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SUBRINGS CONTAINING IDEALS

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All groups in this note are abelian, with addition the group operation. A group belonging to a class $\mathscr{C}$ of abelian groups will be called a $\mathscr{C}$-group. The additive group of a ring R will be donoted R^+, and $E(G)$ will denote the additive group of the ring of endomorphisms of a group G.

In [1], Hirano proved that if a ring R is a subring of a ring S with S^+/R^+ a finite group, then there exists an ideal $I \trianglelefteq S$ such that $I \subseteq R$, and S/I is a finite ring. Arguments used by Hirano will be employed is this note to show that for certain classes of groups $\mathscr{C}$, if R is a subring of a ring S with S^+/R^+ belonging to $\mathscr{C}$, then S possesses a left ideal I_ℓ, and a right ideal I_r, both contained in R, such that R^+/I_ℓ^+, and S^+/I_r^+ belong to $\mathscr{C}$. If the class $\mathscr{C}$ consists entirely of finitely generated groups, then the one sided ideals I_ℓ and I_r are replaced by a single two sided ideal I.

Definition. A non-empty class $\mathscr{C}$ of groups will be called a finite-like class if $\mathscr{C}$ is closed with respect to subgroups, epimorphic images, extensions of $\mathscr{C}$-groups by $\mathscr{C}$-groups, and $E(G)$ belong to $\mathscr{C}$ for every $\mathscr{C}$-group G.

Examples of finite-like classes of groups are the class of: finite groups, finitely generated groups, bounded groups, groups G which do not possess an element of order p, for every prime p belonging to a set of primes P.

Theorem 1. *Let $\mathscr{C}$ be a finite-like class of groups, and let R be a subring of a ring S such that S^+/R^+ belongs to $\mathscr{C}$. Then S possesses a left ideal I_ℓ, and a right ideal I_r, both contained in R, such that S^+/I_ℓ^+, and S^+/I_r^+ belong to $\mathscr{C}$.*

Proof. Let $\varphi: R^+ \to E(S^+/R^+)$ be the homomorphism defined by $\varphi(a)(s+R^+) = as+R^+$ for all $a \in R$, and $s \in S$. Let $K_1 = \ker \varphi$. Clearly $K_1 = \{a \in R \mid aS \subseteq R\}$, and R^+/K_1 is a $\mathscr{C}$-group. Similarly, it follows that the group $K_2 = \{a \in R \mid Sa \subseteq R\}$ satisfies $R^+/K_2 \in \mathscr{C}$. Put $K = K_1 \cap K_2$. Since R^+/K is isomophic to a subgroup of $(R^+/K_1) \oplus (R^+/K_2)$, it follows that R^+/K is a $\mathscr{C}$-group. Let $I_\ell = K+SK$, and $I_r = K+KS$. Clearly I_ℓ is a left ideal in S, and $I_\ell \subseteq R$. Since R^+/I_ℓ^+ is an epimorphic image of R^+/K, it follows that R^+/I_ℓ^+ is a $\mathscr{C}$-group. Similarly, I_r satisfies the desired properties.

An immediate consequence of Theorem 1 is:

Corollary 2. *Let $\mathscr{C}$ be a finite-like class of groups, and let S be a ring such that S^+ does not belong to $\mathscr{C}$, and $aS = Sa = S$ for all $a \in S$, $a \neq 0$. If R is a subring of S with S^+/R^+ a $\mathscr{C}$-group, then $S = R$.*

Corollary 3. *Let $\mathscr{C}$ be a finite-like class of groups. Let S be a ring such that S^+ does not belong to $\mathscr{C}$, and let R be a subring of S with unity, satisfying $aR = Ra = R$ for all $a \in R$, $a \neq 0$. If S^+/R^+ belongs to $\mathscr{C}$, then S is a ring direct sum $S = R \oplus T$, with T^+ a $\mathscr{C}$-group.*

Proof. Theorem 1 yields that S possesses a left ideal I_ℓ, and right ideal I_r, both contained in R, such that S^+/I_ℓ^+, and S^+/I_r^+ belong to $\mathscr{C}$. Since S^+ is not a $\mathscr{C}$-group, both I_ℓ and I_r are non-zero ideals. Let $a \in I_\ell$, $a \neq 0$. Then $R = Ra \subseteq I_\ell$, and so $I_\ell = R$. Similarly, $I_r = R$. Therefore R is a two-sided ideal in S. Let e be the unity in R, and $s \in S$. Then $es = ese = se$, and so e is a central idempotent in S. Put $T = \{s - se \mid s \in S\}$. Since $R = Se$, it follows that $S = R \oplus T$, with T^+ a $\mathscr{C}$-group.

Definition. A class of groups consisting only of finitely generated groups, and closed with respect to epimorphic images, and finite direct sums will be called a finitely generated class.

It it easy see that a class of groups $\mathscr{C}$ is a finitely generated class if and only if $\mathscr{C}$ is a finite-like class, and every group belonging to $\mathscr{C}$ is finitely generated.

Theorem 4. *Let $\mathscr{C}$ be a finitely generated class of groups. Let R be a subring of a ring S such that S^+/R^+ belongs to $\mathscr{C}$. Then S possesses a two-sided ideal I contained in R such that S^+/I^+ is a $\mathscr{C}$-group.*

Proof. Let $K = \{a \in R \mid aS \subseteq R\}$. Then R^+/K belongs to $\mathscr{C}$ as was shown in the proof of Theorem 1. Let $\{a_1 + R^+, \cdots, a_n + R^+\}$ be a finite set of generators for S^+/R^+. For each $1 \leq i \leq n$ define $\varphi_i : K \to E(S^+/R^+)$ via $\varphi_i(a)(s + R^+) = a_i as + R^+$ for all $a \in K$, and $s \in S$. Put $L_i = \ker \varphi_i$, and $L = \bigcap_{i=1}^n L_i$. It is readily seen that K/L belongs to $\mathscr{C}$, that $LS \subseteq R$, and $SL \subseteq R$. For any element $x \in S$, there exist integers m_i, $1 \leq i \leq n$, and $b \in R$ such that $x = b + \sum_{i=1}^n m_i a_i$. Let $a \in L$, and $s \in S$. Since $bas \in R$, and $a_i as \in R$ for all $1 \leq i \leq n$, it follows that $xas \in R$, i.e., $SLS \subseteq R$. Put $I = L + SL + LS + SLS$. Clearly I is an ideal in S contained in R, and S^+/I^+ belongs to $\mathscr{C}$.

The following consequences of Theorem 4 are the counterparts of Corollaries 1-4 in [1]. The proofs are easy, and essentially the same as those given in [1].

Corollary 5. *Let $\mathscr{C}$ by a finitely generated class of group, and let S be a simple ring such that S^+ does not belong to $\mathscr{C}$. If R is a subring of S with S^+/R^+ belonging to $\mathscr{C}$, then $S = R$.*

Corollary 6. *Let $\mathscr{C}$ be a finitely generated class of groups, and let S be a ring such that S^+ does not belong to $\mathscr{C}$. If R is a subring such that R is simple, R possesses a unity, and S^+/R^+ belongs to $\mathscr{C}$, then S is a ring direct sum $S = R \oplus T$ with T^+ belong to $\mathscr{C}$.*

Corollary 7. *Let $\mathscr{C}$ be a finitely generated class of groups, and let S be a ring such that T^+ does not belong to $\mathscr{C}$ for every non-zero epimorphic image T of S. Let d be a derivation on S. If $im(d)$ belongs to $\mathscr{C}$, then $d = 0$.*

Corollary 8. *Let $\mathscr{C}$ be a finitely generated class of groups, and let S be a ring such that T^+ does not belong to $\mathscr{C}$ for every non-zero epimorphic image T of S. Let d be the inner derivation on S induced by an element x in S. If $im(d)$ belongs to $\mathscr{C}$ then x is contained in the center of S.*

Reference

[1] Y. Hirano: On extensions of rings with finite additive index. Math. J. Okayama Univ. **32** (1990), 93—95.

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