Math. J. Okayama Univ. 33 (1991), 189－200

G-STRUCTURES ON SPHERES

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1. Introduction. In this paper we consider the problem of determining G-structures on the standard n-sphere S^n. More precisely, let G_n denote either the special orthogonal group $SO(n)$, the special unitary group $SU(n)$ or the symplectic group $Sp(n)$. Given a closed connected subgroup H of G_n we ask whether or not the principal bundle

$$G_n \to G_{n-1} \to G_{n+1}/G_n \qquad (*)$$

admits a reduction of the structure group to H.

The problem has been solved in significant cases by Adams, Atiyah, Todd, Walker, Steenrod, Leonard, Önder and Dibag. We extend the above results using the classification of compact, connected Lie groups G which act transitively and effectively on S^n. We will consider the quaternionic, complex and real cases of the problem separately.

First, we consider the quaternionic case, $G_n = Sp(n)$:

$$Sp(n) \to Sp(n+1) \to S^{4n+3} \qquad (1).$$

P. Leonard [16, Theorem I. C] has obtained a solution to the general case for $n \not\equiv 11 \bmod 12$; there is not a reduction to any subgroup H of $Sp(n)$ for $n \not\equiv 11 \bmod 12$. Moreover, Sigrist and Suter [21] obtained a final solution for H the standard subgroup $Sp(n-k)$, $1 \leqq k < n$. Let c_k be the k-th quaternionic James number. In general we can find the integer k such that $c_k | n+1$ and $c_{k+1} \nmid n+1$ for any n. Then the principal bundle (1) can be reduced to $Sp(n+1-k)$, and cannot be reduced to $Sp(n-k)$ (see [22]). We consider a subgroup H of $Sp(n+1-k)$, then we have the following:

Theorem 1. *For $c_k | n+1$ and $c_{k+1} \nmid n+1$, the principal bundle* (1) *cannot be reduced to any proper subgroup H of $Sp(n+1-k)$.*

Note that, for $k = 1$, the condition $c_k | n+1$ and $c_{k+1} \nmid n+1$ can be rewritten as $n \not\equiv 23 \bmod 24$ since $c_1 = 1$ and $c_2 = 24$ (see [13]). Consequently, Theorem 1 is a stronger result than the results of Leonard [16, Theorem I. C].

Second, we consider the complex case, $G_n = SU(n)$:

$$SU(n) \to SU(n+1) \to S^{2n+1} \qquad (2).$$

The results of Atiyah and Todd [3, Theorem 1.7] and also of Adams and Walker [2, Theorem 1.2] completely solve the problem for H the standard subgroup $SU(n-k)$, $1 \leqq k < n$. P. Leonard [16, Theorem I. B] has obtained a solution to the general cases for n even; there is not a reduction to any subgroup H of $SU(n)$ for n even.

Besides, if n is odd, T. Önder [20, Theorem 4.1] has obtained the complete solution for H, the standard subgroup $Sp(s)$, $1 < s \leqq \frac{n-1}{2}$.

Let b_k be the complex James number. In general we can find the integer k such that $b_{2k} | n+1$ and $b_{2k+1} \nmid n+1$ for n odd. Then the principal bundle (2) can be reduced to $SU(n+1-2k)$, and cannot be reduced to $SU(n-2k)$ by [13, Theorem 1.4], [2] and [3]. And then the principal bundle (2) can be reduced to $Sp\left(\frac{n+1-2k}{2}\right)$ if and only if $c_k \left| \frac{n+1}{2}\right.$ by [20]. We consider a subgroup H of $SU(n+1-2k)$, then we have the following:

Theorem 2. *For $b_{2k}|n+1$ and $b_{2k+1} \nmid n+1$, if $c_k \nmid \frac{n+1}{2}$ then the principal bundle (2) cannot be reduced to any proper subgroup H of $SU(2q)$, where $2q = n+1-2k$.*

Finally, we consider the real case, $G_n = SO(n)$:

$$SO(n) \rightarrow SO(n+1) \rightarrow S^n \qquad (3).$$

J. F. Adams [1, Theorem 1.1] has obtained a complete solution for H the standard subgroup $SO(n-k)$, $1 \leqq k < n$.

Also, P. Leonard [16, Theorem I. A] has obtained a solution to the general cases for n even; there is not a reduction to any subgroup H of $SO(n)$ for n even unless $n = 6$ and H is $SU(3)$ or $U(3)$.

Besides, if n is odd, I. Dibag [8, Theorem II(ii), Proposition 3.2] and T. Önder [19, Theorem 1.2, Lemma 3.1] have obtained a partial solution for $H = U(s)$ and $Sp(k)$ respectively: For $n = 2m-1 > 4$, $s > \frac{2m-1}{4}$, the principal bundle (3) admits a reduction to $U(s)$ if and only if $\nu_2(b_{m-s}) \leqq \nu_2(m)$. And for $n = 4m-3$, there is no reduction to $Sp(k)$ for $k \neq m-1$, and for $n = 4m-1$, $m > 2$, $k > \frac{4m-1}{8}$, the principal bundle (3) admits a reduction to $Sp(k)$ if and only if $\nu_2(c_{m-k}) \leqq \nu_2(m)$.

Generally, if n is odd, then we have $n \equiv 2^a - 1 \bmod 2^{a+1}$ with some integer $a \geqq 1$. We consider all the cases by dividing into three parts, $a = 1$, $a = 2$ and $a \geqq 3$.

For $n \equiv 2^a - 1 \bmod 2^{a+1}$ with some integer $a \geqq 1$, the principal bundle (3) can be reduced to $SO(n+1-(2a+J))$, and cannot be reduced to $SO(n-(2a+J))$ by [1, Theorem 1.1], where

$$\begin{aligned} J &= 1 \text{ if } a \equiv 0 \bmod 4 \\ &= 0 \text{ if } a \equiv 1 \text{ or } 2 \bmod 4 \\ &= 2 \text{ if } a \equiv 3 \bmod 4. \end{aligned}$$

And we consider a subgroup H of $SO(n+1-(2a+J))$. For $a = 1$ ($n = 4m+1$), there is a reduction to $U(2m)$ by Steenrod [22]. Then we have the following:

Theorem 3. *If $n \equiv 1 \bmod 4$ ($n \neq 1$), the principal bundle* (3) *cannot be reduced to a proper subgroup of $SO(2q)$ except $Spin(7)$ (when $n = 9$) and $Spin(9)$ (when $n = 17$) unless H is $SU(q)$ or $U(q)$, where $2q = n-1$.*

For $a = 2$ ($n = 8m+3$), there is a reduction to $U(4m)$ and $Sp(2m)$ by Steenrod [22]. By Lemmas 5.1 and 5.2, the principal bundle (3) can also be reduced to $SU(4m)$, $Sp(1) * Sp(2m)$ and $U(1) * Sp(2m)$ where $*$ means $\times_{Z_2}$ the equivariant product, $\mathbf{Z}_2 \subset U(1) \subset Sp(1) \subset Sp(2m)$. Then we have the following:

Theorem 4. *If $n \equiv 3 \bmod 8$ ($n \neq 3$), the principal bundle* (3) *cannot be reduced to a proper subgroup of $SO(4m)$ except $Spin(7)$ (when $n = 11$) and $Spin(9)$ (when $n = 19$) unless H is one of the subgroups $U(2m)$, $SU(2m)$, $Sp(m)$, $Sp(1) * Sp(m)$ and $U(1) * Sp(m)$, where $4m = n-3$.*

For $a \geqq 3$ ($n \neq 7$), then the principal bundle (3) can be reduced to $U\left(\frac{n+1-(2a+J)}{2}\right)$ if and only if $2a+J$ is even and $\nu_2(b_r) \leqq a-1$ by [8], where $2r = 2a+J$. Moreover, for $n = 4m-1$, the principal bundle (3) can be reduced to $Sp\left(\frac{n+1-(2a+J)}{4}\right)$ if and only if $2a+J \equiv 0 \bmod 4$ and $\nu_2(c_t) \leqq a-2$, where $4t = 2a+J$ by [19]. Then we have the following:

Theorem 5. *If $n \equiv 2^a - 1 \bmod 2^{a+1}$ with some integer $a \geqq 3$ ($n \neq 7$). Then the principal bundle* (3) *cannot be reduced to any proper subgroup of*

$SO(n+1-(2a+J))$ *except* G_2 (*when* $n=15$) *and* $Spin(9)$ (*when* $n=23$).

We do not know whether or not the principal bundle (3) admits a reduction to G_2, $Spin(7)$ or $Spin(9)$ in Theorems 3, 4 and 5.

The paper is organized as follows. In Section 2, we recall results due to Borel, Montgomery, Samelson and Yasukura on the classification of compact, connected Lie groups G which act on S^n transitively and effectively through the standard action of $SO(n+1)$. We also introduce some notation in Section 2. We prove Theorems 1, 2, 3, 4 and 5 in Sections 3, 4, 5, 6 and 7 respectively.

The results of this paper are based on the author's master dissertation at Okayama University. The author wishes to thank Professor M. Mimura for suggesting the problem and for many helpful suggestions, and also Dr. N. Iwase for helpful suggestions for proof of Proposition 5.3 and for kindness to read the manuscript during the preparation of this paper, and also Professor H. Ōshima for helpful conversations.

2. Notations. Compact, connected Lie groups G which act transitively and effectively on S^n through the standard action of $SO(n+1)$ are classified as follows:

Theorem 2.1. *Suppose that a compact, connected Lie group G which acts transitively and effectively on S^n through the standard action of $SO(n+1)$. Then,*

(a) *for n even,* $G=SO(n+1)$ *or exceptional group* G_2 ($n=6$).

(b) *for n odd,* $G=SO(n+1)$, $SU(q)$, $U(q)$ (*where* $n=2q-1$); $Sp(m)$, $U(1)*Sp(m)$, $Sp(1)*Sp(m)$ (*where* $n=4m-1$); $Spin(7)$ ($n=7$) *or* $Spin(9)$ ($n=15$).

Each group has the unique orthogonal representation up to orthogonal automorphism, which is the standard inclusion map to $SO(n+1)$.

By using this with Lemma 4.4 in [16], we need only to treat the standard inclusions if we consider the problem of the reduction of the structure group to the groups in Theorem 2.1.

This classification was firstly observed by F. Uchida and the proof of the theorem appears in the paper of Yasukura [24 as Theorem 4.8 with cohomogeneity = 1], based on the results of Montgomery and Samelson [18, Theorem I] and also on Borel [4, Theorem III], [5, Theorem 3].

The relations among these groups are as follows:

$$(2.2)\qquad \begin{array}{ccccc} Sp(m) & \overset{i_e}{<} & U(1)*Sp(m) & \overset{i'_m}{<} & Sp(1)*Sp(m) \\ \wedge k_m & & \wedge & & \wedge g_m \\ SU(2m) & \overset{i_{2m}}{<} & U(2m) & \overset{j'_{2m}}{<} & SO(4m), \end{array}$$

$$(2.3)\qquad \begin{array}{ccccc} SU(3) & < & G_2 & < & SO(7) \\ & & \wedge & & \wedge \\ & & Spin(7) & < & SO(8) \\ & & \wedge & & \wedge \\ & & Spin(9) & < & SO(16) \end{array}$$

where the notation $H < G$ means that H is a subgroup of G and maps are all standard inclusions (see [24] and [25]). Note that $Spin(7)$ is not contained in $SU(4)$ or $Sp(2)$ for dimensional reasons and that $U(1) * Sp(m)$ and $Sp(1) * Sp(m)$ are not contained in $SU(2m)$ (see Propositions 2.32 and 2.53 in [25]). Finally, we remark that $Spin(9)$ has the possibility to be contained in $SU(8)$ and $Sp(4)$.

We will use the following notation to indicate the standard inclusion maps: i_n^k: $G_n \to G_{n+k}$, j_n: $SU(n) \to SO(2n)$, where $G_n = SO(n)$, $SU(n)$ or $Sp(n)$. Note that $j_{2n} \circ k_n = g_n \circ i'_n \circ i_e$ and $j_n = j'_n \circ i_n$.

By a subgroup of G_n we will mean a closed connected subgroup.

If G is a Lie group, X a CW-complex and ξ a principal fiber bundle with structure group G over the suspension SX of X, then ξ is classified by a map c: $X \to G$ or its adjoint map c': $SX \to BG$, where BG is the classifying space for G (see [12]). We will speak of either map as a classifying map for ξ.

In this paper, $a|b$ means that a divides b. Finally, if p is a prime integer and n an integer, then $\nu_p(n)$ denotes the highest power of p dividing n.

3. Proof of Theorem 1. If the principal bundle (1) can be reduced to subgroup H of $Sp(n+1-k)$, then H must act transitively and effectively on $S^{4(n-k)+3} = Sp(n+1-k)/Sp(n-k)$ through $Sp(n+1-k)$ by Lemma 3.2 in [16]. By Theorem 2.1, H must be one of the groups $Sp(n+1-k)$ or $Spin(9)$ (when $n = 4$, $k = 1$). Note that $Sp(n+1-k)$ is not proper subgroup of $Sp(n+1-k)$, so we need only to consider about $Spin(9)$.

Suppose that the principal bundle (1) with $n = 4$ can be reduced to $Spin(9)$. Let c: $S^{19} \to BSp(4)$ be the classifying map of the principal bundle (1) and let Bj: $BSpin(9) \to BSp(4)$ be the classifying map induced from an inclusion map $j\colon Spin(9) \to Sp(4)$. Then there is a map $f\colon S^{19} \to BSpin(9)$ such that $c \simeq Bj \circ f$. By applying π_{19} on the fibration $S^{19} \xrightarrow{c} BSp(4) \to BSp(5)$, we get the following exact sequence of abelian groups (see [7], [9]):

$$\mathbf{Z} \xrightarrow{c_*} \mathbf{Z}_{9!} \to 0,$$

where $c_* = Bj_* \circ f_*$. Thus Bj_* must be surjective. Let us recall that $\pi_{19}(BSpin(9)) \cong \mathbf{Z}_{2835} + \mathbf{Z}_{16} + \mathbf{Z}_8 + \mathbf{Z}_2$ (see [17]). Since the 2-primary component of $\mathbf{Z}_{9!}$ is $\mathbf{Z}_{128}$ and the 2-primary part of $\pi_{19}(BSpin(9))$ is $\mathbf{Z}_{16} + \mathbf{Z}_8 + \mathbf{Z}_2$, we have the exact sequence at the prime 2:

$$\mathbf{Z}_{16} + \mathbf{Z}_8 + \mathbf{Z}_2 \xrightarrow{Bj'_*} \mathbf{Z}_{128} \to 0,$$

where Bj'_* is the restriction of Bj_* to the 2-primary component. Hence we have that Bj_* is not surjective. It is a contradiction. Consequently, we deduce that the classifying map c cannot factor through $BSpin(9)$. Therefore the principal bundle (1) cannot be reduced to $Spin(9)$.

4. Proof of Theorem 2. If the principal bundle (2) can be reduced to a subgroup H of $SU(n+1-2k)$, H must act transitively and effectively on $S^{2n-4k+1} = SU(n+1-2k)/SU(n-2k)$ through $SU(n+1-2k)$ by Corollary 3.2 in [16]. By Theorem 2.1, H must be one of the groups $SU(2q)$, $Sp(q)$ or $Spin(9)$ (when $n = 8$), where $2q = n+1-2k$.

By Önder (see [20]), the principal bundle (2) can be reduced to $Sp(q)$ if and only if $c_k | r+1$, where $2r = n-1$.

We now consider the principal bundle (2) with $n = 8$;

$$SU(8) \to SU(9) \to S^{17},$$

and a subgroup $Spin(9)$ of $SU(8)$. Suppose that the principal bundle (2) can be reduced to $Spin(9)$. Let c be the classifying map of the principal bundle (2). Then there is a map $f\colon S^{17} \to BSpin(9)$ such that $c \simeq Bj \circ f$, where $Bj\colon BSpin(9) \to BSU(8)$ is the adjoint inclusion map induced by an inclusion map $j\colon Spin(9) \to SU(8)$. By a quite similar argument in the proof of Theorem 1, one can deduce that the classifying map c cannot factor through $BSpin(9)$, since $\pi_{19}(BSpin(9)) \cong \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ (see [17]), $\pi_{17}(BSU(8)) \cong \mathbf{Z}_{8!}$ and $\pi_{17}(BSU(9)) \cong 0$ (see [7]). Therefore the principal bundle (2) cannot be reduced to $Spin(9)$.

5. Proof of Theorems 3. Suppose that $n \equiv 1 \bmod 4$. If the principal bundle (3) admits a reduction to a subgroup H of $SO(n-1)$, then H must act transitively and effectively on $S^{n-2} = SO(n-1)/SO(n-2)$ through $SO(n-1)$ by Corollary 3.2 in [16]. So by Theorem 2.1, H must be one of the groups $SO(4m)$, $SU(2m)$, $U(2m)$, $Sp(m)$, $U(1) * Sp(m)$, $Sp(1) * Sp(m)$, $Spin(7)$ ($n = 9$) or $Spin(9)$ ($n = 17$), where $4m = n-1$. We check one by one whether or not the principal bundle (3) admits a reduction to above groups. Note that $SO(4m)$ is not a proper subgroup, so there is no need to consider the case for $H = SO(4m)$.

The reduction to $SU(2m)$ is possible by Theorem 24.4 in [22]. If $q \geqq 2$ and $n \geqq 3$, then there is an isomorphism of homotopy groups i_{n*}: $\pi_{n-1}(SU(q)) \to \pi_{n-1}(U(q))$ induced by i_n: $SU(n) \to U(n)$. Then we recall the following well known lemma.

Lemma 5.1. *Let $n \geqq 3$ and $k = n-2q \geqq 0$. If $q \geqq 2$, then the following are equivalent:*
(*a*) *The principal bundle* (3) *can be reduced to $SU(q)$.*
(*b*) *The principal bundle* (3) *can be reduced to $U(q)$.*

Proof. Clearly (*a*) implies (*b*). We show that (*b*) implies (*a*). If the principal bundle (3) can be reduced to $U(q)$, then there is a map f: $S^{n-1} \to U(q)$ such that $c \simeq i_{2q}^{k} \circ j_q \circ f$, where c: $S^{n-1} \to SO(n)$ is the classifying map of the principal bundle (3). There is a map f': $S^{n-1} \to SU(q)$ such that $f \simeq i_n \circ f'$, since there exists the isomorphism i_{n*}: $\pi_{n-1}(SU(q)) \to \pi_{n-1}(U(q))$. So, there is a relation $c \simeq i_{2q}^{k} \circ j_q' \circ f'$, since $j_q \circ i_q = j_q'$. This means that the principal bundle (3) can be reduced to $SU(q)$. Q.E.D.

By the assumption that $n \neq 1$ and $n \equiv 1 \bmod 4$, we have $2m \geqq 2$. So by Lemma 5.1, the principal bundle (3) can be reduced to $U(2m)$.

Next we consider the reduction to $Sp(m)$, $U(1) * Sp(m)$ and $Sp(1) * Sp(m)$. If $m = 1$, which shows $n = 5$, then $Sp(1) = SU(2)$, $U(1) * Sp(1) = U(2)$ and $Sp(1) * Sp(1) = SO(4)$. Therefore the principal bundle (3) admits a reduction to such groups and we need only to consider the case for $m \geqq 2$, which means that $n \geqq 8$. We recall the following lemma which is proved by Leonard [16].

Lemma 5.2. *For $n \geqq 8$ and $r = n-4m \geqq 0$. If $m \geqq 2$, then the following are equivalent:*
(*a*) *The principal bundle* (3) *can be reduced to $Sp(m)$.*

(b) *The principal bundle* (3) *can be reduced to* $U(1) * Sp(m)$.

(c) *The principal bundle* (3) *can be reduced to* $Sp(1) * Sp(m)$.

Proof. Obviously (a) implies (b), and (b) implies (c) by (2.2). So, we are left to prove that (c) implies (a).

If the principal bundle (3) can be reduced to $Sp(1) * Sp(m)$, then there is a map f: $S^{n-1} \to Sp(1) * Sp(m)$ such that $c \simeq i^r_{4m} \circ g_m \circ f$, where c is the classifying map of the principal bundle (3). Since we have an isomorphism of homotopy groups $p_*: \pi_{n-1}(Sp(1) \times Sp(m)) \to \pi_{n-1}(Sp(1) * Sp(m))$ induced by the double covering p: $Sp(1) \times Sp(m) \to Sp(1) * Sp(m)$ and since $p \circ g_m = j_{2m} \circ k_m \circ k$ (see [16, Lemma 7.5]), we have the following homotopy commutative diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad c \quad} & SO(n) \\ \Big\downarrow f' \quad \searrow f & & \Big\uparrow i^r_{4m} \\ Sp(1) * Sp(m) & \xrightarrow{g_m} & SO(4m) \\ \Big\uparrow p & & \Big\uparrow j_{2m} \circ k_m \\ Sp(1) \times Sp(m) & \xrightarrow[k]{\quad\quad} & Sp(m), \end{array}$$

where f': $S^{n-1} \to Sp(1) \times Sp(m)$ is such that $f \simeq p \circ f'$. This gives us a reduction to $Sp(m)$ since $c \simeq i^r_{4m} \circ j_{2m} \circ k_m \circ k \circ f'$. Q. E. D.

Thus we need only to consider the reduction to $Sp(m)$. If the principal bundle (3) can be reduced to $Sp(m)$, then there is a map f: $S^{n-1} \to Sp(m)$ such that $c \simeq i^r_{4m} \circ j_{2m} \circ k_m \circ f$. Let us recall that the principal bundle (3) can be reduced to $SU(2m)$, since $2 | n+1$ by [22, Theorem 24.4]. It follows that the principal bundle (3) is equivalent, in $SO(n)$, with the principal bundle

$$SU(2m) \to SU(2m+1) \to S^n \qquad (2)'.$$

Let c': $S^{n-1} \to SU(2m)$ be the classifying map of the principal bundle $(2)'$, then $c \simeq i^r_{4m} \circ j_{2m} \circ c'$. So, we have the following homotopy commutative diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c} & SO(n) \\ c' \Big\downarrow & & \Big\uparrow i^r_{4m} \\ SU(2m) & \xrightarrow[j_{2m}]{} & SO(4m) \end{array}$$

Now we consider the following homotopy exact sequence of the principal bundle (2)':

$$\cdots \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SU(2m)) \to \pi_{n-1}(SU(2m+1)) \cdots.$$

By the results of Theorem 10.4 of Chapter 7 in [12], if we take $h: S^n \to S^n$ to be a map representing a generator of $\pi_n(S^n)$, then $\partial([h]) = [c']$ and $[c']$ is a generator of $\pi_{n-1}(SU(2m)) = \mathbf{Z}_{2m}$: by the fact that $\pi_{n-1}(SU(2m+1)) = 0$.

So we can write $[k_m \circ f] = b[c']$ in $\pi_{n-1}(SU(2m))$ with some integer b.

Proposition 5.3. *If the integer b is defined as above, the order of $\pi_{n-1}(SU(2m)/Sp(m))$ divides b.*

Proof. Let $g: S^n \to S^n$ be a map such that $[g] = b[h]$ in $\pi_n(S^n)$, and let ξ denote the principal bundle (2)'. And we define $g^*(\xi) = (E, p, S^n)$ which is the induced bundle associated with ξ. It follows from the relation $\partial([g]) = \partial(b[h]) = b\partial([h]) = b[c'] = [k_m \circ f]$ that the classifying map of $g^*(\xi)$ is $k_m \circ f$. Therefore we have the following commutative diagram:

$$(*)' \qquad \begin{array}{ccccc} SU(2m) & \longrightarrow & E & \xrightarrow{p} & S^n \\ \downarrow id & & \downarrow g' & & \downarrow g \\ SU(2m) & \to & SU(2m+1) & \underset{\pi}{\to} & S^n, \end{array}$$

where g' is a map induced from g. Note that $g^*(\xi)$ can be reduced to $Sp(m)$ since there exists a map $f: S^{n-1} \to Sp(m)$. So, the fiber bundle

$$SU(2m)/Sp(m) \to E/Sp(m) \xrightarrow{p'} S^n$$

has a cross section $s: S^n \to E/Sp(m)$ such that $id = p' \circ s$. By the commutativity of the diagram $(*)'$, we get the following commutative diagram:

$$(**)' \qquad \begin{array}{ccccc} SU(2m)/Sp(m) & \longrightarrow & E/Sp(m) & \xrightarrow{p'} & S^n \\ \downarrow id & & \downarrow g'' & & \downarrow g \\ SU(2m)/Sp(m) & \to & SU(2m+1)/Sp(m) & \underset{\pi'}{\to} & S^n \end{array} \qquad (2)'',$$

where the bundle map g'' is induced from g' to the orbits. Then we have that $\pi' \circ s' = g$ if we define $s' = g'' \circ s$. Consequently we have that $\partial'([g])$

$= \partial'([\pi' \circ s']) = \partial' \circ \pi'_*([s']) = 0$ in the following exact sequence of homotopy groups which is associated with $(2)''$:

$$\cdots \pi_n(SU(2m+1)/Sp(m)) \xrightarrow{\pi'_*} \pi_n(S^n) \xrightarrow{\partial'} \pi_{n-1}(SU(2m)/Sp(m)) \cdots.$$

Then, by the definition of g, the following relation holds: $0 = \partial'([g]) = \partial'(b[h]) = b\partial'([h])$.

On the other hand, we consider the following diagram of the exact sequence:

$$\begin{array}{ccccccc} & & & & \pi_{n-1}(SU(2m+1)) \cong 0 & & \\ & & & & \downarrow & & \\ \pi_n(S^n) & \xrightarrow{\partial'} & \pi_{n-1}(SU(2m)/Sp(m)) & \to & \pi_{n-1}(SU(2m+1)/Sp(m)) & \xrightarrow{\pi'_*} & \pi_{n-1}(S^n) \cong 0 \\ \wr\Vert & & \wr\Vert & & \downarrow & & \\ \mathbf{Z} & & \mathbf{Z}_{2m!/(m+1,2)} & & \pi_{n-2}(Sp(m)) \cong \mathbf{Z}. & & \end{array}$$

By the exactness at $\pi_{n-1}(SU(2m+1)/Sp(m))$ on the horizontal line, $\pi_{n-1}(SU(2m+1)/Sp(m))$ must be finite. Hence the injection $\pi_{n-1}(SU(2m+1)/Sp(m)) \to \pi_{n-2}(Sp(m))$ is trivial. So the group $\pi_{n-1}(SU(2m+1)/Sp(m))$ is itself trivial. Since $\partial'([h])$ is the generator of $\pi_{n-1}(SU(2m)/Sp(m))$ and $b\partial'([h]) = 0$, the order of $\pi_{n-1}(SU(2m)/Sp(m))$ divides b.

Q. E. D.

Let us recall that $[c] = [i^r_{4m} \circ j_{2m} \circ k_m \circ f] = b[i^r_{4m} \circ j_{2m} \circ c']$ in $\pi_{n-1}(SO(n))$, and $\nu_2(b) \geqq 2$ for $m \geqq 2$, since 4 divides the order of $\pi_{n-1}(SU(2m)/Sp(m)) \cong \mathbf{Z}_{2m!/(m+1,2)}$. Now, $[c] \equiv 0 \bmod 4$ since $\nu_2(b) \geqq 2$. Now, $\pi_{n-1}(SO(n)) \cong \mathbf{Z}_2$ or $\mathbf{Z}_2 + \mathbf{Z}_2$ (see [15]), so $[c] = 0$ in $\pi_{n-1}(SO(n))$. This means that the principal bundle (3) is trivial. This is a contradiction. Therefore the principal bundle (3) cannot be reduced to $Sp(m)$. By Lemma 5.2, we have that the principal bundle (3) also cannot be reduced to $U(1) * Sp(m)$ nor $Sp(1) * Sp(m)$. Thus we have examined the existence of the reduction to subgroup H of $SO(n-1)$ except $Spin(7)$ and $Spin(9)$. Therefore Theorem 3 is completely proved.

6. Proof of Theorem 4. Suppose that $n \equiv 3 \bmod 8$ and $n \neq 3$. By Corollary 3.2 in [16], if the principal bundle (3) can be reduced to subgroup H of $SO(n-3)$, then H must act transitively and effectively on $S^{n-4} = SO(n-3)/SO(n-4)$ through $SO(n-3)$. By Theorem 2.1, H must be one of the groups $SO(4m)$, $SU(2m)$, $U(2m)$, $Sp(m)$, $U(1) * Sp(m)$, $Sp(1) * Sp(m)$, $Spin(7)$ $(n = 11)$ or $Spin(9)$ $(n = 19)$, where $4m = n-3$.

We consider whether or not the principal bundle (3) admits a reduction to the above groups.

The reduction to $SU(2m)$ and $Sp(m)$ are possible by Theorem 24.4 in [22]. So, by Lemmas 5.1 and 5.2, we have that the principal bundle (3) also can be reduced to $U(2m)$, $U(1) * Sp(m)$ and $Sp(1) * Sp(m)$. Therefore we have determined whether or not there is a reduction to such groups except $Spin(7)$ ($n = 11$) and $Spin(9)$ ($n = 19$). This completes the proof of Theorem 4.

7. Proof of Theorem 5. Suppose that $n \equiv 2^a - 1 \bmod 2^{a+1}$ with $a \geqq 3$. If the principal bundle (3) can be reduced to a subgroup H of $SO(n+1-(2a+J))$, then H must act transitively and effectively on $S^{n-(2a+J)} = SO(n+1-(2a+J))/SO(n-(2a+J))$ through $SO(n+1-(2a+J))$ by Corollary 3.2 in [16]. By Theorem 2.1, according to the value of a, H must be one of the following groups:

(a) $SO(n-2a)$ or G_2 (when $n = 15$) for $a \equiv 0 \bmod 4$,

(b) $SO(2q)$, $SU(q)$ or $U(q)$ for $a \equiv 1 \bmod 4$, where $2q = n+1-2a$,

(c) $SO(4t)$, $SU(2t)$, $U(2t)$, $Sp(t)$, $Sp(1) * Sp(t)$, $U(1) * Sp(t)$ or $Spin(9)$ (when $n = 23$) for $a \equiv 2$ or $3 \bmod 4$, where $4t = n+1-(2a+J)$.

This is the complete list of the subgroup H of $SO(n+1-(2a+J))$ to which the structure group can be reduced. By Dibag [8, Theorem II(ii), Proposition 3.2], the principal bundle (3) can be reduced to the subgroup $U\left(\frac{n+1-(2a+J)}{2}\right)$ if and only if $2a+J$ is even and $\nu_2(b_r) \leqq a-1$, where $2r = 2a+J$. If we consider the reduction to $U(2t)$ or $U(q)$, then we only consider the case $a \geqq 5$ with $a \not\equiv 0 \bmod 4$, or $a = 3$ ($n \neq 7$). This means that $2(2a+J) \leqq n+1$. Now $\nu_2(b_a) \geqq a$ by [2] because a is odd or $a \equiv 2 \bmod 4$, so the principal bundle (3) cannot be reduced to $U(q)$ and $SU(q)$ (by Lemma 5.1), where $2q = n+1-2a$. Therefore the principal bundle (3) cannot be reduced to $Sp(t)$ since $n+1-2a \leqq 4t = n+1-(2a+J)$. So the principal bundle (3) cannot be reduced to above groups except G_2 and $Spin(9)$.

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(Received March 15, 1990)