Math. J. Okayama Univ. 33 (1991), 177−188

ON THE ATTACHING MAP IN THE STIEFEL MANIFOLD OF 2-FRAMES

JUNO MUKAI

0. Introduction. Let $\mathbf{F}=\mathbf{R}$(real), $\mathbf{C}$(complex) or $\mathbf{H}$(quaternionic) and $d=\dim_{\mathbf{R}}\mathbf{F}$. Let $\iota_n\in\pi_n(S^n)$ be the identity map, $\eta_n\in\pi_{n+1}(S^n)$ for $n\geq 2$ and $\nu_n\in\pi_{n-3}(S^n)$ for $n\geq 4$ the Hopf maps. Throughout the paper $O_{n,k}(\mathbf{F})$ stands for the Stiefel manifold consisting of orthonormal k-frames in $\mathbf{F}^n$, $Q_{n,k}(\mathbf{F})\subset O_{n,k}(\mathbf{F})$ does for the stunted quasiprojective space and $Q_{2n+1,2}(\mathbf{F}) = S^{2dn-1}\cup_{\omega_n(\mathbf{F})}e^{d(2n+1)-1}$, where $\omega_n(\mathbf{R})=2\iota_{2n-1}$, $\omega_n(\mathbf{C})=\eta_{4n-1}$ and $\omega_n(\mathbf{H})=(2n+1)\nu_{8n-1}$. We have a cellular decomposition :

$$O_{2n+1,2}(\mathbf{F})=Q_{2n+1,2}(\mathbf{F})\cup_{\gamma_n(\mathbf{F})}e^{(4n+1)d-2}.$$

The purpose of the present note is to determine the $(d-k)$-fold suspension $\Sigma^{d-k}\gamma_n(\mathbf{F})\in\pi_{(4n+2)d-k-3}(\Sigma^{d-k}Q_{2n+1,2}(\mathbf{F}))$ for $0\leq k\leq d$. We shall freely use the notation and results of [16], [10] and [11]. We shall also use the EHP-sequences and the information about the (relative) Whitehead products [,]. We denote by $\#\alpha$ the order of α. Our result is stated as follows.

Theorem 1. i) $\#\Sigma^d\gamma_n(\mathbf{F})=2$ *and* $\#\Sigma\gamma_n(\mathbf{C})=2$. ii) $\#\Sigma^k\gamma_1(\mathbf{H})=2$ *for* $1\leq k\leq 3$; $\#\Sigma^k\gamma_n(\mathbf{H})=8$ *for* $n\geq 2$ *and* $k=1$ *or* 2 ; $\#\Sigma^3\gamma_n(\mathbf{H})=4$ *for* $n\geq 2$.

Theorem 2. $\pi_{(4n+1)d-3}(X)\cong K\{\gamma_n(\mathbf{F})\}\oplus\pi_{(4n-1)d-3}(W)$, *where* $X=Q_{2n+1,2}(\mathbf{F})$, $W=O_{2n+1,2}(\mathbf{F})$ *and* $K=\mathbf{Z}$ *if* $d\neq 1$ *or* $d=n=1$; $K=\mathbf{Z}_8$ *if* $d=1$ *and* $n=3$ *or* $n\geq 5$; $K=\mathbf{Z}_4$ *if* $d=1$ *and* $n=2$ *or* 4.

The author wishes to thank M. Mimura for suggesting a problem to determine $\gamma_n(\mathbf{R})$ and reading the first manuscript.

The paper is organized as follows. § 1 is devoted to prepare some lemmas due to James and Toda. § 2 is to summarize the behavior of the J-image of the characteristic element for $O_{2n+1,2}(\mathbf{F})$. §§ 3−5 are devoted to prove the theorems and to determine the generalized Hopf invariant of $\gamma_n(\mathbf{R})$.

1. Some results of James and Toda. Let $X=S^q\cup_\alpha e^n$ for $q\leqq n-1$ and $B=X\cup_\gamma e^{n+q}$, where B is regarded as the q-sphere bundle over S^n [5].

This research was partially supported by Grant-in-Aid for Scientific Research (No. 03640033), Ministry of Education, Science and Culture.

Let $i: S^q \to X$, $j: (X, *) \to (X, S^q)$ be the inclusions and $p: (X, S^q) \to (S^n, *)$ a map collapsing S^q to the base point. Let $\varkappa = \varkappa_n: (CS^{n-1}, S^{n-1}) \to (X, S^q)$ be a characteristic map, where CS^m is a cone on S^m. By (5.1) of [6] and (3.3) of [2], we have

$$(1) \qquad j_*\gamma = (-1)^{nq}[\iota_q, \varkappa].$$

By Lemma 4.4.3 of [1] and by Lemma 2.32 and Corollary 3.6 of [15], we have the following

Lemma 1. *Let $\beta \in \pi_{n-1}(SO_{q+1})$ be the characteristic element for B and $\theta \in \pi_{n+q}(S^{q+1})$ an element obtained from β by the Hopf construction. Then $\Sigma\gamma = \pm(\Sigma i)_*\theta$ and $H(\theta) = \pm\Sigma^{q+1}\alpha$.*

We denote by $\hat{\alpha} \in \pi_{k+1}(CS^n, S^n)$ for $\alpha \in \pi_k(S^n)$ an element satisfying $\partial'\hat{\alpha} = \alpha$, where $\partial': \pi_{k+1}(CS^n, S^n) \to \pi_k(S^n)$ is the boundary isomorphism. We denote by $\Sigma': \pi_r(X, S^q) \to \pi_{r+1}(\Sigma X, S^{q+1})$ the relative suspension homomorphism [15]. By Theorem 2.1 of [3], we have an exact sequence for $t = n+2q+3k-2$ $(k \geq 0)$:

$$(2) \qquad \begin{aligned} &\pi_t(\Sigma^k X, S^{q+k}) \xrightarrow{(\Sigma^k p)_*} \cdots \to \pi_r(\Sigma^k X, S^{q+k}) \xrightarrow{(\Sigma^k p)_*} \\ &\pi_r(S^{n+k}) \xrightarrow{H'} \pi_{r-n-k}(S^{q+k}) \xrightarrow{Q} \pi_{r-1}(\Sigma^k X, S^{q+k}) \to \cdots, \end{aligned}$$

where $H' = (\Sigma^k\alpha)_*\Sigma^{-n-k}H$ and $Q(\ \ ) = [\ , (\Sigma')^k\varkappa]$.

Lemma 2. i) $\operatorname{Ker}\{(\Sigma^k i)_*: \pi_r(S^{q+k}) \to \pi_r(\Sigma^k X)\} = (\Sigma^k\alpha)_*\pi_r(S^{n+k-1})$ *for $r = n+q+k-1$ if $k = 0$ or $k \geq 2$.*

ii) $\operatorname{Ker}(\Sigma i)_* = \{[\iota_{q+1}, \Sigma\alpha]\} + (\Sigma\alpha)_*\pi_{n+q}(S^n)$.

Proof. i) for $k = 0$ is just (3.2) of [6]. Recall $\operatorname{Ker}(\Sigma^k i)_* = \operatorname{Im}\partial$, where $\partial: \pi_{r+1}(\Sigma^k X, S^{q+k}) \to \pi_r(S^{q+k})$ is the connecting map. Since $\pi_{r+1}(\Sigma^k X, S^{q+k}) \cong \pi_{r-1}(S^{n+k})$ for $k \geq 2$ and $\partial((\Sigma')^k\varkappa \circ \hat{\beta}) = \Sigma^k\alpha \circ \beta$ for $\beta \in \pi_t(S^{n+k-1})$, we have the assertion for $k \geq 2$.

By (2), we have $\pi_{n+q+1}(\Sigma X, S^{q+1}) = \mathbf{Z}\{[\iota_{q+1}, \Sigma'\varkappa]\} \oplus \pi_{n+q+1}(S^{n+1})$. This leads us to ii) and completes the proof.

As is well known, we have the following

Remark 1. i) Let G be a group generated by $\Delta(\iota_{4n-1}) = [\iota_{2n-1}, \iota_{2n-1}]$. Then $G = 0$ if $n = 1, 2$ or 4 and $G = \mathbf{Z}_2$ if otherwise. We have a short exact sequence

(3) $$0 \to G\{\Delta(\iota_{4n-1})\} \hookrightarrow \pi_{4n-3}(S^{2n-1}) \xrightarrow{\Sigma} \pi_{4n-2}(S^{2n}) \to 0$$

which is split if $n = 1, 2, 4$ or n is not a power of 2.

ii) We have

(4) $$\pi_{4n-1}(S^{2n}) \cong \mathbf{Z}\{\Delta(\iota_{4n-1})\} \oplus \Sigma\pi_{4n-2}(S^{2n-1}) \textit{ for } n = 3 \textit{ or } n \geq 5.$$

By Propositions 2.7 and 2.2 of [16], $H(\Delta(\iota_{4n+1})) = \pm 2\iota_{4n-1}$ and $H(2\iota_{2n} \circ \Delta(\iota_{4n+1})) = \Sigma(2\iota_{2n-1} \wedge 2\iota_{2n-1}) \circ H(\Delta(\iota_{4n+1})) = \pm 8\iota_{4n-1}$. So, by (4), we have $[\iota_{2n}, 2\iota_{2n}] \notin (2\iota_{2n})_*\pi_{4n-1}(S^{2n})$. By this and [12], we have the following

Remark 2. $[\iota_{q+1}, \Sigma\alpha] \notin (\Sigma\alpha)_*\pi_{n+q}(S^n)$ for some n, where $X = Q_{2n+1,2}(\mathbf{F})$, $\alpha = \omega_n(\mathbf{F})$ and $q = 2dn-1$.

Let $\mathbf{R}P^n$ be the real n-dimensional projective space and $\mathbf{R}P^n_k = \mathbf{R}P^n/\mathbf{R}P^{k-1}$ the stunted space.

Lemma 3. $\pi_{(4n+1)d-3}(X, S^{2dn-1}) \cong \pi_{(4n+1)d-3}(S^{(2n+1)d-1}) \oplus L\{[\iota_{2dn-1}, \varkappa]\}$, *where* $X = Q_{2n+1,2}(\mathbf{F})$, $\varkappa = \varkappa_{(2n+1)d-1}$ *and* $L = \mathbf{Z}$ *if* $d \neq 1$ *or* $d = n = 1$; $L = \mathbf{Z}_4$ *if* $d = 1$ *and* $n = 3$ *or* $n \geq 5$; $L = \mathbf{Z}_2$ *if* $d = 1$ *and* $n = 2$ *or* 4.

Proof. First we shall give a proof in the real case. By (2), we have an exact sequence for $n \geq 2$:

$$\pi_{4n-1}(S^{2n}) \xrightarrow{H'} \pi_{2n-1}(S^{2n-1}) \xrightarrow{Q} \pi_{4n-2}(X, S^{2n-1}) \xrightarrow{p_*} \pi_{4n-2}(S^{2n}) \to 0.$$

By (4), $\operatorname{Im} H' \cong 4\mathbf{Z}$ for $n = 3$ or $n \geq 5$. $\operatorname{Im} H' \cong 2\mathbf{Z}$ for $n = 2$ or 4. So we have a short exact sequence

(5) $$0 \to L \hookrightarrow \pi_{4n-2}(X, S^{2n-1}) \xrightarrow{p_*} \pi_{4n-2}(S^{2n}) \to 0.$$

We set $m = 2n-1$ and $\alpha = \Delta(\iota_{2m+1})$. By (2.18) of [15], we have $\hat{\alpha} = [\iota_m, \hat{\iota}_m]$, where $\hat{\iota}_m$ coincides with the identity map of (CS^m, S^m). So, by (2.16–18) of [15] or by (3.4–6) of [2], we have $\varkappa\hat{\alpha} = \varkappa_*[\iota_m, \hat{\iota}_m] = [2\iota_m, \varkappa] = 2[\iota_m, \varkappa]$. Let $\beta \in \pi_{2m-1}(S^m)$ be an element such that $\#\beta = \#\Sigma\beta$. Then $p_*(\varkappa \circ \hat{\beta}) = \Sigma\beta$ and $\#(\varkappa \circ \hat{\beta}) = \#\Sigma\beta$. Therefore, if (3) is split, so is (5).

Suppose that (3) is not split. Then there exists an element $\beta \in \pi_{2m-1}(S^m)$ such that $2\beta = \Delta(\iota_{2m+1})$ and $\#\Sigma\beta = 2$. Since $2(\varkappa\hat{\beta}) = \varkappa\hat{\alpha} = 2[\iota_m, \varkappa]$, we have $\#\delta = 2$ and $p_*\delta = \Sigma\beta$ for $\delta = \varkappa\hat{\beta} - [\iota_m, \varkappa]$. So (5) is also split in this case. This leads us to the assertion of the real case except for $n = 1$.

We have $X = \mathbf{R}P^2$ and $O_{3,2}(\mathbf{R}) = \mathbf{R}P^3$ if $d = n = 1$. So, by use of the homotopy exact sequence of a pair (X, S^1), we have $\pi_2(X, S^1) \cong \mathbf{Z}\{\gamma_1(\mathbf{R})\} \oplus \mathbf{Z}\{\kappa\}$. Since $p_*\kappa = \iota_2$ and $j_*\gamma_1(\mathbf{R}) = [\iota_1, \kappa]$ by (1), we have the splitting of (5).

For $d = 2$ or 4, we have, by (2), a short exact sequence for $r = (4n+1)d-3$:

$$0 \to \pi_{2dn-1}(S^{2dn-1}) \xrightarrow{Q} \pi_r(X, S^{2dn-1}) \xrightarrow{p_*} \pi_r(S^{(2n+1)d-1}) \to 0.$$

Since $\Sigma\colon \pi_{r-1}(S^{(2n+1)d-2}) \to \pi_r(S^{(2n+1)d-1})$ is isomorphic onto, the sequence is split. This completes the proof.

By (11.8) and Theorem 11.7 of [16], we have the following

Lemma 4. *There exists a mapping* $\delta\colon \Sigma^{n-1}\mathbf{R}P_n^{n+k-1} \to S^n$ *such that* Ker $\{\Sigma^k\colon \pi_i(S^n) \to \pi_{i+k}(S^{n+k})\} = \delta_*\pi_i(\Sigma^{n-1}\mathbf{R}P_n^{n+k-1})$ *for* $i \leq 3n-3$. *In the 2-components, the assertion holds for* $i \leq 4n-4$.

By Proposition 7.10 of [4], $Q_{n,k}(\mathbf{F})$ is a stable retract of $O_{n,k}(\mathbf{F})$. Especially we have $\Sigma^{d+1}\gamma_n(\mathbf{F}) = 0$.

Hereafter, by abuse of notation, we often use the inclusion i and the projection p to denote $\Sigma^r i$ and $\Sigma^s p$ for integers r and s, respectively.

Let $\sigma_n \in \pi_{n+7}(S^n)$ for $n \geq 8$ be the Hopf map and ι_X the identity class of $X = Q_{2n+1,2}(\mathbf{F})$. Then $X \wedge X$ is homotopy equivalent to a mapping cone

$$\Sigma^{2dn-1}X \cup_{\lambda_n(\mathbf{F})} C(\Sigma^{(2n+1)d-2}X),$$

where $\lambda_n(\mathbf{F}) = \iota_X \wedge \omega_n(\mathbf{F})$.

In the 2-components, stable Toda brackets $\langle 2\iota, \eta, 2\iota\rangle$, $\langle \eta, \nu, \eta\rangle$ and $\langle \nu, 8\iota, \nu\rangle$ consist of single elements η^2, ν^2 and 8σ, respectively. By this and by Lemma 3.5 and Theorem 3.6 of [16] and by their proofs, we have the following

Lemma 5. $\lambda_n(\mathbf{R}) = i\eta_{4n-2}p$, $\lambda_n(\mathbf{C}) = 3ai\nu_{8n-2}p$ *and* $\lambda_n(\mathbf{H}) = 15bi\sigma_{16n-2}p - (\Sigma^{8n-1}\tilde{\theta})p$ *for* $n \geq 1$ *and odd integers a and b, where* $\tilde{\theta}$ *is a coextension of* $\theta = 2\Sigma^3\omega_n(\mathbf{H})$ *with respect to* $\omega_n(\mathbf{H})$.

2. The *J*-image of the characteristic element. Let $\gamma'_n(\mathbf{F}) \in \pi_{d(n+1)-2}(O_n(\mathbf{F}))$ be the characteristic map [11], where $O_n(\mathbf{F}) = O_n$, U_n or Sp_n according as $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. Let $J\colon \pi_k(O_n(\mathbf{F})) \to \pi_{k+dn}(S^{dn})$ be the J-homomorphism and $j_n(\mathbf{F}) = J(\gamma'_n(\mathbf{F})) \in \pi_{(2n+1)d-2}(S^{dn})$. Then $j_n(\mathbf{F})$ is an element obtained from the characteristic element $\gamma'_n(\mathbf{R})$, $c\gamma'_n(\mathbf{C})$ or $rc\gamma'_n(\mathbf{H})$ by the Hopt construction, where $r: U_n \hookrightarrow SO_{2n}$ and $c: Sp_n \hookrightarrow SU_{2n}$ are the canonical maps. We recall the following relations: $j_n(\mathbf{R}) = \Delta(\iota_{2n+1}) = \pm [\iota_n, \iota_n]$, $\Sigma j_n(\mathbf{C}) = j_{2n+1}(\mathbf{R})$, $\Sigma^2 j_n(\mathbf{H}) = j_{2n+1}(\mathbf{C})$, $H(j_n(\mathbf{C})) = (n-1)\eta_{4n-1}$ and $H(j_n(\mathbf{H})) = \pm(n+1)\nu_{8n-1}$. By Lemma 1, we have

$$\Sigma\gamma_n(\mathbf{F}) = \pm i_* j_{2n}(\mathbf{F}),\ \Sigma^d\gamma_n(\mathbf{F}) = i_*\Delta(\iota_{2d(2n+1)-1}) \text{ and } H(j_{2n}(\mathbf{F})) = \pm\Sigma^{2dn}\omega_n(\mathbf{F}). \tag{6}$$

By [8], [14] and [16], we have

$$\Delta(\eta_{2n+1}) \neq 0 \text{ if and only if } n = 4, 5 \text{ or } n \not\equiv 3 \bmod 4 \text{ and } n \geq 8\ ;\ \Delta(\eta^2_{2n-1}) \neq 0 \text{ if and only if } n = 4 \text{ or } n \equiv 0, 1 \bmod 4 \text{ and } n \geq 6. \tag{7}$$

We denote by (a, b) the greatest common divisor of integers a and b.

Lemma 6. i) *In the 2-component, there exists an element* $\lambda \in \pi_{16n-1}(S^{8n-3})$ *such that* $\pm(2n+1)\Delta(\nu_{16n+1}) = 2j_{2n}(\mathbf{H}) - \Sigma^3\lambda$ *and* $H(\lambda) = \nu^2_{16n-7}$. *There exists* $\lambda' \in \pi_{16n-3}(S^{8n-5})$ *such that* $2\lambda = \Sigma^2\lambda'$ *and* $H(\lambda') \equiv \varepsilon_{16n-11} \bmod \eta_{16n-11}\sigma_{16n-10}$. *We set* $\lambda = \nu_5\sigma_8$ *and* $\lambda' = \pm\varepsilon'$ *for* $n = 1$.

ii) $\# j_{2n}(\mathbf{C}) = 2$; $\# j_{2n+1}(\mathbf{C}) = 4$ *and* $2j_{2n+1}(\mathbf{C}) = \Delta(\eta_{8n+5})$ *for* $n \geq 2$; $\#\Sigma j_{2n}(\mathbf{H}) = 8$ *and* $4\Sigma j_{2n}(\mathbf{H}) = \Delta(\eta^2_{16n+3})$; $\# j_{2n}(\mathbf{H}) = 24/(3, 2n+1)$ *and* $\{12/(3, 2n+1)\} j_{2n}(\mathbf{H}) = j_{4n}(\mathbf{C}) \circ \eta^2_{16n}$.

Proof. i) for $n \geq 2$ is obtained from Lemma 11.17 and Proposition 11.15 of [16]. For $n = 1$, the assertion holds [16].

We recall that $\pi_{4n}(SO_{4n}) \cong (\mathbf{Z}_2)^2$ or $(\mathbf{Z}_2)^3$ according as n is odd or even [7]. Since $j_n(\mathbf{C}) = J(r\gamma'_n(\mathbf{C}))$ and $H(j_{2n}(\mathbf{C})) = \eta_{8n-1}$, we have the first of ii).

Since $\pi_6(SO_6) = 0$, we have $j_3(\mathbf{C}) = 0$. We consider an anti-commutative diagram between exact sequences for $n \geq 2$:

$$\begin{array}{ccccc} \pi_{4n+3}(S^{4n+2}) & \xrightarrow{\partial} & \pi_{4n+2}(SO_{4n+2}) & \xrightarrow{i_*} & \pi_{4n+2}(SO_{4n+3}) \\ \downarrow\Sigma^{4n+3} & & \downarrow J & & \downarrow J \\ \pi_{8n+6}(S^{8n+5}) & \xrightarrow{\Delta} & \pi_{8n+4}(S^{4n+2}) & \xrightarrow{\Sigma} & \pi_{8n+5}(S^{4n+3}). \end{array}$$

By [7], $\pi_{4n+2}(SO_{4n+2+k}) \cong \mathbf{Z}_{2(2-k)}$ for $k = 0$ or 1 and $2r\gamma'_{2n+1}(\mathbf{C}) = \partial\eta_{4n+2}$. So we have $2j_{2n+1}(\mathbf{C}) = \Delta(\eta_{8n+5})$. By (7), we have the second of ii).

We recall that $\pi_{8n+2}(SO_{8n}) = \{\gamma'_{8n}(\mathbf{R}) \circ \nu_{8n-1}, rc\gamma'_{2n}(\mathbf{H})\} \cong \mathbf{Z}_{24} \oplus \mathbf{Z}_8$ and $\pi_{8n+2}(SO_{8n+1}) = \mathbf{Z}_8\{r'c\gamma'_{2n}(\mathbf{H})\}$ [11]. $J(\gamma'_{8n}(\mathbf{R}) \circ \nu_{8n-1}) = j_{8n}(\mathbf{R}) \circ \nu_{16n-1} = \pm \Delta(\nu_{16n+1})$, $J(rc\gamma'_{2n}(\mathbf{H})) = j_{2n}(\mathbf{H})$ and $J(r'c\gamma'_{2n}(\mathbf{H})) = \Sigma j_{2n}(\mathbf{H})$. By Theorem 4 of [11], $\{12/(3,2n+1)\}j_{2n}(\mathbf{H})=j_{4n}(\mathbf{C})\circ\eta^2_{16n}$ and $\{12/(3,2n+1)\}\Sigma j_{2n}(\mathbf{H})=\Sigma j_{4n}(\mathbf{C})\circ\eta^2_{16n+1}=\Delta(\eta^2_{16n+3})$. By (7) and an EHP-sequence

$$\pi_{16n+5}(S^{16n+3})\xrightarrow{\Delta}\pi_{16n+3}(S^{8n+1})\xrightarrow{\Sigma}\pi_{16n+4}(S^{8n+2}),$$

we have the rest of ii). This completes the proof.

Let G_k be the k-th stable homotopy group of spheres. By [14] and Lemma 6, we have the following

Remark 3. i) $\pi_{8n}(S^{4n})\cong G_{4n}\oplus\mathbf{Z}_2\{\Delta(\eta_{8n+1})\}\oplus\mathbf{Z}_2\{j_{2n}(\mathbf{C})\}$; $\pi_{8n-1}(S^{4n-1})\cong G_{4n}\oplus\mathbf{Z}_2\{\delta\}$, where $H(\delta)=\eta^2_{8n-3}$ and $\Sigma\delta=\Delta(\eta_{8n+1})$.

ii) $\pi_{16n+4}(S^{8n+2})\cong G_{8n+2}\oplus\mathbf{Z}_4\{j_{4n+1}(\mathbf{C})\}$; $\pi_{16n+3}(S^{8n+1})\cong G_{8n+2}\oplus\mathbf{Z}_8\{\Sigma j_{2n}(\mathbf{H})\}$; $\pi_{16n+2}(S^{8n})\cong G_{8n+2}\oplus(\mathbf{Z}_8\oplus\mathbf{Z}_{24})\{\Delta(\nu_{16n+1}),\ j_{2n}(\mathbf{H})\}$; $\pi_{16n+2-k}(S^{8n-k})\cong G_{8n+2}\oplus\mathbf{Z}_8\{\Sigma^{3-k}\lambda\}$ for $k=1$ or 2.

3. The complex or quaternionic case. Hereafter we set $X=Q_{2n+1,2}(\mathbf{F})$ and $\gamma=\gamma_n(\mathbf{F})$.

Proposition 7. $\#\Sigma\gamma=\#\Sigma^2\gamma=2$ *for* $\mathbf{F}=\mathbf{C}$.

Proof. By (6), $\Sigma^2\gamma=i_*\Delta(\iota_{8n-3})$. Assume that $\Sigma^2\gamma=0$. Then, by Lemma 2, there exists an element $\beta\in\pi_{8n-1}(S^{4n})$ satisfying $\Delta(\iota_{8n+3})=\eta_{4n+1}\circ\Sigma^2\beta$. So we have $j_{2n}(\mathbf{C})=\eta_{4n}\circ\Sigma\beta+a\Delta(\eta_{8n+1})$ for $a=0$ or 1. Apply H to this relation. Then $\eta_{8n-1}=0$ and this is a contradiction. Therefore $\Sigma^2\gamma\neq 0$ and $\#\Sigma^2\gamma=2$.

By (6) and Lemma 6, $2\Sigma\gamma=2i_*j_{2n}(\mathbf{C})=0$. So we have $\#\Sigma\gamma=2$. This completes the proof.

Hereafter in this section, we shall deal with the quaternionic case.

Lemma 8. $\#\Sigma\gamma=\#\Sigma^2\gamma$ *and* $8\Sigma\gamma=0$.

Proof. In an EHP-sequence

$$\pi_{16n+4}(\Sigma(\Sigma X\wedge\Sigma X))\xrightarrow{\Delta}\pi_{16n+2}(\Sigma X)\xrightarrow{\Sigma}\pi_{16n+3}(\Sigma^2 X),$$

the left group is isomorphic to $\pi_{16n+4}((S^{16n+1}\cup_{(2n+1)\nu_{16n+1}}e^{16n+5})\vee S^{16n+5})\cong\mathbf{Z}_{(24,2n+1)}\{i\,\nu_{16n+1}\}$ by Lemma 5. Hence Σ is monomorphic if $2n+1\equiv 1$ or 2 mod 3 and so is on the 2-component if $2n+1\equiv 0$ mod 3. By Lemma 6, $\#\Sigma j_{2n}(\mathbf{H})=8$ and $8j_{2n}(\mathbf{H})=0$ if $2n+1\equiv 0$ mod 3. Therefore we have $8\Sigma\gamma=0$.

This completes the proof.

Proposition 9. i) $\#\Sigma^4\gamma = 2$.
ii) $\#\Sigma^k\gamma = 2$ *for* $n = 1$ *and* $1 \leq k \leq 3$.
iii) $\#\Sigma^3\gamma = 4$ *for* $n \geq 2$.

Proof. By use of the homotopy exact sequence of a pair $(\Sigma^5 X, S^{8n+4})$, we have $\pi_{16n+7}(\Sigma^5 X) \cong \mathbf{Z}\{i\Delta(\iota_{16n+9})\} \oplus K$, where K is a finite abelian group. In an EHP-sequence

$$\begin{array}{ccccc} \pi_{16n+7}(\Sigma^5 X) & \xrightarrow{H} & \pi_{16n+7}(\Sigma(\Sigma^4 X \wedge \Sigma^4 X)) & \xrightarrow{\Delta} & \pi_{16n+5}(\Sigma^4 X), \\ \cup\!\!\!\!\oplus & & \wr\| & & \\ \mathbf{Z}\{i\Delta(\iota_{16n+9})\} & & \pi_{16n+7}(S^{16n+7}) & & \end{array}$$

$H(i\Delta(\iota_{16n+9})) = \pm 2i\iota_{16n+7}$. So we have $\#(i\Delta(\iota_{16n+7})) = 2$. So, by (6), we have i).

By i) of Lemma 6, $2j_2(\mathrm{H}) \equiv 3\nu_8 \circ \sigma_{11} \pm 3\Delta(\nu_{17})$. So, by Lemma 8 and its proof, $2i_*j_2(\mathrm{H}) = 0$ and $2\Sigma^k\gamma = 0$ for $n = 1$ and $1 \leq k \leq 3$. So, by i), we have ii).

By Lemmas 6, 8 and i), $\#\Sigma^3\gamma = 2$ or 4. Assume that $2\Sigma^3\gamma = 2i_*\Sigma^2 j_{2n}(\mathrm{H}) = 0$. Then, by (4) and Lemma 2, there exists an element $\alpha \in \pi_{16n-2}(S^{8n-1})$ satisfying $2\Sigma^2 j_{2n}(\mathrm{H}) = (2n+1)\nu_{8n+2} \circ \Sigma^6\alpha$. So $2\Sigma j_{2n}(\mathrm{H}) \equiv (2n+1)\nu_{8n+1} \circ \Sigma^5\alpha \mod \Delta(\eta^2_{16n+3}) = 4\Sigma j_{2n}(\mathrm{H})$. Therefore $\pm 2j_{2n}(\mathrm{H}) = (2n+1)\nu_{8n} \circ \Sigma^4\alpha + x\Delta(\nu_{16n+1})$ for an integer x. Since $2(2n+1)\nu_{16n-1} = 2H(j_{2n}(\mathrm{H})) = \pm 2x\nu_{16n-1}$, we have $x \equiv \pm(2n+1) \mod 12$. By Lemma 6, we have $\Sigma^3\lambda = \pm(2n+1)\nu_{8n} \circ \Sigma^4\alpha$ since $4H(j_{2n}(\mathrm{H})) \neq 0$. By use of the EHP-sequences, we have $\pm\lambda \equiv (2n+1)\nu_{8n-3} \circ \Sigma\alpha \mod \Delta(\nu^2_{16n-5})$. Applying H to this relation, we have $\nu^2_{16n-7} \equiv 0 \mod H(\Delta(\nu^2_{16n-5})) = 2\iota_{16n-7} \circ \nu^2_{16n-7} = 0$. This is a contradiction and hence we have iii). This completes the proof.

Proposition 10. $\#\Sigma^k\gamma = 8$ *if* $n \geq 2$ *and* $k = 1$ *or* 2.

Proof. By Lemma 8, it suffices to work in the 2-components and to prove the assertion for $k = 2$. By Lemma 8 and Proposition 9, $\#\Sigma^2\gamma = 4$ or 8. Assume that $4\Sigma^2\gamma = 4i_*\Sigma j_{2n}(\mathrm{H}) = 0$. Then, by Lemmas 2 and 6, there exists an element $\alpha \in \pi_{16n}(S^{8n+1})$ satisfying $\Sigma^6\lambda' = \nu_{8n+1} \circ \Sigma^3\alpha$. By (7), $\Delta(\eta_{16n-3}) \neq 0$ and $\Delta(\eta^2_{16n-5}) \neq 0$. So, by use of the EHP-sequences, there exists an element $\beta \in \pi_{16n-4}(S^{8n-3})$ satisfying $\alpha = \Sigma^4\beta$. By an EHP-sequence

$$\pi_{16n+4}(S^{16n+1}) \xrightarrow{\Delta} \pi_{16n+2}(S^{8n}) \xrightarrow{\Sigma} \pi_{16n+3}(S^{8n+1}),$$

$\Sigma^5\lambda' - \nu_{8n}\circ\Sigma^6\beta = a\Delta(\nu_{16n+1})$ for an integer a. Applying H to this relation, we have $\pm 2a\nu_{16n-1} = 0$ and $a = 4b$ for an integer b. By Lemma 6, $4b\Delta(\nu_{16n+1}) = -2b\Sigma^5\lambda'$. So we have $(1+2b)\Sigma^5\lambda' = \nu_{8n}\circ\Sigma^6\beta$. Since $\Sigma: \pi_{16n+k}(S^{8n-2+k}) \to \pi_{16n+k+1}(S^{8n-1+k})$ is monomorphic for $k = 0$ or 1, $(1+2b)\Sigma^3\lambda' = \nu_{8n-2}\circ\Sigma^4\beta$. We set $m = 8n-5$. By Lemma 4, there exists a mapping $\delta: \Sigma^{m-1}\mathbf{R}P_m^{m+2} \to S^m$ such that Ker $\{\Sigma^3: \pi_{2m+7}(S^m) \to \pi_{2m+10}(S^{m+3})\} = \delta_*\pi_{2m+7}(\Sigma^{m-1}\mathbf{R}P_m^{m+2})$. $\mathbf{R}P_m^{m+2} = \Sigma^{m-3}\mathbf{R}P_3^5$ and $\pi_{2m+7}(\Sigma^{m-1}\mathbf{R}P_m^{m+2}) = \pi_{11}^S(\mathbf{R}P_3^5)$ (the stable group). Therefore we have

$$(8) \qquad (1+2b)\lambda' - \nu_m\circ\Sigma\beta \in \delta_*\pi_{11}^S(\mathbf{R}P_3^5).$$

Recall $\mathbf{R}P_3^5 = (S^3 \cup_{2\iota_3} e^4) \cup_{i\eta_3} e^5$. By [9], $\pi_9^S(\mathbf{R}P^2) = \mathbf{Z}_2\{\widetilde{8\sigma}\} \oplus \mathbf{Z}_2\{i\eta\sigma\} \oplus \mathbf{Z}_2\{i\varepsilon\}$. By use of a cofibre sequence starting with $i\eta_3$, we have an exact sequence

$$\mathbf{Z}_{16}\{\sigma\} \xrightarrow{(i\eta)_*} \pi_9^S(\mathbf{R}P^2) \xrightarrow{i'_*} \pi_{11}^S(\mathbf{R}P_3^5) \xrightarrow{p'_*} \mathbf{Z}_2\{\nu^2\} \to 0,$$

where $i': \Sigma^2\mathbf{R}P^2 \hookrightarrow \mathbf{R}P_3^5$ and $p': \mathbf{R}P_3^5 \to S^5$ are the canonical maps. Let $\tilde{\nu}'$ be an element of the Toda bracket $\langle i', i\eta, \nu\rangle \subset \pi_{11}^S(\mathbf{R}P_3^5)$. Then $2\tilde{\nu}'\nu \in \langle i', i\eta, \nu^2\rangle\circ 2\iota = -i'\langle i\eta, \nu^2, 2\iota\rangle \supset i''\langle\eta, \nu^2, 2\iota\rangle \ni i''\varepsilon \bmod i''\eta\sigma = 0$, where $i'' = i'\circ i: S^3 \hookrightarrow \mathbf{R}P_3^5$. So we have $2\tilde{\nu}'\nu = i\varepsilon$ and $\pi_{11}^S(\mathbf{R}P_3^5) = \mathbf{Z}_2\{i'\widetilde{8\sigma}\} \oplus \mathbf{Z}_4\{\tilde{\nu}'\nu\}$.

On the other hand, $H(\delta) \in [\Sigma^{2m-4}\mathbf{R}P_3^5, S^{2m-1}] \cong \{\mathbf{R}P_3^5, S^3\}$. We recall that $\{\mathbf{R}P^2, S^1\} = \mathbf{Z}_2\{\eta p\}$ and $\{\mathbf{R}P^2, S^0\} = \mathbf{Z}_4\{\bar{\eta}\}$, where $\bar{\eta}$ is an extension of η. By use of the above cofibre sequence, we have an exact sequence

$$0 \leftarrow \mathbf{Z}_2\{\eta p\} \xleftarrow{i'^*} \{\mathbf{R}P_3^5, S^3\} \xleftarrow{p'^*} \mathbf{Z}_2\{\eta^2\} \xleftarrow{(i\eta)^*} \mathbf{Z}_4\{\bar{\eta}\}.$$

Let $\bar{p} \in \{\mathbf{R}P_3^5, S^4\}$ be an extension of p with respect to $i\eta$. Then $\{\mathbf{R}P_3^5, S^3\} = \mathbf{Z}_2\{\eta\bar{p}\}$. $\eta\bar{p}\circ i'\widetilde{8\sigma} = \eta p\widetilde{8\sigma} = 8\eta\sigma = 0$ and $\eta\bar{p}\circ\tilde{\nu}'\nu \in \eta\circ\langle p, i\eta, \nu\rangle\circ\nu \subset \eta\circ G_4\circ\nu = 0$. So we have $(\eta\bar{p})_*\pi_{11}^S(\mathbf{R}P_3^5) = 0$. Applying H to (8), we have $(1+2b)\, H(\lambda') \in H(\delta)_*\pi_{11}^S(\mathbf{R}P_3^5) \subset (\eta\bar{p})_*\pi_{11}^S(\mathbf{R}P_3^5) = 0$. By Lemma 6, $H(\lambda') \equiv \varepsilon_{2m-1} \bmod \eta_{2m-1}\sigma_{2m}$. This is a contradiction and completes the proof.

4. The real case. We set $Y = \Sigma^{2n-3}\mathbf{R}P^2$ for $n \geq 2$, $X = Q_{2n+1,2}(\mathbf{R}) = \Sigma^{2n-2}\mathbf{R}P^2$ and $\gamma = \gamma_n(\mathbf{R})$ for $n \geq 1$.

Proposition 11. *$\#\Sigma\gamma = 2$ for $n \geq 1$.*

Proof. By (6), $\Sigma\gamma = i_*\Delta(\iota_{4n+1})$. Assume that $\Sigma\gamma = 0$. Then, by Lemma 2, we have $\Delta(\iota_{4n+1}) \in \{2\Delta(\iota_{4n+1})\} + (2\iota_{2n})_*\pi_{4n-1}(S^{2n})$. Applying H to this relation, we have $\pm 2\iota_{4n-1} \in \{4\iota_{4n-1}\} + \{8\iota_{4n-1}\}$. This is a contradiction and completes the proof.

By Propositions 7, 9, 10 and 11, we have completed the proof of Theorem 1.

We recall that $\pi_{4n-2}(\Sigma(Y \wedge Y)) = \mathbf{Z}_4\{\tilde{i}'\}$ and $2\tilde{i}' = i'\eta_{4n-3}$ for $n \geq 2$, where $i' : S^{4n-3} \hookrightarrow \Sigma(Y \wedge Y)$ is the inclusion [10].

Lemma 12. *Let $n \geq 2$. Then $2\gamma = \pm\Delta(\Sigma^2\tilde{i}')$, $\#\gamma = 4$ for even n and $\#\gamma = 8$ for odd n.*

Proof. First we shall show $\#\gamma = 4$ for $n = 2$ or 4. We consider a commutative diagram between exact sequences :

$$\begin{array}{ccccccc} \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{j_*} & \pi_6(X, S^3) & \xrightarrow{\partial} & \pi_5(S^3) \\ \| & & \downarrow i''_* & & \downarrow p_* & & \| \\ \pi_6(S^3) & \xrightarrow{i'_*} & \pi_6(V_{5,2}) & \xrightarrow{p'_*} & \pi_6(S^4) & \xrightarrow{\partial'} & \pi_5(S^3). \end{array}$$

By (2) and Lemma 3, we have $\pi_6(X, S^3) = \mathbf{Z}_2\{[\iota_3, \kappa]\} \oplus \mathbf{Z}_2\{\kappa\hat{\eta}_3^2\}$ and $\pi_7(X, S^3) = \mathbf{Z}_2\{[\eta_3, \kappa]\} \oplus \mathbf{Z}_2\{\kappa\hat{\nu'}\}$. $\partial[\iota_3, \kappa] = 2[\iota_3, \iota_3] = 0$, $\partial(\kappa\hat{\eta}_3^2) = 2\iota_3 \circ \eta_3^2 = 0$, $\partial[\eta_3, \kappa] = [\eta_3, 2\iota_3] = 0$ and $\partial(\kappa\hat{\nu'}) = 2\iota_3 \circ \nu' = 2\nu'$. So we have $\mathrm{Im}\ i_* \cong \mathbf{Z}_2$ and j_* is epimorphic. We recall that $\pi_6(V_{5,2}) \cong \mathbf{Z}_2$[13] and $\partial'\eta_4^2 = 2\iota_3 \circ \eta_3^2 = 0$. So we have $\pi_6(V_{5,2}) = \mathbf{Z}_2\{i''_*\tilde{\eta}_3\eta_5\}$ and $i'_*\nu' = 0$. Therefore we have $i_*\nu' = a\gamma$ for $a = 1$ or 2. By (1) and Lemma 3, we have $0 = aj_*\gamma = a[\iota_3, \kappa]$ and hence we have $a = 2$, $2\gamma = i\nu'$ and $\pi_6(X) = \mathbf{Z}_4\{\gamma\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$.

By (2) and Lemma 3, we have $\pi_{14}(X, S^7) = \mathbf{Z}_2\{[\iota_7, \kappa]\} \oplus \mathbf{Z}_2\{\kappa\hat{\nu}_7^2\}$ and $\pi_{15}(X, S^7) = \mathbf{Z}_2\{[\eta_7, \kappa]\} \oplus \mathbf{Z}_8\{\kappa\hat{\sigma'}\}$. The connecting map ∂ is trivial except for the following : $\partial(\kappa\hat{\sigma'}) = 2\iota_7 \circ \sigma' = 2\sigma'$. So, by a parallel argument to the above, we have $2\gamma = i_*\sigma'$ and $\pi_{14}(X) = \mathbf{Z}_4\{\gamma\} \oplus \mathbf{Z}_2\{\tilde{\nu}_7^2\}$ for $n = 4$. We note that $\pi_{14}(V_{9,2}) \cong \mathbf{Z}_2$ [13].

By Proposition 11 and an EHP-sequence

$$\pi_{4n}(\Sigma(X \wedge X)) \xrightarrow{\Delta} \pi_{4n-2}(X) \xrightarrow{\Sigma} \pi_{4n-1}(\Sigma X),$$

we have $2\gamma = a\Delta(\Sigma^2\tilde{i}')$ for an integer a. If a is even, $2\gamma = (a/2)i_*\Delta(\eta_{4n-1})$. So we have $2\gamma = 0$ for $n = 2$ or 4 and $2[\iota_{2n-1}, \kappa] = 2j_*\gamma = 0$ for $n = 3$ or $n \geq 5$ by (1). This contradicts the above and Lemma 3. Hence we have the first assertion.

By (7), $\Delta(\eta_{4n-1})$ is trivial for even n and nontrivial for odd n. So $2\Delta(\Sigma^2\tilde{i}') = i_*\Delta(\eta_{4n-1}) = 0$ and $4\gamma = 0$ for even n. By (1) and Lemma 3, $2\gamma \neq 0$. This leads us to the second assertion.

For odd n, it suffices to show $i_*\Delta(\eta_{4n-1}) \neq 0$. By [10], we have $i_*\Delta(\eta_{11}) = i_*\nu_5\eta_8^2 \neq 0$. Assume that it is trivial for $n \geq 5$. Then, by Lemma 2, there exists an element $\beta \in \pi_{4n-2}(S^{2n-1})$ satisfying $\Delta(\eta_{4n-1}) = 2\iota_{2n-1} \circ \beta$. By (7), $\Delta(\eta_{4n-3}) \neq 0$ for odd $n \geq 5$. So we have $\beta = \Sigma\beta'$ for some $\beta' \in \pi_{4n-3}(S^{2n-2})$. Therefore $\Delta(\eta_{4n-1}) = 2\Sigma\beta'$ and $\Delta(\eta_{4n-1}^2) = 2\Sigma\beta' \circ \eta_{4n-2} = 0$. By (7), $\Delta(\eta_{4n-1}^2) \neq 0$ for odd $n \geq 5$. This is a contradiction and completes the proof.

We set $X = Q_{2n+1,2}(\mathbf{F})$, $W = O_{2n+1,2}(\mathbf{F})$, $r = 2dn-1$ and $s = 2r+d-1 = (4n+1)d-3$. We consider a commutative diagram among exact sequences for $n \geq 2$:

$$
\begin{array}{ccccccc}
 & & \pi_s(W,X) & \xrightarrow[\cong]{\Sigma''} & \pi_r(S^r) & & \\
 & & \downarrow\partial'' & & \downarrow Q & & \\
\pi_{s-1}(S^r) & \xrightarrow{i_*} & \pi_{s-1}(X) & \xrightarrow{j_*} & \pi_{s-1}(X, S^r) & \xrightarrow{\partial} & \pi_{s-2}(S^r) \\
\| & & \downarrow i''_* & & \downarrow p_* & & \| \\
\pi_{s-1}(S^r) & \xrightarrow{i'_*} & \pi_{s-1}(W) & \xrightarrow{p'_*} & \pi_{s-1}(S^{r+d}) & \xrightarrow{\partial'} & \pi_{s-2}(S^r) \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where $\Sigma'' = \Sigma^{-(r+d-1)} \circ p''_*$ for the canonical map $p'' : (W, X) \to (S^s, *)$. By (1), Lemmas 3 and 12, we have Theorem 2.

5. Determination of $H(\gamma_n(\mathbf{R}))$. We shall show $H(\gamma) = \pm\tilde{i}'$, where $\gamma = \gamma_n(\mathbf{R})$ and $n \geq 2$. We set $Y = \Sigma^{2n-3}\mathbf{R}P^2$ and $X = \Sigma Y$.

Lemma 13. *Let $n \geq 2$. Then we have*

i) $i_*\Delta(\eta_{4n-3}) = 0$;

ii) $\mathrm{Im}\{\Sigma' : \pi_{4n-3}(Y, S^{2n-2}) \to \pi_{4n-2}(X, S^{2n-1})\} \cong \pi_{4n-2}(S^{2n})$.

Proof. By (7), $\Delta(\eta_{4n-3}) = 0$ for $n = 2$ or 4. So we have i) for $n = 2$ or 4. It suffices to prove $H(\gamma) \neq 0$ for $n = 3$ or $n \geq 5$ since $2\tilde{i}' = H(c\gamma)$ for $c = 1$ or 2 implies $i_*\Delta(\eta_{4n-3}) = \Delta(2\tilde{i}') = \Delta(H(c\gamma)) = 0$.

We consider an anti-commutative diagram :

$$\begin{array}{ccccc} \pi_{4n-3}(Y) & \xrightarrow{\Sigma} & \pi_{4n-2}(X) & \xrightarrow{H} & \pi_{4n-2}(\Sigma(Y\wedge Y)) \\ \downarrow j_* & & \downarrow j'_* & & \\ \pi_{4n-3}(Y, S^{2n-2}) & \xrightarrow{\Sigma'} & \pi_{4n-2}(X, S^{2n-1}) & & \\ \downarrow p_* & & \downarrow p'_* & & \\ \pi_{4n-3}(S^{2n-1}) & \xrightarrow{\Sigma} & \pi_{4n-2}(S^{2n}) & & . \end{array}$$

Assume that $H(\gamma)=0$. Then there exists an element $\beta \in \pi_{4n-3}(Y)$ such that $\gamma=\Sigma\beta$. Therefore, by (1), $[\iota_{2n-1}, \kappa]=j'_*\gamma=-\Sigma'(j_*\beta)$. By Theorem 2.1 of [3], we have an exact sequence for $n\geq 3$:

$$\pi_{2n-1}(S^{2n-2}) \xrightarrow{Q} \pi_{4n-3}(Y, S^{2n-2}) \xrightarrow{p_*} \pi_{4n-3}(S^{2n-1}) \to 0,$$

where $Q(\ )=[\ ,\kappa']$ and $\kappa'=(\Sigma')^{-1}\kappa$ is a generator of $\pi_{2n-1}(Y, S^{2n-2})\cong$ **Z**. By the above diagram and Lemma 3, $\Sigma(p_*j_*\beta)=-p'_*\Sigma'(j_*\beta)=0$. So we have $p_*j_*\beta=a\Delta(\iota_{4n-1})$ for $a=0$ or 1 and $p_*(2j_*\beta)=0$. Therefore we have $2j_*\beta=bQ(\eta_{2n-2})$ for $b=0$ or 1. By [15], $\Sigma'(2j_*\beta)=0$ and hence we have $2[\iota_{2n-1}, \kappa]=0$. This contradicts Lemma 3 and completes the proof of i).

By the lower square of the above diagram, p_*, Σ are epimorphic and p'_* is a split epimorphism. This leads us to ii) and completes the proof.

Proposition 14. *$H(\gamma)=\pm\tilde{\iota}'$ for $n\geq 2$.*

Proof. It suffices to prove that Σ: $\pi_r(Y)\to\pi_{r+1}(X)$ for $r=4n-4$ is monomorphic. We consider the suspension homomorphism between exact sequences up to sign :

$$\begin{array}{ccccccc} \pi_{r+1}(Y, S^{2n-2}) & \xrightarrow{\partial} & \pi_r(S^{2n-2}) & \xrightarrow{i_*} & \pi_r(Y) & \xrightarrow{j_*} & \pi_r(Y, S^{2n-2}) \\ \downarrow\Sigma' & & \downarrow\Sigma & & \downarrow\Sigma & & \downarrow\Sigma' \\ \pi_{r+2}(X, S^{2n-1}) & \xrightarrow{\partial'} & \pi_{r+1}(S^{2n-1}) & \xrightarrow{i'_*} & \pi_{r+1}(X) & \xrightarrow{j'_*} & \pi_{r+1}(X, S^{2n-1}). \end{array}$$

By Theorem 2.1 of [3], $\pi_r(Y, S^{2n-2})\cong \mathbf{Z}\{[\iota_{2n-2}, \kappa']\}\oplus\pi_r(S^{2n-1})$ and $\pi_{r+1}(X, S^{2n-1})\cong\pi_{r+1}(S^{2n})$ for $n\geq 2$. Since $\pi_r(Y)$ is finite, $j_*\alpha$ for $\alpha\in\pi_r(Y)$ belongs to the second direct summand. The left Σ has the kernel $\Delta\pi_{r+2}(S^{4n-3})=\{\Delta(\eta_{4n-3})\}$ and $\partial'[\iota_{2n-1}, \kappa]=2[\iota_{2n-1}, \iota_{2n-1}]=0$. So, by chasing the diagram and using Lemma 13, we conclude that Σ: $\pi_r(Y)\to\pi_{r+1}(X)$ is monomorphic. This completes the proof.

REFERENCES

[1] J. F. ADAMS: On the non-existence of elements of Hopf invariant one, Ann. of Math. 72 (1960), 20–104.

[2] A. L. BLAKERS and W. S. MASSEY: Products in homotopy theory, Ann. of Math. 58 (1953), 295–324.

[3] I. M. JAMES: On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford (2) 5 (1954), 260–70.

[4] I. M. JAMES: The Topology of Stiefel Manifolds, London Math. Soc. Lecture Note 24, Cambridge, 1976.

[5] I. M. JAMES and J. H. C. WHITEHEAD: The homotopy theory of sphere bundles over spheres (I), Proc. London Math. Soc. (3) 4 (1954), 196–218.

[6] I. M. JAMES and J. H. C. WHITEHEAD: The homotopy theory of sphere bundles over spheres (II), Proc. London Math. Soc. (3) 5 (1955), 148–66.

[7] M. A. KERVAIRE: Some nonstable homotopy groups of Lie groups, Illinois J. Math. 4 (1960), 161–69.

[8] M. MAHOWALD: Some Whitehead products in S^n, Topology 4 (1965), 17–26.

[9] J. MUKAI: Stable homotopy of some elementary complexes, Mem. Fac. Sci. Kyushu Univ. A 20 (1966), 266–82.

[10] J. MUKAI: A remark on Toda's result about the suspension order of the stunted real projective space, Mem. Fac. Sci. Kyushu Univ. A 42 (1988), 87–94.

[11] J. MUKAI: Remarks on homotopy groups of symmetric spaces, Math. J. Okayama Univ. 32 (1990), 159–64.

[12] Y. NOMURA: Note on some Whitehead products, Proc. Japan Acad. 50 (1974), 48–52.

[13] G. F. PAECHTER: The groups $\pi_r(V_{n,m})$ (I), Quart. J. Math. Oxford (2) 7 (1956), 249–68.

[14] S. THOMEIER: Einige Ergebnisse über Homotopiegruppen von Sphären, Math. Ann. 164 (1966), 225–50.

[15] H. TODA: Generalized Whitehead products and homotopy groups of spheres, J. Inst. Poly. Osaka City Univ. 3 (1952), 43–82.

[16] H. TODA: Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Studies 49, Princeton, 1962.

DEPARTMENT OF MATHEMATICS
FACULTY OF LIBERAL ARTS
SHINSHU UNIVERSITY
MATSUMOTO, NAGANO PREF. 390, JAPAN

(Received January 29, 1991)