,19,21,23
16-1,2,5,6,9,11,17,18,19,20,22
17-2,3,6,7,10,12,18,19,20,21,23
18-1,3,4,7,8,11,13,19,20,21,22
19-2,4,5,8,9,12,14,20,21,22,23
20-1,3,5,6,9,10,13,15,21,22,23
21-1,2,4,6,7,10,11,14,16,22,23
22-1,2,3,5,7,8,11,12,15,17,23
23-1,2,3,4,6,8,9,12,13,16,18.

Aut(24) *is of order* 253 *and it is generated by*

σ =(2,3,5,9,17,10,19,14,4,7,13)
(6,11,21,18,12,23,22,20,16,8,
15)

and

τ =(1,2,3,4,5,6,7,8,9,10,11,12,13,
14,15,16,17,18,19,20,21,22,
23).

Remark. In [1] we carelessly missed one Hadamard matrix which was corrected in [2]. So we have changed the numbering of Hadamard matrices and Hadamard 3-designs as follows:

*H*59 → *H*60;

*H*59 is now the matrix found by Kimura in [2].

*D*129 → *D*130;

*D*129 is now the Hadamard 3-design involved in *H*59. This is the only Hadamard 3-design involved in *H*59, because *H*59 has a transitive automorphism group.

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