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A NOTE ON $\mathscr{E}(\mathbf{H}P^n)$ FOR $n \leqq 4$

NORIO IWASE*, KEN-ICHI MARUYAMA and SHICHIRO OKA**

Projective spaces are constructed for real, complex and quaternionic numbers, classically, as based CW-complexes. We work in the category of based CW-complexes and based mappings and denote by $[X, Y]$ the homotopy set of mappings from X to Y and by $\mathscr{E}(X)$ the group of all invertible elements in $[X, X]$ with monoid structure by composition.

Problem. *Determine the group* $\mathscr{E}(\mathbf{F}P^n)$ *for* $\mathbf{F} = \mathbf{R}, \mathbf{C}$ *or* $\mathbf{H}$, $n \geq 1$.

When $n = 1$, it is trivial, since $\mathbf{F}P^1$ is a sphere of dimension $\dim_{\mathbf{R}} \mathbf{F}$ and we have

$$[\mathbf{F}P^1, \mathbf{F}P^1] \cong End(\mathbf{Z}) \text{ as monoids classified by the mapping degree,}$$

where $End(\mathbf{Z}) \cong \mathbf{Z}^\times$ with the monoid structure by multiplication. Thus $\mathscr{E}(\mathbf{F}P^1) \cong Aut(\mathbf{Z}) \cong \mathbf{Z}/2$, where $\mathbf{Z}/m$ is the cyclic group of order m. So, we may assume that $n \geq 2$.

In the real case, $\mathbf{R}P^k$ is the k-skeleton of the Eilenberg-MacLane complex $\mathbf{R}P^\infty = K(\mathbf{Z}/2, 1)$ and we have the following split surjection of monoids:

$$(0.1) \qquad [\mathbf{R}P^n, \mathbf{R}P^n] \xrightarrow{\pi} End(\mathbf{Z}/2) \cong \mathbf{Z}/2^\times.$$

Homotopical computations show that $\pi^{-1}(1) \cong (1+2Z)^\times$ and the natural homomorphism $\mathscr{E}(\mathbf{R}P^n) \to Aut(\pi_n(\mathbf{R}P^n)) \cong Aut(\mathbf{Z}) \cong \mathbf{Z}/2$ gives an isomorphism (see [1]).

In the complex or quaternionic case, the cells in $\mathbf{F}P^n$ are concentrated in even dimensions. Thus the restriction to $\mathbf{F}P^{n-1}$ of a self mapping of $\mathbf{F}P^n$ gives a monoid homomorphism r_n: $[\mathbf{F}P^n, \mathbf{F}P^n] \to [\mathbf{F}P^{n-1}, \mathbf{F}P^{n-1}]$. Hence we have $r_n(\mathscr{E}(\mathbf{F}P^n)) \subseteq \mathscr{E}(\mathbf{F}P^{n-1})$.

In the complex case, $\mathbf{C}P^k$ is the $2k$-skeleton of $\mathbf{C}P^\infty = K(\mathbf{Z}, 2)$ and hence we have

(0.2) $[\mathbf{C}P^n, \mathbf{C}P^n] \cong End(\mathbf{Z})$ as monoids and $\mathscr{E}(\mathbf{C}P^n) \cong Aut(\mathbf{Z}) \cong \mathbf{Z}/2$

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** Professor Oka has died in 1984 and left some notes on this topic.

as groups.

Moreover r_n gives an isomorphism of the monoids and the groups.

In the quaternionic case, unlike the above cases, $\mathbf{H}P^\infty = BS^3$ has non-zero homotopy groups in any dimensions higher than 3. By an easy computation, one can show that

(0.3) $\mathscr{E}(\mathbf{H}P^2) \cong \mathbf{Z}/2$ and $r_2 : \mathbf{Z}/2 \cong \mathscr{E}(\mathbf{H}P^2) \to \mathscr{E}(\mathbf{H}P^1) \cong \mathbf{Z}/2$ is trivial.

We shall show this in the proof of our result stated as follows:

Theorem 0.4*.** (1) *$\mathscr{E}(\mathbf{H}P^3) \cong \mathbf{Z}/2\times\mathbf{Z}/2$ and r_3 is surjective.*
(2) *$\mathscr{E}(\mathbf{H}P^4) \cong \{1\}$ or $\mathbf{Z}/2$ and r_4 is injective.*

This illustrates a difference from the real or complex case.

To explain our method, we need some notation. Let us denote by $Map_*(X, Y)$ the space of (based) mappings from X to Y and by $C_f = C_f(X, Y)$ the subspace of mappings homotopic to f in $Map_*(X, Y)$ (with the base point f).

We denote by i_k: $\mathbf{H}P^k \to \mathbf{H}P^n$ the canonical inclusion for $\infty \geq n \geq k \geq 1$ and by p_k: $Map_*(\mathbf{H}P^k, \mathbf{H}P^n) \to Map_*(\mathbf{H}P^{k-1}, \mathbf{H}P^n)$ the restriction fibration, which maps C_{f_k} to $C_{f_{k-1}}$, where f_k is the restriction to $\mathbf{H}P^k$ of a mapping f in $Map_*(\mathbf{H}P^n, \mathbf{H}P^n)$.

The key lemma to Theorem 0.4 is given in § 2 and is stated as follows.

Lemma 2.1. *If λ is odd, then the restriction to $\mathbf{H}P^1$ induces the following split surjection.*

$$\pi_1(C_{f_n}) \to \pi_1(C_{f_1}) \cong \mathbf{Z}/2.$$

Let $\tilde{p}_k$ be the restriction of p_k to $\widetilde{C}_{f_k} = (p_2 \cdots p_k)^{-1}(C_{f_1})$ (with the base point f_k). Let us recall that when $n = \infty$ and f is not null-homotopic, the tower of fibrations $\{\tilde{p}_k\}$ has the inverse limit $C_f(\mathbf{H}P^\infty, \mathbf{H}P^\infty)$ weakly equivalent to $SO(3)$ (see [2]) and that there is a homotopy spectral sequence associated with a tower of fibrations $\{\tilde{p}_k\}$, namely,

Theorem 0.5. *Let f be a self mapping of $\mathbf{H}P^\infty$. Then there is a un-*

*** Professor Oka has shown the result (1) on $\mathbf{H}P^3$ and some similar result to (2) on $\mathbf{H}P^4$.

stable homotopy spectral sequence $\{(E^r_{4s,t}, d^r)\}$, $d^r: E^r_{4s,t} \to E^r_{4s+4r,t+4r-1}$, *converging to* $E_0\pi_{t-4s}(C_f)$ ($\cong E_0\pi_{t-4s}(SO(3))$ *unless* f *is null-homotopic*), *whose* E^1*-term is given as follows:*

$$E^1_{4s,t} \cong \pi_{t-1}(S^3),\ t \geq 0,$$
$$D^1_{4s,t} \cong \pi_{t-4s}(\widetilde{C}_{f_s}(\mathrm{H}P^s, \mathrm{H}P^\infty), f_s),\ t \geq 4s,$$
$$d^1_{4s,t}(\ell) = \pm(s\ell\circ\nu_{t-1} \pm \lambda\nu'_3\circ\Sigma^3\ell),\ t \geq 4s \text{ and } t > 4,$$

where λ *is the mapping degree of* f_1 *and* ν'_3 *is the Brakers-Massey element which generates* $\pi_6(S^3) \cong \mathbf{Z}/12$.

Remark. (1) *The first summand appearing in the expression of* $d^1(\ell)$ *is nothing but the composition of* ℓ *with* $t-4s$ *fold suspension of* $s\nu_{4s}$, *which is the attaching mapping of the top cell of* $\mathrm{H}P^{s+1}/\mathrm{H}P^{s-1} = S^{4s}\cup_{s\nu_{4s}} e^{4s+4}$ (see [6]). *Thus* d^1 *is a homomorphism if* $t \neq 4$.

(2) $d^1_{4,4}$ *is given by the formula* $d^1_{4,4}(m\nu'_3) = \pm((1/2)m(m+2\lambda-1)\nu'_3)$. *The last term is equal to* $\pm((1/2)(m+\lambda)(m+\lambda-1)\nu'_3)$, *since* $\lambda(\lambda-1) = 0$ mod 24 (see Fact 2 in § 2).

(3) *We do not determine the second differential, which would be described in terms of Toda brackets.*

On the degree λ, the following fact has been known (see Sullivan [7] and Mislin [3]).

Fact 1. *The following three conditions are equivalent for an arbitrary integer* λ.

i) *The composition of the self mapping on* S^4 *of degree* λ *with the inclusion* $i_1: S^4 = \mathrm{H}P^1 \hookrightarrow \mathrm{H}P^\infty$ *is extendable to* $\mathrm{H}P^\infty$.

ii) *The self mapping on* S^3 *of degree* λ *is a loop mapping.*

iii) $\lambda = 0$ *or an odd square number.*

We could not give a conjecture to Problem in general case, but the following

Conjecture. (1) $\mathscr{E}(\mathrm{H}P^4) \cong \{1\}$.

(2) *The image of* $p_{k*}: \pi_1(C_{f_k}) \to \pi_1(C_{f_{k-1}})$ *is isomorphic to* $\pi_1(SO(3))$ *for all* $k \geq 2$.

We will show Theorem 0.5 in a slightly stronger form in Section 1 (without assuming $n = \infty$) and calculate the homotopy groups of $\widetilde{C}_{f_k}$ in Section 2, and in Section 3 we prove Theorem 0.4.

§ 1. **Proof of Theorem 0.5.** From now on, we fix a positive integer $n \geq 2$.

Let f_n be in $Map_*(\mathbf{H}P^n, \mathbf{H}P^n)$. We denote by f_k the restriction of f_n to $\mathbf{H}P^k$ for $k < n$ and by λ the mapping degree of the compression of f_1 into $\mathbf{H}P^1$.

The cofibre sequence $S^{4k-1} \to \mathbf{H}P^{k-1} \xhookrightarrow{i_{k-1}} \mathbf{H}P^k \to S^{4k}$ induces the fibre sequence

$$F(f_k) \to \widetilde{C}_{f_k} \to \widetilde{C}_{f_{k-1}} \to \Omega^{4k-1}(\mathbf{H}P^n), \tag{1.1}$$

where $F(f_k) = p_n^{-1}(f_{k-1})$ (with the base point f_k) and hence $F(f_1) = \widetilde{C}_{f_1}$. Then $F(f_k)$ is homotopy equivalent to $\Omega^{4k}(\mathbf{H}P^n)$, which acts on the total space. Thus we obtain the following long homotopy exact sequence:

$$\begin{aligned}
&\cdots \xrightarrow{\partial} \pi_q(F(f_k)) \xrightarrow{j_{k*}} \pi_q(C_{f_k}) \to \pi_q(C_{f_{k-1}}) \to \cdots \\
(1.2)\qquad &\xrightarrow{\partial} \pi_1(F(f_k)) \xrightarrow{j_{k*}} \pi_1(C_{f_k}) \to \pi_1(C_{f_{k-1}}) \to \\
&\xrightarrow{\partial} \pi_0(F(f_k)) \xrightarrow{j_{k*}} \pi_0(\widetilde{C}_{f_k}) \to \pi_0(\widetilde{C}_{f_{k-1}}) \xrightarrow{\partial} \pi_{4k-1}(\mathbf{H}P^n),\ q \geq 1,
\end{aligned}$$

where $j_{k*}: \pi_0(F(f_k)) \to \pi_0(\widetilde{C}_{f_k})$ is injective if $\ker j_{k*} = 0$, and is surjective when $k = 2$, since $\widetilde{C}_{f_1} = C_{f_1}$ is connected.

Since $\mathbf{H}P^k/\mathbf{H}P^{k-1} = S^{4k}$ coacts on $\mathbf{H}P^k$, $F(f_k)$ have the homotopy type of $F(*_k) \simeq \Omega^{4k}\mathbf{H}P^n$, where $*_k$ denotes the trivial constant mapping: $\mathbf{H}P^k \to \mathbf{H}P^n$. This implies that $\pi_q(C_{f_1}) = \pi_q(F(f_1)) \cong \pi_{q+4}(\mathbf{H}P^n)$ and $\pi_q(F(f_k)) \cong \pi_{q+4k}(\mathbf{H}P^n)$ for $1 \leq k \leq n$ and $q \geq 0$. We remark that the latter group is isomorphic with $\pi_{q+4k-1}(S^3)$, provided that $4m+2 \geq q+4k$.

Let us recall that the first differential d^1 is given by the composition $\partial \circ j_{k*}: \pi_{q+4k}(\mathbf{H}P^n) \cong \pi_q(F(f_k)) \to \pi_q(C_{f_k}) \to \pi_{q-1}(F(f_{k+1})) \cong \pi_{q+4k+3}(\mathbf{H}P^n)$. Then we obtain the following proposition by modifying the proof of [5, Theorem 3.3].

Proposition 1.3. *Let $1 \leq k+1 \leq n$. The following equation holds in $\pi_{q+4k+3}(\mathbf{H}P^n)$.*

$$d^1(\ell) = \pm k\ell \circ \nu_{q+4k} \pm \lambda \circ [i_1, \ell] \text{ for } \ell \in \pi_{q+4k}(\mathbf{H}P^n).$$

Proof. Let $Y_k = S^q \times \mathbf{H}P^k$. Then Y_k can be decomposed as $\{Y_{k-1} \cup$

$\mathrm{H}P^k\} \cup e^{q+4k}$. Hence there is the co-action $\mu_0: Y_k \to Y_k \vee S^{q+4k}$ of S^{q+4k} on Y_k with co-axes (Id, χ), where Id denotes the identity of Y_k and χ; $Y_k \to S^{q+4k}$ denotes the canonical collapsion which has degree 1.

We denote by $pr_t: X_1 \times X_2 \to X_t$ the canonical projection to t-th factor and by $\Lambda: X \times Y \to X \times Y/(X \vee Y) = X \wedge Y$ the canonical collapsion. Composing μ_0 with $pr_2 \vee \iota_{q+4k}$, we get a mapping μ: $Y_k \to \mathrm{H}P^k \vee S^{q+4k}$, which is extendable to a mapping $\tilde{\mu}$: $Y_k \cup \mathrm{H}P^{k+1} \to \mathrm{H}P^{k+1} \vee S^{q+4k}$ by putting $\tilde{\mu}|_{\mathrm{H}P^{k+1}} = I_{k+1}$ the identity. Then $\tilde{\mu}$ has the co-axes $(pr_2, \tilde{\chi})$, where $\tilde{\chi}$ is the extension of χ by putting $\tilde{\chi}|_{\mathrm{H}P^{k+1}} = I_{k+1}$.

We can now give a description of d^1, namely, $d^1(\ell) = \partial \circ j_{k*}(\ell)$ is given as follows:

$$d^1(\ell) = \nabla \circ (f_{k+1} \vee \ell) \circ \tilde{\mu} \circ \beta,$$

where β is the attaching mapping of the top cell of $Y_{k+1} = \{Y_k \cup \mathrm{H}P^{k+1}\} \cup_\beta e^{q+4k+4}$ and ∇: $\mathrm{H}P^n \vee \mathrm{H}P^n \to \mathrm{H}P^n$ is the folding mapping.

Let us consider the composition $\tilde{\mu} \circ \beta$: $S^{q+4k+3} \to \mathrm{H}P^{k+1} \vee S^{q+4k}$, which is in $\pi_{q+4k+3}(\mathrm{H}P^{k+1} \vee S^{q+4k}) \cong \pi_{q+4k+3}(\mathrm{H}P^{k+1}) \oplus \pi_{q+4k+3}(S^{q+4k}) \oplus \mathbf{Z}[i_1, \iota_{q+4k}]$.

The first factor is given by $pr_1 \circ \tilde{\mu} \circ \beta = pr_2 \circ \beta$ and pr_2 is extendable to Y_{k+1} in which β has to be trivial. Thus $pr_1 \circ \tilde{\mu} \circ \beta = 0$.

The second factor is given by $pr_2 \circ \tilde{\mu} \circ \beta = \tilde{\chi} \circ \beta = \chi' \circ \Lambda \circ \beta$, where χ': $\Sigma^q \mathrm{H}P^k \to S^{q+4k}$ is the mapping of degree 1. Here $\Lambda \circ \beta$ is nothing but the attaching mapping of the top cell of $\Sigma^q \mathrm{H}P^{k+1}$, which is known to be $\Sigma^q(k\nu_{4k}) = k\nu_{q+4k}$ up to sign by [6]. Thus $pr_2 \circ \tilde{\mu} \circ \beta = k\nu_{q+4k}$.

To determine the third factor, we choose an integer c so that the third factor is $c[i_1, \iota_{q+4k}]$. Then $\tilde{\mu} \circ \beta$ is homotopic to $c[i_1, \iota_{q+4k}]$ in the space $Z_{k+1} = \mathrm{H}P^{k+1} \vee S^{q+4k} \cup_{k\nu_{q+4k}} e^{q+4k+4}$.

Hence we have the relation $u_4 \cdot v_{q+4k} = \pm c v_{q+4k+4}$ in the ring $H^*(Z_{k+1} \cup_{c[i_1, \iota_{q+4k}]} e^{q+4k+4}; \mathbf{Z})$, where u_4 is the element in dimension 4 corresponding to the ring generator of $H^*(\mathrm{H}P^{k+1}; \mathbf{Z})$, v_{q+4k} is the element corresponding to the generator of $H^{q+4k}(S^{q+4k}; \mathbf{Z})$ and v_{q+4k+4} is the generator in dimension $q+4k+4$.

On the other hand, $Y_{k+1} \cong \{Y_k \cup \mathrm{H}P^{k+1}\} \cup_\beta e^{q+4k+4}$ has the integral cohomology ring isomorphic to $H^*(\mathrm{H}P^{k+1}; \mathbf{Z}) \otimes H^*(S^q; \mathbf{Z}) \cong \mathbf{Z}[u_4]/(u_4^{k+2}) \otimes \wedge(v_q)$, where v_q is the element corresponding to the generator of $H^q(S^q; \mathbf{Z})$.

There is the following homotopy commutative diagram of cofibre sequences.

$$\begin{array}{ccccc} S^{q+4k+3} & \xrightarrow{\beta} & Y_k \cup \mathrm{H}P^{k+1} & \longrightarrow & \{Y_k \cup \mathrm{H}P^{k+1}\} \bigcup_\beta e^{q+4k+4} = Y_{k+1} \\ \| \downarrow & & \tilde{\mu} \downarrow & & \\ S^{q+4k+3} & \xrightarrow{c[i_1,\ \iota_{q+4k}]} & Z_{k+1} & \longrightarrow & Z_{k+1} \bigcup_{c[i_1, \iota_{q+4k}]} e^{q+4k+4}. \end{array}$$

Then the homotopy commutativity of the diagram yields a mapping ω: $Y_{k+1} \to Z_{k+1} \bigcup_{c[i_1, \iota_{q+4k}]} e^{q+4k+4}$ inducing an injection onto a direct summand in integral cohomology, since the axis $\tilde{\chi}$ has degree 1. Hence we obtain the following equation.

$$\omega*(u_4) = u_4$$
$$\omega*(v_{q+4k}) = (\deg \chi) u_4^k \cdot v_q = u_4^k \cdot v_q$$

Then it follows that $\omega^*(u_4 \cdot v_{q+4k}) = u_4^{k+1} \cdot v_q$, which is a generator in dimension $q+4k+4$, and hence $c = \pm 1$. Thus $d^1(\ell) = \nabla_* \circ (f_{k+1} \vee \ell)_*(0, \pm k \nu_{q+4k}, \pm [i_1, \iota_{q+4k}]) = \pm \ell_*(k \nu_{q+4k}) \pm [f_{k+1} \circ i_1, \ell] = \pm k \ell \nu_{q+4k} \pm \lambda [i_1, \ell]$. This implies the proposition.

When $1 \leq k+1 \leq n$, the condition $q \leq 3$ implies that $4n-4k-1 \geq q \geq 0$. In that case, $\pi_{q+4k}(\mathrm{H}P^n) \cong \pi_{q+4k}(\mathrm{H}P^\infty)$ and the latter group is isomorphic to $\pi_{q+4k-1}(S^3)$ by taking adjoints. Then we may assume that $n = \infty$ without any loss of generality.

Since the adjoint of a Whitehead product is the Samelson product of the adjoints up to sign, the adjoint of $[i_1, ad(\ell)]$ is $\langle ad(i_1), \ell \rangle = \langle \iota_3, \ell \rangle = \langle \iota_3, \iota_3 \rangle \circ \Sigma^3 \ell = \nu_3' \circ \Sigma^3 \ell$, up to sign.

Thus we have obtained the following

Corollary 1.4. *If $q \leq 3$ or more generally $4n-4k-1 \geq q \geq 0$, then the following equation holds in $\pi_{q+4k+2}(S^3)$:*

$$d^1(\ell) = \pm k \ell \circ \nu_{q+4k-1} \pm \lambda \nu_3' \circ \Sigma^3 \ell, \ \textit{for}\ \ell \in \pi_{q+4k-1}(S^3),\ q+4k > 4.$$

This completes the proof of Theorem 0.5.

§ 2. Homotopy groups of $\widetilde{C}_{f_2}$, $\widetilde{C}_{f_3}$ and $\widetilde{C}_{f_3}$. There is the following well-known fact on the degree λ of f_1.

Fact 2. *The following three conditions are equivalent for an arbitrary integer λ.*

i) *The composition of the self mapping on S^4 of degree λ with the* *inclusion* $i_1: S^4 = \mathbf{H}P^1 \hookrightarrow \mathbf{H}P^2$ *is extendable to* $\mathbf{H}P^2$.

ii) *The self mapping on* S^3 *of degree* λ *is an H-mapping.*

iii) $\lambda(\lambda-1) \equiv 0 \mod 24$.

One can show this by using the injectivity of suspension $E: [X, S^3] \to [\Sigma X, S^4]$ and the following homotopy formula (see [9]) for any integer k and composable elements α and β in the unstable homotopy groups of spheres:

$$(k\beta)\circ\alpha = k(\beta\circ\alpha)+\binom{k}{2}[\beta,\beta]\circ h_0(\alpha)-\binom{k+1}{3}[[\beta,\beta],\beta]\circ h_1(\alpha),$$

where h_t denotes the t-th Hopf-Hilton invariant.

From Fact 2, it follows that $\lambda(\lambda-1) \equiv 0 \mod 24$. Thus $\lambda = 1$ when f_n is a homotopy equivalence. So, from now on, we make an additional assumption that λ is odd.

In this section, we consider homotopy groups only in dimensions < 3. Hence $E^1_{4s,t} = \pi_{t-4s}(F(f_s)) \cong \pi_{t-1}(S^3)$ for the dimensional reasons.

First we introduce the following information from the infinite term.

Lemma 2.1. *If* λ *is odd, then the restriction to* $\mathbf{H}P^1$ *induces the following split surjection*:

$$\pi_1(C_{f_n}) \to \pi_1(C_{f_1}) \cong \mathbf{Z}/2.$$

Proof. The compositions with f_n and f_1 induce the following commutative diagram:

$$\begin{array}{ccccc} C_{I_n} & \longrightarrow & C_{i_1}(\mathbf{H}P^1, \mathbf{H}P^n) & \longleftarrow & C_{I_1} \\ f_{n\#}\downarrow & & f_{1\#}\downarrow & & \lambda I_{1\#}\downarrow \\ C_{f_n} & \longrightarrow & C_{f_1} & \longleftarrow & C_{\lambda I_1} \end{array}$$

where I_k denotes the identity mapping of $\mathbf{H}P^k$, $k \geq 1$.

Let us recall that $\pi_1(C_{f_1}) \cong \pi_4(S^3) \cong \mathbf{Z}/2$. The mapping $\lambda I_{1\#}$ induces the multiplication by λ in the homotopy groups, and hence an isomorphism of π_1, since λ is odd and π_1 is isomorphic to $\mathbf{Z}/2$.

Thus we may suppose that $f_n = I_n$. Let us recall that the action of $Aut(S^3)$ on S^3 is represented by the isomorphism ϕ: $\mathbf{R}P^3 \to Aut(S^3)$ given by $\phi([g])(x) = gxg^{-1}$ for $g, x \in S^3$, which is linear, leave the unit 1 fixed and preserves metric and orientation, and hence can be identified with the canonical action of $1 \oplus SO(3) \subset SO(4)$. Then the action induces that of

$SO(3) \cong Aut(S^3)$ on $\mathbf{H}P^n$, which is represented by a mapping ϕ_n: $SO(3) \to C_{\iota_n} \subseteq C_{\iota_n}(\mathbf{H}P^n, \mathbf{H}P^n)$. Then the mapping

$$\tilde{\phi}_1: SO(3) \xrightarrow{\phi_1} C_{\iota_1}(\mathbf{H}P^1, \mathbf{H}P^n) \subseteq C_{\iota_1}(\mathbf{H}P^1, \mathbf{H}P^\infty) \simeq C_{\iota_3}(S^3, S^3)$$

is given by the formula $\tilde{\phi}_1(g)(x) = g(x)$ for $x \in S^3$. Through the homotopy equivalence $C_{\iota_3}(S^3, S^3) \simeq C_0(S^3, S^3) \subset Q(S^0)$, we may regard $\tilde{\phi}_1$ as the restriction to $SO(3)$ of the J-mapping: $SO \to Q(S^0)$. Thus the homomorphism $\tilde{\phi}_{1*}$ is an isomorphism and so is

$$\phi_{1*}: \pi_1(SO(3)) \xrightarrow{\phi_{n*}} \pi_1(C_{\iota_n}(\mathbf{H}P^n, \mathbf{H}P^n)) \to \pi_1(C_{\iota_1}(\mathbf{H}P^1, \mathbf{H}P^n)),$$

since $\pi_1(C_{\iota_1}(\mathbf{H}P^1, \mathbf{H}P^n)) \cong \pi_1(C_{\iota_3}(S^3, S^3))$, for the dimensional reasons. This implies the lemma.

Then by using the homotopy exact sequence (1.2), we obtain the following

Proposition 2.2. *If λ is odd, then $d^1_{4s,t}$: $E^1_{4s,t} \to E^1_{4s+4,t+3}$ is a zero homomorphism when $(4s, t) = (4, 5), (8, 8), (8, 9), (8, 10)$ or $(16, 16)$; an isomorphism when $(4s, t) = (4, 6)$ or $(12, 13)$; and an injection when $(4s, t) = (12, 12)$. At the prime 2, $d^1_{4,7}$ is a zero mapping if λ is odd.*

Proof. By [8], there exist the following equations:

(2.3.1) $\eta_3\nu_4 = \nu'_3\eta_6$ generates $\pi_7(S^3) \cong \mathbf{Z}/2$, ([8, (5.9)])
(2.3.2) $\eta_3\nu'_4 = 0$, ([8, Lemma 5.7])
(2.3.3) $\sigma'''\nu_{12} = \eta_5^2\varepsilon_7 \bmod 4(\nu_5\sigma'_8)$, ([8, (6.2)])
(2.3.4) $\eta_5^2\varepsilon_7 = 2\varepsilon'_5 \equiv 0 \bmod 4$, ([8, Lemma 6.6])
(2.3.5) $H(\nu'_3\mu_6) = \eta_5\mu_6$, ([8, P. 75])
(2.3.6) $H(\nu'_3\eta_6\varepsilon_7) = 4\nu_5\sigma_8$, ([8, P. 75])
(2.3.7) $H(\mu_3) = \sigma'''$, ([8, P. 54])
(2.3.8) $\pi_9(S^3 ; 2) = 0$, ([8, Propositions 5.11])
(2.3.9) $\pi_{10}(S^3) = \mathbf{Z}/15$,
(2.3.10) $\pi_{11}(S^3) = \mathbf{Z}/2\cdot\varepsilon_3$, ([8, Theorem 7.1])
(2.3.11) $\pi_{12}(S^3) = \mathbf{Z}/2\cdot\mu_3 \oplus \mathbf{Z}/2\cdot\eta_3\varepsilon_4$, ([8, Theorem 7.2])
(2.3.12) $\pi_{14}(S^3 ; 2) = \mathbf{Z}/4 \oplus \mathbf{Z}/2\cdot\varepsilon_3\nu_{11} \oplus \mathbf{Z}/2\cdot\nu'_3\varepsilon_6$, ([8, Theorem 7.4])
(2.3.13) $E\pi_{14}(S^2 ; 2) = 0$, ([8, P. 75])
(2.3.14) $\pi_{15}(S^3) = \mathbf{Z}/2\cdot\nu'_3\mu_6 \oplus \mathbf{Z}/2\cdot\nu'_3\eta_6\varepsilon_7$. ([8, Theorem 7.6])

(2.3.15) $\pi_{15}(S^5; 2) = \mathbf{Z}/8 \cdot \nu_5\sigma_8 \oplus \mathbf{Z}/2 \cdot \eta_5\mu_6$, ([8, Theorem 7.3])

where H: $\pi_q(S^p) \to \pi_q(S^{2p-1})$ and E: $\pi_q(S^p) \to \pi_{q+1}(S^{p+1})$ denote the Hopf invariant and the suspension homomorphisms in the EHP-sequence of James [4].

By Corollary 1.4, Equation (2.3.1) implies that $d^1_{4,5} = 0$.

Equations (2.3.1) and (2.3.2) imply that $\eta_3^2\nu_5 = \eta_3\nu'_4\eta_7 = 0$. Hence by Corollary 1.4, we obtain that $d^1_{4,6}$ is an isomorphism and that $d^1_{8,9} = 0$, since $d^1 \circ d^1 = 0$.

Equation (2.3.8) implies that $d^1_{4,7} = 0$ at the prime 2.

Equations (2.3.1) and (2.3.9) imply that $d^1_{8,8} = 0$ and Equations (2.3.8) and (2.3.11) imply that $d^1_{8,10} = 0$.

By Corollary 1.4, Equations (2.3.10) and (2.3.12) imply that $d^1_{12,12}$ is an injection to the subgroup generated by $\varepsilon_3\nu_{11} + \nu'_3\varepsilon_6$.

If $d^1_{12,13}$ is an isomorphism, then $d^1_{16,16} = 0$, since $d^1 \circ d^1 = 0$. So we are left to show that $d^1_{12,13}$ is an isomorphism.

As in [8], Equation (2.3.13) implies that H: $\pi_{15}(S^3) \to \pi_{15}(S^5; 2)$ is injective. By [8, Proposition 2.2] with Equations (2.3.7), (2.3.3) and (2.3.4), we have that

$$H(\mu_3\nu_{12}) = H(\mu_3)\nu_{12} = \sigma'''\nu_{12} = \eta_5^2\varepsilon_7 = 2\varepsilon'_5 \equiv 0 \bmod 4,$$

Then by Corollary 1.4 with the equation $H \circ E = 0$ and Equations (2.3.5) and (2.3.6), it follows that, for $(a, b) \in \mathbf{Z} \times \mathbf{Z}$,

$$\begin{aligned} H \circ d^1(a\mu_3 + b\eta_3\varepsilon_4) &= aH(d^1(\mu_3)) + bH(d^1(\eta_3\varepsilon_4)) \\ &= aH(\nu'_3\mu_6 + \mu_3\nu_{12}) + bH(\nu'_3\eta_6\varepsilon_4 + \eta_3\varepsilon_4\nu_{12}) \\ &= a\{H(\nu'_3\mu_6) + H(\mu_3\nu_{12})\} + b\{H(\nu'_3\eta_6\varepsilon_4) + H(\eta_3\varepsilon_4\nu_{12})\} \\ &= a\eta_5\mu_6 + 2a\varepsilon'_5 + 4b\nu_5\sigma_8 \\ &\equiv a\eta_5\mu_6 \bmod 4. \end{aligned}$$

Let us assume that $H \circ d^1(a\mu_3 + b\eta_3\varepsilon_4) = 0$. Then it follows that $a \equiv 0 \bmod 2$ by Equation (2.3.15) and that $a\mu_3 = 0$ by Equation (2.3.11). Hence $H(a \cdot d^1(\mu_3) + b \cdot d^1(\eta_3\varepsilon_4)) = 4b\nu_5\sigma_8 = 0$ and $4b \equiv 0 \bmod 8$ by Equation (2.3.15). Then by Equation (2.3.12), it follows that $b\eta_3\varepsilon_4 = 0$. Thus we obtain that $H \circ d^1_{12,13}$ is injective. Therefore by (2.3.11) and (2.3.14), so is $d^1_{12,13}$: $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2$. This implies that d^1 is an isomorphism.

Proposition 2.4. *If λ is odd, then there is the following isomorphisms:*

(1) $\pi_0(\widetilde{C}_{f_2}) \cong \mathbf{Z}/2$, *if* $n \geq 2$.

(2) $\pi_1(C_{f_2}) \cong \mathbf{Z}/2,\quad$ *if* $n \geq 2$,
(3) $\pi_2(C_{f_2}) \cong 0$ *or* $\mathbf{Z}/3$, *if* $n \geq 2$.

For $n \geq 2$, *the restriction to* $\mathbf{H}P^1$ *induces the following split surjections:*

(4) $p_{2*} : \pi_0(\widetilde{C}_{f_2}) \to \pi_0(C_{f_1}) \cong *$,
(5) $p_{2*} : \pi_1(C_{f_2}) \cong \pi_1(C_{f_1}) \cong \mathbf{Z}/2$,
(6) $p_{2*} : \pi_2(C_{f_2}) \to \pi_2(C_{f_1}) \cong 0$,
(7) $p_{2*} : \pi_3(C_{f_2}\,;2) \to \pi_3(C_{f_1}\,;2) \cong \mathbf{Z}/4$.

Proof. Since $F(f_1) = \widetilde{C}_{f_1}$, the connecting homomorphism $\partial: \pi_q(\widetilde{C}_{f_1}) \to \pi_{q-1}(F(f_2))$ can be identified with $d^1_{4,q+4}: \pi_q(F(f_1)) = E^1_{4,q+4} \to E^1_{8,q+7} = \pi_{q-1}(F(f_2))$, which is a zero mapping when $q = 1$ and an isomorphism when $q = 2$ by Proposition 2.2. At the prime 2, we also have that d^1 is a zero mapping when $q = 3$ by Proposition 2.2. Then by (1.2) with $k = 2$, we obtain the following short exact sequences:

$$\pi_3(C_{f_2}\,;2) \xrightarrow{p_{2*}} \pi_3(C_{f_1}\,;2) \to 0$$
$$\pi_2(F_{f_2}) \xrightarrow{j_{2*}} \pi_2(C_{f_2}) \to 0$$
$$0 \to \pi_1(C_{f_2}) \xrightarrow{p_{2*}} \pi_1(C_{f_1}) \to 0$$
$$0 \to \pi_0(F(f_2)) \xrightarrow{j_{2*}} \pi_0(\widetilde{C}_{f_2}) \to *$$

By (2.3.8) and (2.3.1), $\pi_2(F_{f_2})$ and $\pi_0(F_{f_2})$ are isomorphic to $\mathbf{Z}/3$ and $\mathbf{Z}/2$, respectively. Also we have that $\pi_1(C_{f_1})$ is isomorphic to $\mathbf{Z}/2$. This implies the proposition.

Proposition 2.5. *If* λ *is odd, then there are following isomorphisms:*

(1) $\pi_0(\widetilde{C}_{f_3}) \cong \mathbf{Z}/2\times\mathbf{Z}/2,\quad$ *if* $n \geq 3$,
(2) $\pi_1(C_{f_3}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$, *if* $n \geq 3$.

For $n \geq 3$, *the restriction to* $\mathbf{H}P^2$ *induces the following surjections:*

(3) $p_{3*} : \pi_0(\widetilde{C}_{f_3}) \to \pi_0(\widetilde{C}_{f_2}) \cong \mathbf{Z}/2$,
(4) $p_{3*} : \pi_1(C_{f_3}) \to \pi_1(C_{f_2}) \cong \mathbf{Z}/2$,
(5) $p_{3*} : \pi_2(C_{f_3}) \to \pi_2(C_{f_2}) \cong 0$ *or* $\mathbf{Z}/2$,

where the lower sequence has a splitting.

Proof. By Proposition 2.4 (3) and (2.3.11), there are no non-trivial homomorphism: $\pi_2(C_{f_2}) \to \pi_1(F(f_3)) \cong \pi_{12}(S^3)$. From the proof of Proposition 2.4 (2), the connecting homomorphism $\partial: \pi_0(\widetilde{C}_{f_2}) \to \pi_{10}(S^3)$ can be identified with $d^1_{8,8}: \pi_0(F(f_2)) \cong E^1_{8,8} \cong E^1_{12,11} \cong \pi_0(\widetilde{C}_{f_2}) \to \pi_{10}(S^3)$ and is trivial by Proposition 2.2. Thus by Lemma 2.1 and the exactness of (1.2), we obtain the following short exact sequences:

$$\xrightarrow{j_{3*}} \pi_2(C_{f_3}) \xrightarrow{p_{3*}} \pi_2(C_{f_2}) \to 0,$$
$$0 \to \pi_1(F(f_3)) \xrightarrow{j_{3*}} \pi_1(C_{f_3}) \xrightarrow{p_{3*}} \pi_1(C_{f_2}) \to 0,$$
$$0 \to \pi_0(F(f_3)) \xrightarrow{j_{3*}} \pi_0(\widetilde{C}_{f_3}) \longrightarrow \pi_0(\widetilde{C}_{f_2}) \to *,$$

where the upper sequence admits a splitting by Lemma 2.1 and the exactness of the lower sequence means that the successive quotient of the middle set by the action of the group in left-hand-side coincides with the set in right-hand-side. Then Propositions 2.2 and 2.4 imply the proposition.

Proposition 2.6. *If λ is odd, then there are following isomorphism:*

(1) $$\pi_0(\widetilde{C}_{f_4}) \cong 0 \text{ or } \mathbf{Z}/2, \text{ if } n \geq 4.$$

For $n \geq 3$, the restriction to HP^2 induces the following injection:

(2) $$p_{4*}: \pi_0(\widetilde{C}_{f_4}) \to \pi_0(\widetilde{C}_{f_3}) \cong \mathbf{Z}/2\times\mathbf{Z}/2,$$
(3) $$Im\{p_{4*}: \pi_1(\widetilde{C}_{f_4}) \to \pi_1(\widetilde{C}_{f_3})\} \cong \mathbf{Z}/2.$$

Proof. By Proposition 2.2, we obtain that $\partial \circ j_{3*} = d^1_{12,13}: \pi_1(F(f_3)) \to \pi_0(F(f_4))$ is an isomorphism and hence $\partial: \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong \pi_1(C_{f_3}) \to \pi_0(F(f_4)) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ is a split surjection with kernel isomorphic to $\mathbf{Z}/2$. Hence by the exactness of (1.2), we obtain the following (split) short exact sequences:

$$0 \to \mathrm{Im}\{p_{4*}: \pi_1(C_{f_4}) \to \pi_1(C_{f_3})\} \to \mathbf{Z}/2 \to 0$$
$$0 \to \pi_0(\widetilde{C}_{f_4}) \to \ker\{\partial: \pi_0(\widetilde{C}_{f_3}) \to \pi_{14}(S^3)\} \to *$$

From the proof of Proposition 2.5, the connecting homomorphism $\partial: \pi_0(\widetilde{C}_{f_3}) \to \pi_{14}(S^3)$ can be identified with some extension of $d^1_{12,12}$. Since $d^1_{12,12}$ is injective and $\pi_0(F(f_3))$ acts on $\pi_0(\widetilde{C}_{f_3})$, at most two elements can exist in the kernel of ∂. This imply the proposition.

§ 3. **Proof of Theorem 0.4.** Let f be a self homotopy equivalence of HP^n and let f_1 be the restriction to HP^1, $2 \leq m$. Then by Fact 2, the mapping degree of f_1 is 1, and hence $\mathscr{E}(HP^n)$ is naturally identified with

$\pi_0(\widetilde{C}_{l_n})$.

Let us assume that $n=2$. Then Proposition 2.4 (1) implies that $\mathscr{E}(\mathbf{H}P^2)\cong\mathbf{Z}/2$ as sets. Also Proposition 2.4 (4) implies that $p_{2*}=r_2$: $\mathscr{E}(\mathbf{H}P^2)\to\mathscr{E}(\mathbf{H}P^1)$ is trivial.

Let us assume that $n=3$. Then Proposition 2.5 (2) implies that $\mathscr{E}(\mathbf{H}P^3)\cong\mathbf{Z}/2\times\mathbf{Z}/2$ as sets. Also Proposition 2.5 (3) implies that $p_{3*}=r_3$: $\mathscr{E}(\mathbf{H}P^3)\to\mathscr{E}(\mathbf{H}P^2)$ is surjective.

Let us assume that $n=4$. Then Proposition 2.6 (1) implies that $\mathscr{E}(\mathbf{H}P^4)\cong *$ or $\mathbf{Z}/2$ as sets. Also Proposition 2.6 (2) implies that $p_{4*}=r_4$: $\mathscr{E}(\mathbf{H}P^4)\to\mathscr{E}(\mathbf{H}P^3)$ is injective.

A group with two elements is isomorphic to $\mathbf{Z}/2$ the cyclic group of order 2. Thus we have shown (0.3) and the part (2) of Theorem 0.4.

Let us recall that there are two possibilities for a group with four elements; to be isomorphic to $\mathbf{Z}/4$ or to $\mathbf{Z}/2\times\mathbf{Z}/2$. So, we are left to determine the (abelian) group extension

$$1\to\mathbf{Z}/2\to\mathscr{E}(\mathbf{H}P^3)\to\mathscr{E}(\mathbf{H}P^2)\to 1. \tag{3.1}$$

Let $\alpha\colon S^{15}\to\mathbf{H}P^3$ be the attaching mapping of the top cell of $\mathbf{H}P^4$. Let us recall that α is a fibration with fibre an H-space S^3 and hence we have the short exact sequence

$$\pi_{15}(S^{15})\xrightarrow{\alpha*}\pi_{15}(\mathbf{H}P^3)\xrightarrow{i_3*}\pi_{15}(\mathbf{H}P^\infty)\ (\cong\pi_{14}(S^3)),$$

with a splitting $\pi_{14}(S^3)\xrightarrow{E}\pi_{15}(S^4)\xrightarrow{i_1*}\pi_{15}(\mathbf{H}P^3)$.

Let $\partial\colon\pi_0(\widetilde{C}_{l_3})\to\pi_{15}(\mathbf{H}P^4)\cong\pi_{14}(S^3)$ be the connecting homomorphism associated to the fibration $\widetilde{C}_{l_4}\to\widetilde{C}_{l_3}$ with fibre $\Omega^{16}(\mathbf{H}P^4)$. Then ∂ is given by composition with α from the right.

Let $h\colon\mathbf{H}P^3\to\mathbf{H}P^3$ be an extension of the generator h_2 of $\mathscr{E}(\mathbf{H}P^2)$. Then the composition $h\circ\alpha$ is in the group $\pi_{15}(\mathbf{H}P^3)$. Hence we have

$$h\circ\alpha\simeq a\alpha+i_1\circ\Sigma\xi,$$

for some integer a and an element $\xi\in\pi_{14}(S^3)$.

Since the restriction of h to $\mathbf{H}P^1$ has degree 1, h is rationally homotopic to the identity mapping. Hence a has to be 1. Then we have that $h\circ h\circ\alpha$ is homotopic to $\alpha+2i_1\circ\Sigma\xi$, since $h\circ i_1\simeq i_1$. Thus we obtain the following equation:

$$\partial(h\circ h)=2\xi. \tag{3.2}$$

On the other hand, $h_2 \circ h_2$ is homotopic to the identity, the homotopy class of $h \circ h$ lies in the image of j_{3*} from $\pi_0(F(I_3)) \cong \pi_{11}(S^3)$. Thus we may suppose that $h \circ h = j_3(\gamma)$ for some $\gamma \in \pi_{11}(S^3)$. Then by (3.2), we have that $d^1_{12,12}(\gamma) = 2\xi$, where d^1 is the first differential: $\pi_0(F(I_3)) \xrightarrow{j_{3*}} \pi_0(\widetilde{C}_{I_3}) \xrightarrow{\partial} \pi_{14}(S^3)$.

As was seen in the proof of Proposition 2.2, $d^1_{12,12}$ is injective with its image isomorphic to $\mathbf{Z}/2$ generated by $\varepsilon_3\nu_{11} + \nu'_3\varepsilon_6$ which is not divisible by 2. Hence 2ξ must be 0, which implies that $\gamma = 0$, since $d^1_{12,12}$ is injective. This implies that $h \circ h$ is homotopic to the identity mapping. Thus the extension (3.1) is trivial.

This completes the proof of Theorem 0.4.

REFERENCES

[1] J. C. BECKER and D. H. GOTTLIEB: Coverings of fibrations, Compositio Math. 26 (1973), 119−128.

[2] B. DWYER and G. MISLIN: On the homotopy types of the components of $Map_*(BS^3, BS^3)$, in "Algebraic Topology, Barcelona 1986" Lecture Notes in Math. 1298, Springer Verlag, Berlin, 1987, pp. 82−89.

[3] G. MISLIN: The homotopy classification of self-maps of infinite quaternionic projective space, Quarterly J. Math. Oxford 38 (1987), 245−257.

[4] I. M. JAMES: Suspension triad of a sphere, Ann. of Math. 63 (1956), 407−429.

[5] I. M. JAMES: Note on cup products, Proc. Amer. Math. Soc. 84 (1957), 374−383.

[6] I. M. JAMES: Spaces associated with Stiefel manifolds, Proc. London Math. Soc. (3) 9 (1959), 115−140.

[7] D. SULLIVAN: Generics in homotopy theory and the Adams conjecture, Ann. of Math. 100 (1974), 1−79.

[8] H. TODA: Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Studies 49, Princeton University Press, Princeton, 1962.

[9] G. W. WHITEHEAD: Elements of Homotopy Theory, GTM 61, Springer Verlag, New York, Heidelberg, Berlin, 1978.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION, CHIBA UNIVERSITY
CHIBA, 260 JAPAN

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