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WEYL'S TYPE CRITERION FOR GENERAL DISTRIBUTION MOD 1 AND ITS APPLICATIONS

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1. Introduction. Let (x_n), $n = 1, 2, \dots$, be a sequence of real numbers, let $\{u\}$ denote the fractional part of the real number u and let $\chi([a, b) ; t)$ be the indicator function of the interval $[a, b)$. A sequence (x_n) is said to be *uniformly distributed* (*mod* 1) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi([0, t) ; \{x_n\}) = t$$

for each $t \in (0, 1]$.

In 1916, Weyl [13] has proposed the following necessary and sufficient condition that a given sequence of real numbers is uniformly distributed mod 1, which is now called Weyl's criterion for uniform distribution mod 1 : The sequence (x_n) of real numbers is uniformly distributed (mod 1) if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(\nu x_n) = 0 \text{ for all integers } \nu \neq 0,$$

where $e(u) = e^{2\pi i u}$.

Schoenberg [12], in 1928, first generalized the concept of uniform distribution (mod 1) to that of general distribution (asymptotic distribution) (mod 1). He obtained many results including a generalization of Weyl's criterion. Later, many mathematicians [6] have generalized Weyl's criterion to general distribution (mod 1) and obtained various forms of generalized Weyl's criteria. But these criteria did not have their sufficient form in practice and in theoretical point of view. The first author found and proved a very natural generalization of Weyl's criterion (Theorem 2) and it seems very curious for the authors that this criterion has not been seen before in any references. But among these references, only Helmberg [3] reached the nearest point to this criterion, but his point of view and his concepts were about numerical computations of integrals.

In this paper, we propose a new Weyl's criterion for general distribution (mod 1) and consider some generalized discrepancies. In general we usually use the well-known Erdös-Turán's theorem [2] and the LeVeque's inequality [7] to estimate the discrepancies. So we give here generalizations of the Erdös-Turán's theorem and the LeVeque's inequality respectively (Theorem 8, 9) in relation to Theorem 2.

A real-valued function $\mu(x)$ defined on $[0, 1]$ is called a *distribution function* (*mod* 1) if $\mu(x)$ is non-decreasing, left-continuous and satisfies $\mu(0) = 0$ and $\mu(1) = 1$. A positive Lebesgue measurable function $w(x)$ defined on $[0, 1]$ that is equal to $\mu'(x)$ a.e., is called the *density function of* $\mu(x)$. Let $\mu(x)$ be a distribution function (mod 1). The sequence (x_n) is said to be *μ-distributed* (*mod* 1) if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi([0, x) ; \{x_n\}) = \mu(x) \tag{1}$$

for each $x \in (0, 1]$.

Schoenberg [12] proved that if $\mu(x)$ is a continuous distribution function (mod 1), then the sequence (x_n) is μ-distributed (mod 1) if and only if for every real-valued continuous function $f(x)$ defined on $[0, 1]$, the following relation holds:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\}) = \int_0^1 f(x)d\mu(x). \tag{2}$$

We remark that this result is valid if the sufficient condition is replaced by the following condition: for every real-valued Riemann-Stieltjes integrable function $f(x)$ with respect to $\mu(x)$, defined on $[0, 1]$, the relation (2) holds.

2. Weyl's type criterion. In this section we give a Weyl's type criterion for μ-distribution (mod 1). The next theorem is a preliminary result.

Theorem 1. *Let $\mu(x)$ be an absolutely continuous distribution function (mod 1) with a positive Riemann-integrable density function $w(x)$. Then, the sequence (x_n) is μ-distributed (mod 1) if and only if for every real-valued continuous function $f(x)$ defined on $[0, 1]$, the following relation holds:*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{f(\{x_n\})}{w(\{x_n\})} = \int_0^1 f(x)dx. \tag{3}$$

Proof. It is well-known that since the function $\mu(x)$ is absolutely continuous on the interval $[0, 1]$, for every $x \in [0, 1]$,

$$\mu(x) = \int_0^x w(t)dt.$$

Then, we obtain

$$\int_0^1 \frac{f(x)}{w(x)} d\mu(x) = \int_0^1 \frac{f(x)}{w(x)} w(x) dx = \int_0^1 f(x) dx \tag{4}$$

for every continuous function $f(x)$ defined on $[0, 1]$. Suppose that the sequence (x_n) is μ-distributed (mod 1). Let $f(x)$ be a real-valued continuous function defined on $[0, 1]$. It follows from the remark of the result of Schoenberg and (4) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{f(\{x_n\})}{w(\{x_n\})} = \int_0^1 \frac{f(x)}{w(x)} d\mu(x) = \int_0^1 f(x) dx.$$

Conversely, suppose that for every real-valued continuous function $f(x)$ defined on $[0, 1]$, the relation (3) holds. Let $h(x)$ be a Riemann-integrable function defined on $[0, 1]$ and let ε be a given positive number. By the definition of the Riemann-integrable function, there exist two step functions $h_1(x)$ and $h_2(x)$ such that $h_1(x) \leqq h(x) \leqq h_2(x)$ for all $x \in [0, 1]$ and

$$\int_0^1 (h_2(x) - h_1(x)) dx \leqq \varepsilon/3.$$

Then there exist two continuous functions $f_1(x)$ and $f_2(x)$ such that $f_1(x) \leqq h_1(x)$ and $h_2(x) \leqq f_2(x)$ for all $x \in [0, 1]$ and

$$\int_0^1 (h_1(x) - f_1(x)) dx \leqq \varepsilon/3 \text{ and } \int_0^1 (f_2(x) - h_2(x)) dx \leqq \varepsilon/3,$$

so that

$$\int_0^1 (f_2(x) - f_1(x)) dx \leqq \varepsilon.$$

Hence, we have

$$\begin{aligned}
\int_0^1 h(x)dx - \varepsilon &< \int_0^1 h(x)dx - \int_0^1 (h(x) - f_1(x))dx \\
&= \int_0^1 f_1(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{f_1(\{x_n\})}{w(\{x_n\})} \\
&\leqq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{h(\{x_n\})}{w(\{x_n\})} \leqq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{f_2(\{x_n\})}{w(\{x_n\})} \\
&= \int_0^1 f_2(x)dx = \int_0^1 (f_2(x) - h(x))dx + \int_0^1 h(x)dx
\end{aligned}$$

$$\leqq \varepsilon+\int_0^1 h(x)dx,$$

so that the relation (3) holds for every Riemann-integrable function $h(x)$. Let $f(x)$ be a real-valued continuous function defined on $[0, 1]$. Since the function $f(x)w(x)$ is Riemann-integrable, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{f(\{x_n\})w(\{x_n\})}{w(\{x_n\})} = \int_0^1 f(x)w(x)dx,$$

so that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\}) = \int_0^1 f(x)d\mu(x).$$

From the result of Schoenberg, it follows that the sequence (x_n) is μ-distributed (mod 1).

The next corollary follows immediately.

Corollary 1. *Let $\mu(x)$ satisfy the same conditions as in Theorem* 1. *Then, the sequence (x_n) is μ-distributed* (*mod* 1) *if and only if for every complex-valued continuous function $f(x)$ defined on* $\mathbf{R}$ *with period* 1, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{f(\{x_n\})}{w(\{x_n\})} = \int_0^1 f(x)dx.$$

Now, we will show the next theorem: a generalization of Weyl's criterion.

Theorem 2. *Let $\mu(x)$ satisfy the same conditions as in Theorem* 1. *Then, the sequence (x_n) is μ-distributed* (*mod* 1) *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{e(\nu x_n)}{w(\{x_n\})} = \delta_{\nu,0} \text{ for all } \nu\in\mathbf{Z}, \tag{5}$$

where $\delta_{m,n}=1$ if $m=n$, $\delta_{m,n}=0$ otherwise.

Proof. The necessity follows from Theorem 1. Conversely, we suppose that the sequence (x_n) posseses the property (5). Let $f(x)$ be a complex-valued continuous function defined on $\mathbf{R}$ with period 1. By the Weierstrass approximation theorem, for any positive number $\varepsilon>0$, there exists a trigonometric polynomial $P(x)$, that is, a finite linear combination of functions of the type $e(\nu x)$, $\nu \in \mathbf{Z}$, with complex coefficient, such that

$$\sup_{0 \leqq x \leqq 1} |f(x) - P(x)| < \varepsilon.$$

Hence, we have

$$\begin{aligned}
&\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \frac{f(x_n)}{w(\{x_n\})} - \int_0^1 f(x)dx \right| \\
&\leqq \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \frac{|f(x_n) - P(x_n)|}{w(\{x_n\})} \\
&\quad + \lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \frac{P(x_n)}{w(\{x_n\})} - \int_0^1 P(x)dx \right| + \int_0^1 |P(x) - f(x)|dx \\
&\leqq \lim_{N\to\infty} \frac{\varepsilon}{N} \sum_{n=1}^{N} \frac{1}{w(\{x_n\})} + \varepsilon = 2\varepsilon
\end{aligned}$$

for any $\varepsilon > 0$. The last equality follows from (5) with $\nu = 0$. Therefore, from Corollary 1, it follows that the sequence (x_n) is μ-distributed (mod 1).

3. Applications. In this section we give some interesting applications of Weyl's type criterion. Let (p_n), $n = 1, 2, 3, \ldots$, be a sequence of non-negative real numbers with $p_1 > 0$ and put $s_n = p_1 + p_2 + \cdots + p_n$. Then the sequence (x_n) is (R, p_n)*-uniformly distributed* (*mod* 1) if

$$\lim_{N\to\infty} \frac{1}{s_N} \sum_{n=1}^{N} p_n \chi([0, x) ; \{x_n\}) = x$$

for all $0 < x \leqq 1$.

Theorem 3. *Let θ be a positive irrational number. Let $g(x)$ be a periodic function defined on $\mathbf{R}$ with period 2θ and let $g(x)$ satisfy the following conditions: $g(0) = 0$, $g(\theta) = 1$, $g(\theta - x) = g(\theta + x)$ for all $-\theta \leqq x \leqq \theta$, $g(x)$ is continuous and strictly increasing on $[0, \theta]$, and $g(x)$ is differentiable on $(0, \theta)$. Let $G(x)$ be an inverse function of $g(x)$. Then, the sequence $(g(n))$ is μ-distributed mod 1, where $\mu(x) = (1/\theta)G(x)$ for $0 \leqq x \leqq 1$. Furthermore, the sequence $(g(n))$ is $(R, \theta|g'(n)|)$-uniformly distributed (mod 1).*

Proof. For any Riemann-integrable function on $[0, 1]$ and any positive integer N, we have

$$\frac{1}{N} \sum_{n=1}^{N} f(\{g(n)\}) = \frac{1}{N} \sum_{n=1}^{N} f(g(n))$$

$$= \frac{1}{N}\sum_{n=1}^{N} f\left(g\left(4\theta\frac{n}{4\theta}\right)\right) = \frac{1}{N}\sum_{n=1}^{N} f\left(g\left(4\theta\left\{\frac{n}{4\theta}\right\}\right)\right). \tag{6}$$

Since the sequence $(n/(4\theta))$ is uniformly distributed (mod 1) (see [6], p.8) and the function $f(g(4\theta x))$ is Riemann-integrable on $[0, 1]$, we obtain, by (6)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(|g(n)|) = \int_0^1 f(g(4\theta x))dx.$$
$$= \left(\int_0^{1/4}+\int_{1/4}^{1/2}+\int_{1/2}^{1}\right)f(g(4\theta x))dx = I_1+I_2+J, \text{ say.}$$

By the change of variable of the integral, we get

$$I_1 = \frac{1}{4\theta}\int_0^1 f(u)G'(u)du.$$

Since the function $g(x)$ is symmetric,

$$I_2 = \int_{1/4}^{1/2} f(g(2\theta-4\theta x)dx.$$

Using the change of variable again, we have

$$I_2 = \frac{1}{4\theta}\int_0^1 f(u)G'(u)du.$$

Because of the periodicity of the function $g(x)$, it follows that

$$J = \int_{1/2}^{1} f(g(4\theta x-2\theta))dx$$
$$= \left(\int_{1/2}^{3/4}+\int_{3/4}^{1}\right)f(g(4\theta x-2\theta))dx = J_1+J_2, \text{ say.}$$

Similarly, using the change of variable of integrals and the property of $g(x)$: $g(\theta-x) = g(\theta+x)$ for all $-\theta \leqq x \leqq \theta$, we obtain

$$J_1 = \frac{1}{4\theta}\int_0^1 f(u)G'(u)du$$

and

$$J_2 = \int_{3/4}^{1} f(g(4\theta-4\theta x))dx = \frac{1}{4\theta}\int_0^1 f(u)G'(u)du.$$

Hence, we get

$$\int_0^1 f(g(4\theta x))\,dx = \int_0^1 f(x)\frac{1}{\theta}G'(x)dx = \int_0^1 f(x)d\left(\frac{G(x)}{\theta}\right)$$

Therefore, from the theorem of Schoenberg [12], it follows that the sequence $(g(n))$ is μ-distributed (mod 1), where $\mu(x) = G(x)/\theta$ for all $0 \leqq x \leqq 1$. Since the density function of $\mu(x)$ is $w(x) = 1/(\theta g'(G(x))$, by Theorem 2, we obtain

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\theta g'(G(g(n)))e(\nu g(n)) = \delta_{\nu,0} \text{ for all } \nu \in \mathbb{Z}.$$

Because of $g'(G(g(n))) = |g'(n)|$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\theta|g'(n)|e(\nu g(n)) = \delta_{\nu,0}. \tag{7}$$

for all $\nu \in \mathbb{Z}$. In case of $\nu = 0$, (7) yields

$$\sum_{n=1}^{N}\theta|g'(n)| \sim N \text{ as } N \to \infty.$$

Putting $s_N = \sum_{n=1}^{N}\theta|g'(n)|$, we have

$$\lim_{N\to\infty}\frac{1}{s_N}\sum_{n=1}^{N}\theta|g'(n)|e(\nu g(n)) = 0$$

for all $\nu \in \mathbb{Z}\backslash\{0\}$. Therefore, according to Weyl's criterion (see [6], p.61) it follows that the sequence $(g(n))$ is $(R, \theta|g'(n)|)$-uniformly distributed (mod 1).

Example 1. The function $g(x) = |\sin x|$ satisfies all conditions with $\theta = \pi/2$. We define the distribution function (mod 1) $\mu(x) = (2/\pi)\text{arc}\sin x$. From Theorem 3 it follows that the sequence $(|\sin n|)$ is μ-distributed (mod 1) and $(R, (\pi/2)|\cos n|)$-uniformly distributed (mod 1).

By the same reasoning as in the proof of Theorem 3, we obtain the next theorem.

Theorem 4. *Let $g(x)$ be a real-valued function with period θ, where θ is a positive irrational number. Let $g(x)$ satisfy the following conditions: $g(0) = 0$, $g(\theta) = 1$, $g(x)$ is continuous and strictly increasing on $[0, 1]$, $g(x)$ is differentiable on $(0, 1)$, and $g(x)$ has an inverse function $G(x)$ on*

$[0, \theta]$. *Then, the sequence* $(g(n))$ *is* μ*-distributed* (*mod* 1) *and* $(R, \theta g'(n))$*-uniformly distributed* (*mod* 1), *where the distribution function* (*mod* 1) $\mu(x) = (1/\theta)G(x)$.

We also state the following theorem without proof.

Theorem 5. *Let* (x_n) *be a sequence in* $[0, 1)$ *which is uniformly distributed* (*mod* 1). *Let* $\psi(x)$ *be a differentiable increasing function on* $[0, 1]$ *with* $\psi(0) = 0$, $\psi(1) = 1$. *Then the sequence* $(\psi(x_n))$ *is* μ*-distributed* (*mod* 1) *and* $(R, \psi'((x_n)))$*-distributed* (*mod* 1), *where* $\mu(x)$ *is the inverse function of* $\psi(x)$.

Example 2. The function $\psi(x) = e^{x\log 2} - 1 = 2^x - 1$ satisfies the conditions of Theorem 5. Let (x_n) be a uniformly distributed (mod 1) sequence. Then the sequence $(2^{|x_n|})$ is μ-distributed (mod 1) and is $(R, (\log 2)2^{|x_n|})$-uniformly distributed (mod 1), where $\mu(x) = \log(x+1)/\log 2$ which is Gaussian metric in continued fraction theory. By Theorem 2, we obtain

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} 2^{|x_n|}e(\nu 2^{|x_n|}) = 0 \text{ for } \nu \in \mathbb{Z},\ \nu \neq 0.$$

4. Some discrepancies. The following definitions of discrepancies can be found in [3], [4] and [6]. Let N be a positive integer, let $\omega = (x_n)$, $n = 1, 2, \ldots, N$ be a finite sequence of real numbers in $[0, 1)$, and let $\mu(x)$ be a distribution function (mod 1) with a density function $w(x)$. Then, the *discrepancy* (*mod* 1) $D_N(\omega\,;\,\mu)$ is defined to be

$$D_N(\omega\,;\,\mu) = \sup_{0\leq a<b\leq 1}\left|\frac{1}{N}\sum_{n=1}^{N}\chi([a, b)\,:\,x_n) - (\mu(b)-\mu(a))\right|$$

and the *starred discrepancy* (*mod* 1) $D_N^*(\omega\,;\,\mu)$ is defined to be

$$D_N^*(\omega\,;\,\mu) = \sup_{0< x\leq 1}\left|\frac{1}{N}\sum_{n=1}^{N}\chi([0, x)\,;\,x_n) - \mu(x)\right|.$$

Furthermore, we introduce some other kind of discrepancies. We define the μ*-discrepancy:*

$$E_N(\omega\,;\,w) = \sup_{0\leq a<b\leq 1}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\chi([a, b)\,;\,x_n)}{w(x_n)} - (b-a)\right|$$

and the *starred* μ*-discrepancy:*

$$E_N^*(\omega\,;\,w)=\sup_{0<x\leqq 1}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\chi([0,x)\,;\,x_n)}{w(x_n)}-x\right|.$$

For an infinite sequence $\xi=(x_n)$, $n=1,2,\ldots$, of real numbers the discrepancies $D_N(\xi\,;\,\mu)$, $D_N^*(\xi\,;\,\mu)$, $E_N(\xi\,;\,w)$ and $E_N^*(\xi\,;\,w)$ are meant to be the discrepancies of the initial segment formed by the first N terms of ξ, respectively.

The next relations between the discrepancies and the starred discrepancies is well-known: $D_n(\xi\,;\,\mu)\leqq 2D_N^*(\xi\,;\,\mu)$ and $E_N(\xi\,;\,w)\leqq 2E_N^*(\xi\,;\,w)$ (cf. [6]). On the other hand, we will get the relations between $D_n(\xi\,;\,\mu)$ and $E_N(\xi\,;\,w)$. We need the two Lemmas which are some generalizations of Koksma's inequality [5] and found in the slight different forms in [3].

Lemma 1 (analogue of Satz 9 in [3]). *Let $w(x)$ be the positive density function of a distribution function (mod 1) $\mu(x)$, and let $f(x)$ be a real-valued function on $[0,1]$ of bounded variation $V(f)$. Then, for all $N\geqq 1$, we have*

$$\left|\frac{1}{N}\sum_{n=1}^{N}\frac{f(\{x_n\})}{w(\{x_n\})}-\int_0^1 f(x)dx\right|\leqq V(f)E_N(\xi;\,w)+|f(1)|\left|1-\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(\{x_n\})}\right|.$$

Lemma 2 ([3], Satz 4). *Let $\mu(x)$ be a continuous distribution function (mod 1) and let $f(x)$ be a real-valued function on $[0,1]$ of bounded variation $V(f)$. Then, for all $N\geqq 1$, we have*

$$\left|\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\})-\int_0^1 f(x)d\mu(x)\right|\leqq V(f)D_N(\xi\,;\,\mu).$$

Applying Lemma 1 and 2, we get the relations between $D_N(\xi\,;\,\mu)$ and $E_N(\xi\,;\,w)$.

Theorem 6. *Let $w(x)$ be a density function of an absolutely continuous distribution function (mod 1) $\mu(x)$, and suppose that $w(x)$ is of bounded variation $V(w)$ on $[0,1]$. Let $\xi=(x_n)$ be a sequence of real numbers and let N be a positive integer. Then we have*

$$D_N(\xi\,;\,\mu)\leqq(V(w)+2\sup_{0\leqq x\leqq 1}w(x))E_N(\xi\,;\,w).$$

Proof. Let $0\leqq a<b\leqq 1$. Applying Lemma 1 with $f(x)=\chi([a,b)\,;\,x)w(x)$, since $f(1)=0$, we get

$$\left|\frac{1}{N}\sum_{n=1}^{N}\chi([a, b) ; \{x_n\})-(\mu(b)-\mu(a))\right|$$
$$=\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\chi([a, b) ; \{x_n\})w(\{x_n\})}{w(\{x_n\})}-\int_a^b w(x)dx\right|$$
$$=\left|\frac{1}{N}\sum_{n=1}^{N}\frac{f(\{x_n\})}{w(\{x_n\})}-\int_0^1 f(x)dx\right| \leqq V(f)E_N(\xi ; w). \tag{8}$$

Since $V(f) = V(\chi ; w) \leqq V(w)+2\sup_{0\leqq x\leqq 1} w(x)$, from (8) we have

$$D_N(\xi ; \mu) \leqq (V(w)+2\sup_{0\leqq x\leqq 1} w(x))E_N(\xi ; w).$$

Theorem 7. *Let $w(x)$ be the density function of an absolutely continuous distribution function (mod 1) $\mu(x)$, and suppose that $1/w(x)$ is of bounded variation $V(1/w)$ on $[0, 1]$. Let $\xi=(x_n)$ be a sequence of real numbers, and let N be a positive integer. Then, we have*

$$E_N(\xi ; w) \leqq (V(1/w)+2\sup_{0\leqq x\leqq 1} 1/w(x))D_N(\xi ; \mu).$$

Proof. Let $0\leqq a<b\leqq 1$. Applying Lemma 2 with $f(x) = \chi([a, b) ; x)/w(x)$. We get

$$\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\chi([a, b) ; \{x_n\})}{w(\{x_n\})}-(b-a)\right| = \left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})-\int_a^b\frac{1}{w(x)}d\mu(x)\right|$$
$$=\left|\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\})-\int_0^1 f(x)d\mu(x)\right| \leqq V(f)D_N(\xi ; \mu).$$

Since $V(f) \leqq V(1/w)+2\sup_{0\leqq x\leqq 1}(1/w(x))$, we get

$$E_N(\xi ; w) \leqq (V(1/w)+2\sup_{0\leqq x\leqq 1}(1/w(x))D_N(\xi ; \mu).$$

5. A generalization of Erdös-Turán's theorem. We shall prove a generalization of Erdös-Turán's theorem ([2]) to μ-distribution. We follow the method that Niederreiter and Philipp ([9]) used to prove the theorem of Erdös and Turán.

Theorem 8. *For any integer $m\geqq 1$, any sequence $\omega=\{x_1, x_2, \ldots, x_N\}$ of real numbers and any positive density function $w(x)$, we have*

$$E_N^*(\omega ; w)$$
$$\leqq \frac{3}{m+1}+\frac{2}{\pi}\sum_{\nu=1}^{m}\left(\frac{1}{\nu}-\frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N}\frac{e(\nu x_n)}{w(\{x_n\})}-\left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(\{x_n\})}-1\right)\right|$$
$$+5\left|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(\{x_n\})}-1\right|+2\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\{x_n\}}{w(\{x_n\})}-\frac{1}{2}\right|.$$

Proof. We may assume that $0 \leqq x_i < 1$ for $i = 1, 2, \ldots, m$. For $0 \leqq x \leqq 1$, we set

$$R_N(x) = \sum_{n=1}^{N}\frac{\chi([0, x) : x_n)}{w(x_n)} - Nx$$

and $\Delta_N(x) = (1/N)R_N(x)$. Here, we extend the function $\Delta_N(x)$ to $\mathbf{R}$ with period 1. Let

$$S_\nu = \frac{1}{N}\sum_{n=1}^{N}\left(\frac{e(\nu x_n)}{w(x_n)} - \frac{1}{w(x_n)} + 1\right)$$

for $\nu \in \mathbf{Z}$. For $\nu \in \mathbf{Z}\backslash\{0\}$ we have

$$\int_0^1 \Delta_N(x)e(\nu x)dx = \frac{1}{N}\int_0^1 R_N(x)e(\nu x)dx$$
$$= \frac{1}{N}\int_0^1\left(\sum_{n=1}^{N}\frac{\chi([0, x) ; x_n)}{w(x_n)} - Nx\right)e(\nu x)dx$$
$$= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}\int_0^1 \chi((x_n, 1] ; x)e(\nu x)dx - \int_0^1 xe(\nu x)dx$$
$$= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}\int_{x_n}^1 e(\nu x)dx - \frac{1}{2\pi i\nu}$$
$$= \frac{1}{-2\pi i\nu}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{e(\nu x_n)}{w(x_n)} - \frac{1}{w(x_n)}\right) - \frac{1}{2\pi i\nu} = \frac{1}{-2\pi i\nu}S_\nu. \tag{9}$$

Let m be a positive integer, and let a be a real number that is determined later. Setting $a_0 = \int_0^1 \Delta_N(x)dx$, by (9) we obtain

$$\sum_{\nu=-m}^{m}{}' (m+1-|\nu|)e(-\nu a)\frac{S_\nu}{-2\pi i\nu}$$
$$= \sum_{\nu=-m}^{m}{}' (m+1-|\nu|)e(-\nu a)\int_0^1 (\Delta_N(x)-a_0)e(\nu x)dx$$
$$= \int_0^1 (\Delta_N(x)-a_0)\left(\sum_{\nu=-m}^{m}{}' (m+1-|\nu|)e(\nu(x-a))\right)dx$$

$$= \int_{-a}^{1-a} (\Delta_N(x+a) - a_0)\Big(\sum_{\nu=-m}^{m}{}' (m+1-|\nu|)e(\nu x)\Big)dx$$
$$= \int_{-1/2}^{1/2} (\Delta_N(x+a) - a_0)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx,$$

where the prime in the sum indicates that $\nu = 0$ is deleted from the range of summation. Hence, we have

$$\left|\int_{-1/2}^{1/2} (\Delta_N(x+a) - a_0)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx\right|$$
$$\leqq \frac{1}{2\pi}\sum_{\nu=-m}^{m}{}' (m+1-|\nu|)\frac{|S_\nu|}{|\nu|} \tag{10}$$

We either have $\Delta_N(x_0) = -E_N^*$ or $\Delta_N(x_0+0) = E_N^*$ for some $x_0 \in [0, 1)$, where $E_N^* = E_N^*(\omega\,;\,w)$. First, we treat the first case. For $x_0 - E_N^* \leqq t \leqq x_0$, we have $\Delta_N(t) \leqq -E_N^* + x_0 - t$. Choosing $a = x_0 - (1/2)E_N^*$, we get

$$-\Delta_N(x+a) \geqq (1/2)E_N^* + x \text{ for } |x| < (1/2)E_N^*.$$

Therefore, since $\int_0^{1/2} \frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx = \frac{m+1}{2}$, we obtain

$$-\int_{-1/2}^{1/2} (\Delta_N(x+a) - a_0)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx$$
$$= -\int_{-1/2}^{1/2} \Delta_N(x+a)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx + a_0\int_{-1/2}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx$$
$$= -\int_{-1/2}^{1/2} \Delta_N(x+a)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx + a_0(m+1)$$
$$= -\left(\int_{(-1/2)E_N^*}^{(1/2)E_N^*} + \int_{-1/2}^{(-1/2)E_N^*} + \int_{(1/2)E_N^*}^{1/2}\right)\Delta_N(x+a)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx + a_0(m+1)$$
$$\geqq \int_{(-1/2)E_N^*}^{(1/2)E_N^*}\left(\frac{1}{2}E_N^* + x\right)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx - 2E_N^*\int_{(1/2)E_N^*}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx$$
$$+ a_0(m+1)$$
$$= E_N^*\int_0^{(1/2)E_N^*}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx - 2E_N^*\int_{(1/2)E_N^*}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx + a_0(m+1)$$
$$= E_N^*\int_0^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx - 3E_N^*\int_{(1/2)E_N^*}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx + a_0(m+1)$$
$$\geqq E_N^*\frac{m+1}{2} - 3E_N^*\int_{(1/2)E_N^*}^{1/2}\frac{1}{4x^2}dx + a_0(m+1)$$
$$\geqq \frac{m+1}{2}E_N^* - \frac{3}{2} + a_0(m+1) \tag{11}$$

From (10) and (11) we obtain

$$\frac{1}{2\pi}\sum_{\nu=-m}^{m}{}'(m+1-|\nu|)\frac{|S_\nu|}{|\nu|} \geqq \frac{m+1}{2}E_N^* - \frac{3}{2} + a_0(m+1),$$

so

$$E_N^* \leqq \frac{3}{m+1} + \frac{2}{\pi}\sum_{\nu=1}^{m}\left(\frac{1}{\nu} - \frac{1}{m+1}\right)|S_\nu| - 2a_0. \tag{12}$$

Since

$$a_0 = \left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)} - 1\right) - \left(\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)} - \frac{1}{2}\right),$$

the desired result follows. Next, we treat the second case: $\Delta_N(x_0+0) = E_N^*$ for some $x_0 \in [0, 1)$. If $x_0 + E_N^* \leqq 1$, then we get the result in exactly the same way as the first case. So, we assume that $1 < x_0 + E_N^*$. For $x_0 < t \leqq 1$, then we have

$$\Delta_N(t) \geqq \Delta_N' + 1 - t,$$

where $0 < \Delta_N' = E_N^* + x_0 - 1 \leqq \Delta_N(1)$. Now, choosing $a = (x_0+1)/2$, we get

$$\Delta_N(x+a) \geqq \frac{E_N^* + \Delta_N'}{2} - x \text{ for } |x| < \frac{E_N^* - \Delta_N'}{2}.$$

Therefore, we have

$$\begin{aligned}
&\int_{-1/2}^{1/2}(\Delta_N(x+a) - a_0)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx \\
&= \left(\int_{(-1/2)(E_N^*-\Delta_N')}^{(1/2)(E_N^*-\Delta_N')} + \int_{-1/2}^{(-1/2)(E_N^*-\Delta_N')} + \int_{(-1/2)(E_N^*-\Delta_N')}^{1/2}\right)\Delta_N(x+a)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx \\
&\quad - a_0(m+1) \\
&\geqq \int_{(-1/2)(E_N^*-\Delta_N')}^{(1/2)(E_N^*-\Delta_N')}\left(\frac{E_N^*+\Delta_N'}{2} - x\right)\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx \\
&\quad + \int_{-1/2}^{(-1/2)(E_N^*-\Delta_N')}(\Delta_N(x+a) + \Delta_N')\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx \\
&\quad + \int_{(1/2)(E_N^*-\Delta_N')}^{1/2}(\Delta_N(x+a) + \Delta_N')\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx - a_0(m+1) \\
&\quad - 2\Delta_N'\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2 \pi x}dx
\end{aligned}$$

$$\geqq \frac{E_N^*+\Delta_N'}{2}\int_{(-1/2)(E_N^*-\Delta_N')}^{(1/2)(E_N^*-\Delta_N')}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx$$

$$+2(-E_N^*+\Delta_N')\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx-a_0(m+1)$$

$$-2\Delta_N'\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx$$

$$\geqq (E_N^*-\Delta_N')\left(\int_0^{1/2}-\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\right)\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx$$

$$-2(E_N^*-\Delta_N')\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx$$

$$-2\Delta_N'\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx-a_0(m+1)$$

$$\geqq (E_N^*-\Delta_N')\frac{m+1}{2}-3(E_N^*-\Delta_N')\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx$$

$$-2\Delta_N'\int_0^{1/2}\frac{\sin^2(m+1)\pi x}{\sin^2\pi x}dx-a_0(m+1)$$

$$\geqq \frac{m+1}{2}(E_N^*-\Delta_N')-3(E_N^*-\Delta_N')\int_{(1/2)(E_N^*-\Delta_N')}^{1/2}\frac{dx}{4x^2}-\Delta_N'(m+1)-a_0(m+1)$$

$$=\frac{m+4}{2}E_N^*-\frac{3}{2}(m+2)\Delta_N'-\frac{3}{2}-a_0(m+1)$$

$$\geqq \frac{m+4}{2}E_N^*-\frac{3}{2}(m+2)\Delta_N(1)-\frac{3}{2}-a_0(m+1), \tag{13}$$

where the last inequality is obtained by $\Delta_N'\leqq\Delta_N(1)$. From (10) and (13), we get

$$\frac{m+4}{2}E_N^*-\frac{3}{2}(m+2)\Delta_N(1)-\frac{3}{2}-a_0(m+1)\leqq\frac{1}{2}\sum_{\nu=-m}^{m}{}'(m+1-|\nu|)\frac{|S_\nu|}{|\nu|},$$

so

$$E_N^*\leqq\frac{3}{m+4}+\frac{2}{\pi}\sum_{\nu=1}^{m}\left(\frac{1}{\nu}-\frac{1}{m+1}\right)|S_\nu|+2a_0+3\Delta_N(1) \tag{14}$$

Consequently, by (12) and (14) we get

$$E_N^*\leqq\frac{3}{m+1}+\frac{2}{\pi}\sum_{\nu=1}^{m}\left(\frac{1}{\nu}-\frac{1}{m+1}\right)|S_\nu|+2|a_0|+3|\Delta_N(1)|.$$

Remark. We can also derive the following similar result by using Berry-Essen's type inequality due to Niederreiter-Phillip [9: Th. 1], Elliot

[1 : Th.2] and Proinov [10 : Th.1, 11 : Th.3]:

$$E_N^*(\omega ; w) \leqq \frac{4}{m+1}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}\right| + \left|1-\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}\right|$$
$$+\frac{4}{\pi}\sum_{\nu=1}^{m}\left(\frac{1}{\nu}-\frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N}\frac{e(\nu x_n)}{w(x_n)}\right|.$$

Our result is different from that of theirs in that we consider that $\sum_{n=1}^{N}\frac{1}{w(x_n)}$ is not necessarily equal to N.

From Theorem 6 and the relation $E_N \leqq 2E_N^*$, the following corollary yields.

Corollary 2. *Let $\mu(x)$ be an absolutely continuous distribution function (mod 1) with a density function $w(x)$ which is of bounded variation on $[0, 1]$. For any integer $m \geqq 1$ and any sequence $\omega = \{x_1, x_2, \ldots, x_N\}$ of real numbers, we have*

$$D_N(\omega ; \mu) \leqq C\left(\frac{1}{m+1}+\sum_{\nu=1}^{m}\frac{1}{\nu}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{e(\nu x_n)}{w(\{x_n\})}\right|\right.$$
$$+(\log m)\left|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(\{x_n\})}-1\right|$$
$$\left.+\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\{x_n\}}{w(\{x_n\})}-\frac{1}{2}\right|\right),$$

where the constant C only depends on $w(x)$.

6. A generalization of LeVeque's inequality. We shall also give the generalization of LeVeque's inequality [7]. Our proof is a modified form of that of Kuipers and Niederreiter ([6], p.111, Theorem 2.4).

Theorem 9. *For any sequence $\omega = \{x_1, x_2, \ldots, x_N\} \subset [0, 1]$ and any distribution function $\mu(x)$ with a positive density function $w(x)$, we have*

$$E_N(\omega ; w) \leqq \left(\frac{6}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{e(\nu x_n)}{w(x_n)}-\left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\right)\right|^2\right.$$
$$+4\max\left(\left(\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}\right)^3, 0\right)$$
$$\left.-4\min\left(\left(\left(\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}\right)-\left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\right)\right)^3, 0\right)\right)^{1/3}.$$

Proof. The function $R_N(x)$ is piecewise linear in $[0, 1]$ and has only finite many discontinuities at $x_1, x_2, \ldots, x_N$ (see the definition of $R_N(x)$ in the proof of Theorem 8). We can expand $R_N(x)$ into a Fourier series and we have on the interval $[0, 1]$ apart from the finitely many points,

$$R_N(x) = \sum_{\nu=-\infty}^{\infty} a_\nu e(\nu x),$$

where $a_\nu = \int_0^1 R_N(x) e(-\nu x) dx$. Now, we have

$$\begin{aligned} a_0 &= \int_0^1 R_N(x) dx = \int_0^1 \left(\sum_{n=1}^{N} \frac{\chi((x_n, 1] ; x)}{w(x_n)} - Nx \right) dx \\ &= \sum_{n=1}^{N} \frac{1}{w(x_n)} \int_0^1 \chi((x_n, 1] ; x) dx - N \int_0^1 x\, dx \\ &= \sum_{n=1}^{N} \frac{1-x_n}{w(x_n)} - \frac{N}{2} = -\sum_{n=1}^{N} \left(\frac{x_n}{w(x_n)} - \frac{1}{w(x_n)} + \frac{1}{2} \right) \end{aligned} \tag{15}$$

and for $\nu \neq 0$, we have

$$\begin{aligned} a_\nu &= \int_0^1 R_N(x) e(-\nu x) dx \\ &= \int_0^1 \left(\sum_{n=1}^{N} \frac{\chi((x_n, 1] ; x)}{w(x_n)} - Nx \right) e(-\nu x) dx \\ &= \sum_{n=1}^{N} \frac{1}{w(x_n)} \int_{x_n}^1 e(-\nu x) dx - N \int_0^1 x e(-\nu x) dx \\ &= \frac{1}{2\pi i \nu} \sum_{n=1}^{N} \left(\frac{e(-\nu x_n)}{w(x_n)} - \frac{1}{w(x_n)} \right) + \frac{N}{2\pi i \nu} \\ &= \frac{1}{2\pi i \nu} \sum_{n=1}^{N} \left(\frac{e(-\nu x_n)}{w(x_n)} - \frac{1}{w(x_n)} + 1 \right). \end{aligned}$$

By Perseval's equality, we obtain

$$\begin{aligned} \int_0^1 R_N^2(x) dx = &\left(\sum_{n=1}^{N} \left(\frac{x_n}{w(x_n)} - \frac{1}{w(x_n)} + \frac{1}{2} \right) \right)^2 \\ &+ \frac{1}{2\pi^2} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n=1}^{N} \left(\frac{e(-\nu x_n)}{w(x_n)} - \frac{1}{w(x_n)} + 1 \right) \right|^2 \end{aligned} \tag{16}$$

We put

$$S_N = \sum_{n=1}^{N} \left(\frac{x_n}{w(x_n)} - \frac{1}{w(x_n)} + \frac{1}{2} \right), \text{ and } T_N(x) = \frac{1}{N} (R_N(x) + S_N)$$

for $0 \leqq x \leqq 1$. The function $T_N(x)$ is a piecewise linear function which consists of straight line segments with a slope -1 and has only finite many discontinuities $x_1, \ldots, x_N$, each of which has a jump by a positive number. We can continue $T_N(x)$ to $T_N^*(x)$ with period 1 over $\mathbf{R}$. Now, we have

$$\begin{aligned}\int_0^1 T_N(x)dx &= \frac{1}{N}\int_0^1 R_N(x)dx+\frac{1}{N}S_N \\ &= -\frac{1}{N}\sum_{n=1}^{N}\left(\frac{x_n}{w(x_n)}-\frac{1}{w(x_n)}+\frac{1}{2}\right) \\ &\quad +\frac{1}{N}\sum_{n=1}^{N}\left(\frac{x_n}{w(x_n)}-\frac{1}{w(x_n)}+\frac{1}{2}\right) = 0.\end{aligned}$$

Therefore, there exist α and $\beta \in (0, 1)$ with $T_N(\alpha) > 0$ and $T_N(\beta) < 0$, except the trivial case that $x_1 = x_2 = \cdots = x_N = 0$. We distinguish the following four cases.

Case I: $\alpha+T_N(\alpha) \leqq 1$ and $\beta+T_N(\beta) \geqq 0$. In this case we have $T_N(x) \geqq -x+\alpha+T_N(\alpha)$ for $\alpha < x \leqq \alpha+T_N(\alpha)$. By the periodicity of $T_N^*(x)$, there exists $\beta_1 \in [\alpha, \alpha+1]$ with $T_N^*(\beta_1) = T_N(\beta)$. We also have $|T_N^*(x)| \geqq x-\beta_1-T_N^*(\beta_1)$ for $\beta_1+T_N^*(\beta_1) \leqq x \leqq \beta_1$. By the property of the graph of $T_N(x)$, the intervals $[\alpha, \alpha+T_N(\alpha)]$ and $[\beta_1+T_N^*(\beta_1), \beta_1]$ can have at most one point in common. Hence, we have

$$\begin{aligned}\int_0^1 T_N^2(x)dx &= \int_0^1 T_N^{*2}(x)dx = \int_\alpha^{\alpha+1} T_N^{*2}(x)dx \\ &\geqq \int_\alpha^{\alpha+T_N(\alpha)} T_N^2(x)dx+\int_{\beta_1+T_N^*(\beta_1)}^{\beta_1} T_N^{*2}(x)dx \\ &\geqq \int_\alpha^{\alpha+T_N(\alpha)} (-x+\alpha+T_N(\alpha))^2dx+\int_{\beta_1+T_N^*(\beta_1)}^{\beta_1}(x-\beta_1-T_N^*(\beta_1))^2dx \\ &= \frac{1}{3}T_N^3(\alpha)+\frac{1}{3}(-T_N^{*3}(\beta_1)) = \frac{1}{3}(T_N^3(\alpha)+(-T_N^3(\beta)) \qquad (17)\end{aligned}$$

Case II: $\alpha+T_N(\alpha) > 1$ and $\beta+T_N(\beta) \geqq 0$. From the property of the graph of the function $T_N(x)$, it is evident that for $\alpha \leqq x < \alpha+T_N(\alpha)-T' = 1$, $T_N(x) \geqq -x+\alpha+T_N(\alpha)$, where $0 < T' = T_N(\alpha)+\alpha-1 \leqq T_N(1)$. Now we have, as in Case I,

$$\begin{aligned}\int_0^1 T_N^2(x)dx &= \int_\alpha^{\alpha+1} T_N^{*2}(x)dx \geqq \int_\alpha^1 T_N^2(x)dx+\int_{\beta_1+T_N^*(\beta_1)}^{\beta_1} T_N^{*2}(x)dx \\ &\geqq \int_\alpha^{\alpha+T_N(\alpha)-T_N(1)} (-x+\alpha+T_N(\alpha))^2dx+\int_{\beta_1+T_N^*(\beta_1)}^{\beta_1}(x-\beta_1-T_N^*(\beta_1))^2dx\end{aligned}$$

$$= -\frac{1}{3}T_N^3(1)+\frac{1}{3}T_N^3(\alpha)+\frac{1}{3}(-T_N^3(\beta))$$

Hence, we have

$$\frac{1}{3}(T_N^3(\alpha)+(-T_N^3(\beta))) \leqq \int_0^1 T_N^2(x)dx+\frac{1}{3}T_N^3(1). \tag{18}$$

Case III: $\alpha+T_N(\alpha) \leqq 1$ and $\beta+T_N(\beta) < 0$. It is shown in the same way as Case II that

$$\frac{1}{3}(T_N^3(\alpha)+(-T_N^3(\beta))) \leqq \int_0^1 T_N^2(x)dx-\frac{1}{3}T_N^3(0) \tag{19}$$

Case IV: $\alpha+T_N(\alpha) > 1$ and $\beta+T_N(\beta) < 0$. We have the next inequality in the same way.

$$\frac{1}{3}(T_N^3(\alpha)+(-T_N^3(\beta))) \leqq \int_0^1 T_N^2(x)dx-\frac{1}{3}T_N^3(0)+\frac{1}{3}T_N^3(1). \tag{20}$$

Consequently, by (17), (18), (19) and (20), we have

$$\frac{1}{3}(T_N^3(\alpha)+(-T_N^3(\beta)))$$
$$\leqq \int_0^1 T_N^2(x)dx-\mathrm{Min}\left(\frac{1}{3}T_N^3(0),\ 0\right)+\mathrm{Max}\left(\frac{1}{3}T_N^3(1),\ 0\right). \tag{21}$$

From a simple inequality $r^3+s^3 \geqq (1/4)(r+s)^3$ for any $r \geqq 0$, $s \geqq 0$ and (21), we obtain

$$\frac{1}{12}(T_N(\alpha)-T_N(\beta))^3$$
$$\leqq \int_0^1 T_N^2(x)dx-\mathrm{Min}\left(\frac{1}{3}T_N^3(0),\ 0\right)+\mathrm{Max}\left(\frac{1}{3}T_N^3(1),\ 0\right). \tag{22}$$

Now, we can easily deduce that (22) holds even for all α and β in $[0, 1]$. By the definition of $T_N(x)$, we have

$$\frac{1}{12}\left(\frac{R_N(\alpha)-R_N(\beta)}{N}\right)^3$$
$$\leqq \int_0^1 T_N^2(x)dx-\mathrm{Min}\left(\frac{1}{3}T_N^3(0),\ 0\right)+\mathrm{Max}\left(\frac{1}{3}T_N^3(1),\ 0\right)$$

for all $\alpha, \beta \in [0, 1]$. Hence, we get

$$\frac{1}{12}E_N^3 \leqq \int_0^1 T_N^2(x)dx+\mathrm{Max}\left(\frac{1}{3}T_N^3(1),0\right)-\mathrm{Min}\left(\frac{1}{3}T_N^3(0),0\right) \quad (23)$$

Now, we compute the right hand side of (23).

$$\int_0^1 T_N^2(x)dx = \frac{1}{N^2}\int_0^1 R_N^2(x)dx+\frac{2}{N^2}S_N\int_0^1 R_N(x)dx+\frac{1}{N^2}S_N^2$$
$$= \frac{1}{N^2}S_N^2+\frac{1}{2\pi^2N^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{n=1}^{N}\left(\frac{e(\nu x_n)}{w(x_n)}-\frac{1}{w(x_n)}+1\right)\right|^2-\frac{2}{N^2}S_N^2+\frac{1}{N^2}S_N^2$$
$$= \frac{1}{2\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\frac{1}{N}\sum_{n=1}^{N}\left(\frac{e(\nu x_n)}{w(x_n)}-\frac{1}{w(x_n)}+1\right)\right|^2$$

by (14) and (15). Combining this and (23), we obtain the theorem.

Remark. We note that some of results in this paper have been announced in [8] without proofs and the estimation of the trigonometric sum (1) in [8] is incorrect, as for the correct form, see Example 1 in this paper.

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