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ON A CONJUGATE ORBIT OF G_2

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Introduction. A study of the topology of the adjoint action orbits of a compact Lie group, was done by R. Bott in [4]. The orbits are homotopy equivalent to CW-complexes with even dimensional cells. The dimension and the number of cells is obtained from the infinitesimal diagram of the group. Since all maximal tori of a compact Lie group are conjugate, all regular orbits are mutually diffeomorphic. For the singular orbits this is not true in general. In this note we exhibit an example of two singular orbits of the exceptional Lie group G_2 with the same cell decomposition that are not homotopy equivalent.

In section two, we project one of these two orbits by the exponential map onto an orbit of the conjugate action and using the property of triality we show that this orbit is a *minimal embedding* of $S^6 \simeq G_2/SU(3)$ in G_2 that generates the homotopy group $\pi_6(G_2) \simeq Z_3$. This fact is interesting when compared to the following elementary theorem of Elie Cartan [7, p.77].

"*If the Lie groups (G, H) form a symetric pair then G/H has a canonical embedding in G as a totally geodesic submanifold.*"

In our example, although $(G_2, SU(3))$ is not a symmetric pair, $S^6 \simeq G_2/SU(3)$ inherits, by submersion from $(G_2$, Killing), a symmetric metric [3].

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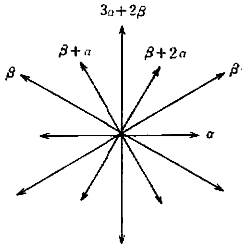
1. Distinction of Adjoint orbits. Let $G_2 = \{A \in SO(8),\ A(xy) = A(x)A(y)$ for all $x, y \in C_a \simeq R^8\}$, where C_a is the algebra of Cayley numbers. The root diagram of the 14 dimensional, compact, simple Lie group G_2 is as follows [11].



We want to look at the Adjoint action $(: xAx^{-1})$ of G_2 on its Lie algebra $\widehat{G_2}$. By Bott's theorem [4], regular orbits are homotopy equivalent to a CW-complex with one cell of dimension zero, one of dimension 12, and two cells in each one of the dimensions 2, 4, 6, 8 and 10. Singular orbits have one cell in dimension 0, 2, 4, 6, 8 and 10. Let H_1, H_2 in $\widehat{G_2}$ be elements corresponding to roots of different norms. If

$$O(H_i) := \{xH_ix^{-1}, x \in G_2\}, \ i = 1, 2$$
$$I(H_i) := \{x \in G_2, xH_i = H_ix\}, \ i = 1, 2$$

we have

$$O(H_i) \simeq G_2/I(H_i), \ i = 1, 2$$

To exhibit the difference between $G_2/I(H_1)$ and $G_2/I(H_2)$ we will use the symmetric pair $(G_2, SO(4))$. An inclusion of $SO(4)$ in G_2 is defined by the following homomorphism of $\mathrm{Spin}(4) \simeq Sp(1) \times Sp(1)$.

$$\theta : Sp(1) \times Sp(1) \to G_2$$
$$(\xi, \eta) \mapsto \theta_{\xi,\eta} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} \eta a \bar{\eta} \\ \xi b \bar{\eta} \end{pmatrix}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ is a representation of a Cayley number by two quaternions with $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} := \begin{pmatrix} ac - \bar{d}b \\ da + b\bar{c} \end{pmatrix}$ defining the Cayley product [14].

Since rank G_2 = rank $SO(4) = 2$, we can get a basis of $\widehat{T}$ where T is a maximal torus in G_2 as follows:

If b_1, b_2 in $\widehat{SO(4)}$ with

$$b_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

then

$$B_1 = \theta_*(b_1) = \begin{array}{c|ccc|cccc|} & i & j & k & e & f & g & h \\ i & 0 & 0 & 0 & & & & \\ j & 0 & 0 & 1 & & 0 & & \\ k & 0 & -1 & 0 & & & & \\ e & & & & 0 & -1 & 0 & 0 \\ f & & & & 1 & 0 & 0 & 0 \\ g & & 0 & & 0 & 0 & 0 & 0 \\ h & & & & 0 & 0 & 0 & 0 \end{array}$$

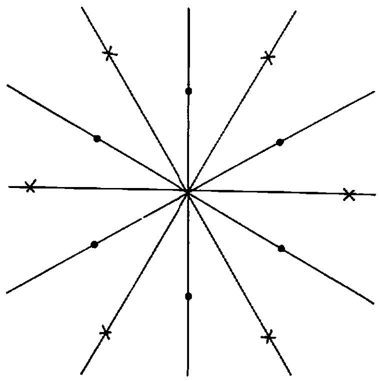
and

$$B_2 = \theta_*(b_2) = \begin{array}{c|ccc|cccc|} & i & j & k & e & f & g & h \\ \hline i & 0 & 0 & 0 & & & & \\ j & 0 & 0 & -1 & & 0 & & \\ k & 0 & 1 & 0 & & & & \\ \hline e & & & & 0 & 0 & 0 & 0 \\ f & & & & 0 & 0 & 0 & 0 \\ g & & 0 & & 0 & 0 & 0 & -1 \\ h & & & & 0 & 0 & 1 & 0 \end{array}$$

If $\langle A, B\rangle = (1/2)$ trace (AB^t) where $A, B \in \widehat{G_2}$, then $\|B_1\|^2 = \|B_2\|^2 = 2$ and $\langle B_1, B_2\rangle = -1$.

Therefore (see the infinitesimal diagram) $H_1 = B_1 + B_2$ and $H_2 = B_2 - B_1$ are elements in distinct singular orbits.

Infinitesimal diagram of G_2



● H_1-orbit × H_2-orbit

Both isotropy subgroups $I(H_1)$ and $I(H_2)$ are isomorphic to $U(2)$: Note first that $\dim I(H_i) = 4$, $i = 1, 2$ (by orbit dimension) and

$$U(2) \simeq \theta(S^1 \times Sp(1)) \subseteq I(H_1),\ U(2) \simeq \theta(Sp(1) \times S^1) \subseteq I(H_2),$$

where

$$S^1 = \{x + iy;\ x, y \in R,\ x^2 + y^2 = 1\} \subseteq Sp(1).$$

The concept of index introduced by Dynkyn in [8] and related to the homotopy group π_3 in [2] will allow us to distinguish between $O(H_1)$ and $O(H_2)$.

Let $\widehat{G_1}$ be a simple subalgebra of a simple algebra $\widehat{G}$. There is in $\widehat{G}$

only one scalar product, up to homothety, such that all automorphisms of $\widehat{G}$ are orthogonal transformations. Fix the scalar product, denoted by $\langle\ ,\ \rangle_{\widehat{G}}$, such that if α is the largest root then $\langle\alpha, \alpha\rangle_{\widehat{G}} = 2$. Define $\langle\ ,\ \rangle_{\widehat{G_1}}$ analogously and observe that there is a $k \in R^+$ such that

$$\langle\ ,\ \rangle_{\widehat{G_1}} = k\langle\ ,\ \rangle_{\widehat{G}}.$$

Now let G_1 be a simple Lie subgroup of the simple Lie group G with corresponding Lie algebras $\widehat{G_1} \subseteq \widehat{G}$.

Theorem ([8], [2 § 8, p.445]). *k is an integer, called the index of $\widehat{G_1}$ in $\widehat{G}$ or of G_1 in G and $\pi_3(G/G_1) = Z_k$.*

To calculate the index of $I(H_1)$ and $I(H_2)$ we note that if

$$Sp(1)\times\{1\} \xrightarrow{\theta_1} G_2$$
$$(\xi, 1) \mapsto \theta_{\xi,1} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ \xi b \end{pmatrix}$$
$$\{1\}\times Sp(1) \xrightarrow{\theta_2} G_2$$
$$(1, \eta) \mapsto \theta_{1,\eta} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} \eta a \bar{\eta} \\ b\bar{\eta} \end{pmatrix}$$

then $\theta_1(Sp(1)\times\{1\}) \subseteq I(H_2)$ and $\theta_2(\{1\}\times Sp(1)) \subseteq I(H_1)$.

As $\theta_*(b_1+b_2) = H_1$ and $\theta_*(b_2-b_1) = H_2$ we have $H_1 \in \theta_*(Sp(1)\widehat{\times}\{1\})$, $H_2 \in \theta_*(\{1\}\widehat{\times} Sp(1))$, $\|H_1\|^2 = 2$ and $\|H_2\|^2 = 6$ (note that $\|(b_1+b_2)\|^2 = \|(b_1-b_2)\|^2 = 2$) which implies that index $I(H_2) = 1$ and index $I(H_1) = 3$, so that $\pi_3(O(H_2)) = \{0\}$ and $\pi_3(O(H_1)) = Z_3$.

Remark. These two singular orbits appear also in [12, p.163–164] without mention of their not being homotopy equivalent.

2. A conjugate orbit. Now we project by the exponential to conjugate orbits of G_2. It is easy to see that $O(\pi H_1)$ projects into a conjugate orbit diffeomorphic to $G_2/SO(4)$. If $A = \theta(1, -1)$ let $\sigma : G_2 \to G_2$, $\sigma(X) = AXA$. Then $\sigma^2 =$ identity and $\tilde{\sigma} : G_2/SO(4) \to G_2$, $\tilde{\sigma}([X]) = X\sigma(X^{-1})$, by the Cartan Theorem we get $G_2/SO(4)$ as a symmetric totally geodesic conjugate orbit.

To investigate the geometry of the other conjugate orbit, let

$$A = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix},\ z = \exp\left(i\frac{2\pi}{3}\right)$$

be in the center of $SU(3)$. We must note that $H_2 - H_1 = -2B_1$, which as a complex matrix has trace zero and therefore belongs to the Cartan subalgebra of $SU(3)$.

Now, by an investigation of the Stiefel diagram of $SU(3)$ [1, p.104], we see that A is obtained as the exponential of a vector which makes an angle of 30° with $H_2 - H_1$. By conjugating maximal tori we get that $O((2\pi/3)H_2)$ projects into the conjugate orbit

$$O(A) \simeq G_2/I(A) \simeq G_2/SU(3) \simeq S^6.$$

Proposition. *The map*

$$\psi : S^6 \simeq G_2/SU(3) \to G_2$$
$$[x] \mapsto xAx^{-1}$$

is a generator of the homotopy group $\pi_6(G_2) \simeq Z_3$.

Proof. We will use the property of triality [6]:

"*for any* $A \in SO(8)$ *there exists* $B, C \in SO(8)$ *such that* $A(xy) = B(x)C(y)$, *for all* $x, y \in C_a \simeq R^8$" (where the products are Cayley multiplication).

Recall that

$\mathrm{Spin}(7) := \{B \in SO(8),\ A(xy) = B(x)C(y)$ for all $x, y \in R^8$ and $A \in SO(7)\}$, is a Lie subgroup of $SO(8)$[W].

The linear transformation $g_\alpha(x) = \alpha x\bar{\alpha}$ with $\alpha, x \in R^8$, $\|\alpha\| = 1$, is in $SO(7)$, because $g_\alpha(1) = 1$. By a Moufang type identity [13] we have

$$g_\alpha(xy) = \alpha(xy)\bar{\alpha} = (\alpha x\alpha^2)((\bar{\alpha})^2 y\bar{\alpha})$$

Therefore f_α with $f_\alpha(x) = \alpha x\alpha^2$ is in Spin(7) and the map

$$f : S^7 \to \mathrm{Spin}(7)$$
$$\alpha \mapsto f_\alpha$$

generates $\pi_7\mathrm{Spin}(7) \simeq Z$, since $\alpha \mapsto g_\alpha$ generates $\pi_7 SO(7) \simeq Z$ [13].

If $\alpha \in S^7$ and $\alpha^3 = 1$ then $\alpha^2 = \bar{\alpha}$ and $f_\alpha(xy) = \alpha(xy)\alpha^2 = (\alpha x\alpha^2)(\bar{\alpha}^2 y\alpha^2) = (\alpha x\alpha^2)(\alpha^4 y\alpha^2) = (\alpha x\alpha^2)(\alpha x\alpha^2) = f_\alpha(x)f_\alpha(y)$; therefore $f_\alpha \in G_2$. Every unitary Cayley number is of the form $\alpha = \cos(t) + J\sin(t)$, $0 \le t \le \pi$, where J is pure imaginary and $\alpha^3 = 1$ if and only if $t = 2\pi/3$ or $t = 0$. Now, f restricted to the parallel $S^6 = \{\cos(t) + J\sin(t),\ t = 2\pi/3\}$ defines a map

$$\begin{aligned} f_1 : S^6 &\to G_2 \\ \alpha &\mapsto f_\alpha \end{aligned}$$

Let $e^7 \subseteq S^7$ be a seven-dimentional cell defined by

$$e^7 = \left\{ \cos(t) + J\sin(t), \ \frac{2\pi}{3} \leq t \leq \pi \right\}.$$

It follows easily that the restriction $\tilde{f}$ of f to e^7 is injective and, as $\partial e^7 = S^6$, $\tilde{f}|_{\partial e^7} = f_1$. By the well known fibration

(1) $$G_2 \cdots \mathrm{Spin}(7) \overset{\pi}{\to} S^7$$

we have $\pi \circ \tilde{f} : e^7 \mapsto S^7$, $\pi \circ \tilde{f} : e^7 - \partial e^7 \to S^7 - \{(1, 0, \ldots, 0)\}$ is bijective, $\pi \circ \tilde{f}(\partial e^7) = (1, 0, \ldots, 0)$ and therefore $\pi \circ \tilde{f} : (e^7, \partial e^7) \to S^7$ is a generator of $\pi_7(S^7) \simeq Z$.

As $\pi \circ f : S^7 \to S^7$ is a map of degree 3 and $\pi_6(\mathrm{Spin}(7)) = \{0\}$, by the exact homotopy sequence of (1), we have

$$\begin{array}{ccccc} \pi_7(\mathrm{Spin}(7)) \overset{\pi}{\to} \pi_7(S^7) & & \xrightarrow{\quad \Delta \quad} & & \pi_6(G_2) \to 0 \\ & \nwarrow \simeq & & \partial \nearrow & \\ & & \pi_7(\mathrm{Spin}(7), G_2) & & \end{array}$$

i) $\pi_*([1]) = [3]$, therefore $\pi_6(G_2) = Z_3$

ii) $\Delta(1) = [1]$, therefore $f_1 : S^6 \to G_2$ is not homotopicaly trivial and so it is a generator of $\pi_6(G_2)$.

Remark. i) was proved by Mimura in [10] using the fact that $\pi_6(S^3) = Z_{12}$. The above approach furnishes also an elementary proof that $\pi_6(SU(4)) \simeq Z_6$ and that $\pi_6(S^3) = Z_{12}$, using the exact homotopy ladder of the principal fibrations over S^7 with total spaces Spin(5), Spin(6) and Spin(7) ([15]).

It remains to prove that the image of f_1 is the conjugate orbit ψ of Λ:

Observe that if $t = 2\pi/3$ and $\alpha = \cos(t) + i\sin(t)$ then $f_\alpha(i) = i$ and therefore $f_\alpha \in SU(3) \subseteq G_2$.

We claim that $f_\alpha = \Lambda$ and for this we must show that $f_\alpha A = Af_\alpha$ for all A in $SU(3)$: $Af_\alpha(x) = A(\alpha x \alpha^2) = A(\alpha)A(x)A^2(\alpha) = \alpha A(x)\alpha^2 = f_\alpha(A(x))$, since $A(\alpha) = \alpha$ by the fact that $A(i) = i$.

Now, we have that for B in G_2 $\psi([B])(x) = B\Lambda B^{-1}(x) = Bf_\alpha B^{-1}(x) = B(\alpha B^{-1}(x)\alpha^2) = B(\alpha)xB^2(\alpha) = f_{B(\alpha)}(x)$. As $B(S^6 = \{\cos(t) + J\sin(t),$

$t = 2\pi i/3\}) = S^6$, we have $\psi(S^6) \subseteq f_1(S^6)$. As f_1 and ψ are embeddings the two sets are equal.

By the Cartan polyhedron of G_2 we have that $O(\Lambda)$ is an isolated orbit and therefore a minimal submanifold [9].

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