

LINEARLY COMPACT DUAL-BIMODULES

Dedicated to Professor Kentaro Murata on his 70th birthday

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Let R and S be rings with identity and ${}_R Q_S$ an (R, S) -bimodule. In the previous paper [3], it is shown that if Q_S is quasi-injective and the canonical ring homomorphism $\lambda: R \rightarrow \text{End}(Q_S)$ is surjective, then the pair of functors

$$H' = \text{Hom}_R(-, Q) : {}_R \underline{M} \rightarrow \underline{N}_S \text{ and } H'' = \text{Hom}_S(-, Q) : \underline{N}_S \rightarrow {}_R \underline{M}$$

defines a duality between ${}_R \underline{M}$ and \underline{N}_S , where ${}_R \underline{M}$ is the full subcategory of $R\text{-mod}$ of finitely generated Q -torsionless R -modules and \underline{N}_S is the full subcategory of $\text{mod-}S$ whose objects are all the S -modules N such that there exists an exact sequence of the form $0 \rightarrow N \rightarrow Q^n \rightarrow Q'$ for some $n > 0$ and some set I .

In this note, we shall give, in the first section, some characterizations of self-cogenerators and then point out that the linearly compactness of a ring is very closed to the existence of some kind of left dual-bimodules. Characterizing these left dual-bimodules, in the second section, we shall show that, for a left dual-bimodule ${}_R Q_S$ with ${}_R Q$ finitely generated, Q_S quasi-injective and λ surjective, Q_S is linearly compact if and only if the duality mentioned above can be extended to a duality between ${}_R \overline{FG}$ and \overline{N}_S (see below for the definition).

1. An S -module Q_S will be called a *self-cogenerator* provided that every right S -module isomorphic to a submodule of a factor module of Q^n , $n = 1, 2, \dots$, is Q -torsionless [7, Definition 3.1]. Trivially each cogenerator in $\text{mod-}S$ is a self-cogenerator. First, we shall give some characterizations of self-cogenerators. As is easily seen, we have

Lemma 1. *Let Q_S be an S -module. Suppose that*

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is an exact sequence of right S -modules such that N' and N'' are Q -torsionless and Q is N -injective. Then N is Q -torsionless.

Lemma 2. *Let Q_S be a quasi-injective S -module. Suppose that every*

factor module of Q is Q -torsionless. Then every factor module of Q^n is also Q -torsionless for $n = 1, 2, \dots$.

Proof. We may show the case where $n = 2$. Let Q' be any submodule of Q^2 and let $p: Q^2 \rightarrow Q$ be the canonical projection. Then the induced homomorphism $\bar{p}: Q^2/Q' \rightarrow Q/p(Q')$ is an epimorphism with $\text{Ker } \bar{p}$ Q -torsionless. By assumption $Q/p(Q')$ is Q -torsionless and hence Q^2/Q' is also Q -torsionless by Lemma 1.

An (R, S) -bimodule Q will be called a *left dual-bimodule* provided that $\ell_R r_Q(A) = A$ for every left ideal A of R and $r_Q \ell_R(Q') = Q'$ for every S -submodule Q' of Q (see [3]). A ring that has the double annihilator property ([1, Exercise 24.11]) will be called a *dual ring*. Hence a dual ring R is a left dual-bimodule regarded as an (R, R) -bimodule. It is also a right dual-bimodule by defining symmetrically. In [3, Lemma 1.3], it is shown that if ${}_R Q_S$ is a left dual-bimodule, then every factor module of Q_S is Q -torsionless. Hence we have

Corollary 3. *Let ${}_R Q_S$ be a left dual-bimodule with Q_S quasi-injective. Then Q_S is a self-cogenerator.*

Proposition 4. *Let ${}_R Q_S$ be an (R, S) -bimodule with Q_S quasi-injective and λ surjective. Then the following conditions are equivalent:*

- (1) Q_S is a self-cogenerator.
- (2) Every factor module of Q^n is Q -torsionless for $n = 1, 2, \dots$.
- (3) Every factor module of Q_S is Q -torsionless.
- (4) $r_Q \ell_R(Q') = Q'$ for every submodule Q' of Q_S .
- (5) $\underline{N}_S = \{N_S \mid 0 \rightarrow N \rightarrow Q^n \text{ is exact for some } n > 0\}$.
- (6) Every submodule of Q^n is Q -reflexive for $n = 1, 2, \dots$.

Proof. (1) \Leftrightarrow (2) and (2) \Leftrightarrow (3) are evident. The equivalence of (3) and (4) follows from [3, Lemma 1.3] and (3) \Leftrightarrow (5) follows from Lemma 2.

(5) \Leftrightarrow (6). Let N_S be a submodule of Q^n . Then $N \in \underline{N}_S$ and $0 \rightarrow N \rightarrow Q^m \rightarrow Q'$ is exact for some $m > 0$ and I . Since Q_S is Q -injective and Q -reflexive, Q is Q^m -injective and Q^m is Q -reflexive. Hence by [3, Lemma 3.1], N must be Q -reflexive.

(6) \Leftrightarrow (1) also follows from [3, Lemma 3.1].

Remarks. (1) The equivalence of (1) and (3) of Proposition 4 has already shown in [5, Lemma 1.1].

(2) In case Q_S is a finitely cogenerated cogenerator, then a right S -module N is finitely cogenerated if and only if there is an $n > 0$ such that $0 \rightarrow N \rightarrow Q^n$ is exact by [1, Exercise 10.3]. For example, each dual-bimodule ${}_R Q_S$ with Q_S injective and λ surjective is a finitely cogenerated cogenerator as an S -module by [3, Proposition 1.8 and Lemma 3.5].

Corollary 5. *For a ring R with R_R injective, the following conditions are equivalent:*

- (1) R_R is a self-cogenerator.
- (2) Every finitely generated right R -module is torsionless.
- (3) Every cyclic right R -module is torsionless.
- (3') Every simple right R -module is torsionless.
- (3'') R_R is a cogenerator.
- (4) $r_R \ell_R(A) = A$ for every right ideal A of R .
- (5) $\underline{N}_R = \{N_R \mid 0 \rightarrow N \rightarrow R^n \text{ is exact for some } n > 0\}$.
- (5') $\underline{N}_R = \{N_R \mid N_R \text{ is finitely cogenerated}\}$.
- (6) Every submodule of R_R^n is reflexive for $n = 1, 2, \dots$.
- (6') Every finitely cogenerated right R -module is reflexive.

Proof. (3) \Leftrightarrow (3') and (3'') \Leftrightarrow (3) are evident and (3') \Leftrightarrow (3'') follows from [1, Proposition 18.15].

Assume (5). Then since (3) and (5) are equivalent, R_R is an injective cogenerator and hence is a finitely cogenerated cogenerator by [6, Satz 3]. Assume (5'). Then since R is in \underline{N}_R , R_R is finitely cogenerated and injective. Hence it is a finitely cogenerated cogenerator again by [6, Satz 3]. Therefore, the equivalence (5) and (5') follows from Remarks (2).

(6) \Leftrightarrow (6') \Leftrightarrow (3') are evident. Hence (6) and (6') are equivalent.

A ring R is a *cogenerator ring* in case both ${}_R R$ and R_R are cogenerators [1, Exercise 24.10].

Corollary 6. *For a ring R with R_R injective, the following conditions are equivalent:*

- (1) R is a dual ring.
- (2) ${}_R R$ and R_R are self-cogenerators.
- (3) R is a cogenerator ring.

Proof. (1) \Leftrightarrow (3). As we shall show in Corollary 11, if R is a dual ring, then R_R injective is equivalent to ${}_R R$ being injective. Hence, from Corollaries 3 and 5 (1) \Leftrightarrow (3) follows.

(3) \Leftrightarrow (2) is evident.

(2) \Leftrightarrow (1). To prove (1) \Leftrightarrow (4) of Proposition 4, it is sufficient to assume that λ is surjective. Hence, in Corollary 5 (1) \Leftrightarrow (4) is always valid. Thus (2) implies (1).

Let N_S be an S -module, $(x_i)_I$ an indexed set of elements of N and $(N_i)_I$ an indexed set of submodules of N . Then the set of congruences $\{x \equiv x_i \pmod{N_i}\}$ is said to be *solvable* (*finitely solvable*), if there is a y in N (a y_F in N for each finite subset F of I) such that $y - x_i \in N_i$ for each i in I ($y_F - x_i \in N_i$ for each i in F).

If every finitely solvable set of congruences in N is solvable, then N will be called *linearly compact* ([7, Definition 2.1]). Using this notion we can characterize left dual-bimodules.

Proposition 7. *Let ${}_R Q_S$ be an (R, S) -bimodule with Q_S linearly compact quasi-injective and λ an isomorphism. Then the following conditions are equivalent:*

- (1) Q is a left dual-bimodule.
- (2) Q_S is a self-cogenerator and has essential socle.

Moreover, if this is the case, ${}_R Q$ is injective and ${}_R R$ is linearly compact.

Proof. (1) \Leftrightarrow (2) follows from Corollary 3 and [3, Proposition 1.8].

(2) \Leftrightarrow (1). By [7, Lemma 3.7] every cyclic left R -module is Q -torsionless. Hence by Proposition 4 and [3, Lemma 1.2] Q is a left dual-bimodule.

The last part follows from (2) by [7, Lemmas 3.5 and 3.7].

Corollary 8. *For a ring R with R_R linearly compact and injective, the following conditions are equivalent:*

- (1) R is a dual ring.
- (2) R_R is a self-cogenerator and has essential socle.
- (3) ${}_R R$ is a self-cogenerator and has essential socle.

Proof. As is remarked in the proof of Proposition 7, (2) implies that ${}_R R$ is linearly compact and injective. Hence, again by Proposition 7, (3) is equivalent to R being a dual ring.

As a consequence of Proposition 7 and [7, Theorem 3.10], we have

Theorem 9. *A ring R is left linearly compact if and only if there exists a left dual-bimodule ${}_R Q_S$ such that Q_S is linearly compact quasi-injective and λ*

is surjective.

2. A subcategory of the module category will be called *finitely closed* if it is closed under submodules, factor modules and finite direct sums [4, p. 465]. Let ${}_R Q_S$ be an (R, S) -bimodule. Following [7], consider the full subcategory of R -mod consisting of all modules isomorphic to factor modules of submodules of R^n for $n = 1, 2, \dots$. This is the full subcategory consisting of all modules isomorphic to submodules of factor modules of R^n for $n = 1, 2, \dots$ and hence is equal to

$$\{ {}_R M \mid 0 \rightarrow M \rightarrow M' \text{ is exact for some } M' \in {}_R FG \}.$$

As is easily seen, this is the smallest one of the finitely closed subcategory containing either R or ${}_R FG$. We shall denote this by ${}_R \overline{FG}$, where ${}_R FG$ means the full subcategory of finitely generated left R -modules.

Similarly, the full subcategory of $\text{mod-}S$ consisting of all modules isomorphic to factor modules of submodules of Q^n for $n = 1, 2, \dots$ coincides with one consisting of all modules isomorphic to submodules of factor modules of Q^n for $n = 1, 2, \dots$. This is the smallest one of the finitely closed subcategory containing either Q or the class of S -modules

$$\{ N_S \mid 0 \rightarrow N \rightarrow Q^n \text{ is exact for some } n > 0 \}.$$

By Proposition 4 this is the smallest one of the finitely closed subcategory containing \underline{N}_S in case Q_S is a quasi-injective self-cogenerator and λ is surjective. Hence we shall denote this by \overline{N}_S . Furthermore, \overline{N}_S also coincides with

$$\{ N_S \mid N' \rightarrow N \rightarrow 0 \text{ is exact for some } N' \in \underline{N}_S \}.$$

We are now ready to characterize those left dual-bimodules mentioned in Theorem 9 by means of a duality.

Theorem 10. *Let ${}_R Q_S$ be a left dual-bimodule with ${}_R Q$ finitely generated and λ surjective. Then the following conditions are equivalent:*

- (1) Q_S is a linearly compact quasi-injective module.
- (2) The pair (H', H'') defines a duality between ${}_R \overline{FG}$ and \overline{N}_S .
- (3) ${}_R Q$ is an injective cogenerator.

Proof. (1) \Leftrightarrow (2) follows from Proposition 7 and [7, Theorem 3.8].

(2) \Leftrightarrow (3). Assume (2). Then since ${}_R Q \in {}_R \overline{FG}$, we can apply [1, Exercise 20.5] to show that, for each $M \in {}_R \overline{FG}$, σ_M is an epimorphism by a

similar way as in [1, Theorem 23.5]. In particular, every cyclic R -module is Q -reflexive by [3, Lemma 1.2] and thus Q_S is quasi-injective by [3, Theorem 3.2]. Furthermore, if $N \in \bar{N}_S$, then $H''(N)$ is in ${}_R\overline{FG}$ and hence $\sigma_{H''(N)}$ is an epimorphism, which shows that $N(\cong H'H''(N))$ is Q -reflexive [1, Proposition 20.14]. In particular, every factor module of Q_S is Q -reflexive. Hence, by [2, Theorem 10], ${}_RQ$ is an injective cogenerator.

(3) \Leftrightarrow (1) follows from [2, Theorem 10] and [7, Theorem 3.6].

The following corollary follows from Theorem 10 and [3, Lemma 3.5].

Corollary 11. *For a dual ring R , the following conditions are equivalent:*

- (1) R_R is linearly compact and is injective.
- (2) ${}_R R$ is linearly compact and is injective.
- (3) ${}_R R_R$ defines a duality between ${}_R\overline{FG}$ and \overline{FG}_R .
- (4) ${}_R R$ is injective.
- (5) R_R is injective.

Finally, we shall remark that a cogenerator ring is also a dual ring satisfying the equivalent condition of Corollary 11 [1, Exercise 24.12].

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(Received October 15, 1991)