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FIRST TWO TERMS IN A MINIMAL INJECTIVE RESOLUTION OF A NOETHER RING

Dedicated to Professor Manabu Harada on his sixtieth birthday

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Let R be a ring and fix $0 \to R \to E_0 \to E_1 \to \ldots$ a minimal injective resolution of ${}_RR$. R is said to be a ring with left dominant dimension $\geqq n$ if the first n terms $E_0, E_1, \ldots, E_{n-1}$ are flat. Hoshino [1] showed that the notion of dominant dimension is right-left symmetric for a (left and right) noether ring. Throughout the present paper we deal with noether rings. So the dominant dimension of a noether ring R is denoted by *dom. dim* R. We should remark that a noether ring R with *dom. dim* $R \geqq 1$ is nothing but a *QF*-3 ring in the sense of Morita [8] and Sato [9], and that a noether ring with *dom. dim* $R \geqq 2$ is artinian. (See [3, Proposition 7]).

The results above and their proofs suggest us that the first two terms E_0 and E_1 are important in studying a noether ring R with *dom. dim* $R \geqq 1$. To state more precisely, we introduce here a slightly stronger notion than that of a cogenerator. A module W is called a finitely embedding cogenerator if any finitely generated module is embedded into a finite product of copies of W, or equivalently, into a direct sum of copies of W. The notion of finitely embedding cogenerator coincides with that of a cogenerator over an artin ring, but not in general.

In the previous paper [3, Theorem 10] we obtained the following result.

Theorem A. *Let R be a noether ring with dom. dim $R \geqq 1$. Then the following conditions are equivalent.*

(1) *$E_0 \oplus E_1$ is a finitely embedding cogenerator.*

(2) *Every finitely generated uniform left R-module, which is torsion, has a nonzero submodule V for which there exists an exact sequence*

$$0 \to L \to F \to V \to 0$$

with F finitely generated free and L reflexive.

(3) *For every finitely generated uniform left R-module, which is torsion, there exists a nonzero submodule V for which any finitely generated projective presentation : $F \to V$ has a kernel, which is a reflexive module. As for torsion theory, we adopt here Lambek torsion theory.*

We did not give a proof in [3] that the condition (3) above is equivalent to the other conditions (1) and (2). But it is immediate by Schanuel's lemma.

Now we are interested in the condition (2) above. More generally, we shall consider the property that, for each element U in a certain class $\mathscr{C}$ of modules, there exists a free presentation $F \to U$ whose kernel is reflexive. Let $\mathscr{C}$ be the class of all simple modules. Then we shall get the following, which is a generalization of [3, Corollary 11].

Theorem B. *Let R be a noether ring with dom.dim $R \geqq 1$. Then the following conditions are equivalent.*

(1) *$E_0 \oplus E_1$ is a cogenerator.*

(2) *Every maximal left ideal is reflexive.*

Recall the definition of right n-Gorenstein ring. A (left and right) noether ring R is said to be a right n-Gorenstein ring if its right self-injective dimension is at most n, that is, inj.dim $R_R \leqq n$. A right n-Gorenstein ring which is also left n-Gorenstein is called an n-Gorenstein ring in the present paper.

Now let $\mathscr{C}$ be the class of all finitely generated left R-modules. Then we shall get another characterization of right 1-Gorenstein ring for a noether ring R not necessarily having the condition *dom.dim* $R \geqq 1$ (Proposition 3).

Next we shall characterize any indecomposable summand of $E_0 \oplus E_1$ as follows.

Theorem C. *Let R be a noether ring with dom.dim $R \geqq 1$. Then the following conditions for an injective indecomposable left module U are equivalent.*

(1) *U is isomorphic to an indecomposable summand of $E_0 \oplus E_1$.*

(2) *There exists a reflexive (irreducible) left ideal I of R with $U \cong E(R/I)$, the injective hull of R/I.*

Further if R is artinian, then we can take I as a maximal left ideal.

This is a non-commutative version of the result by Matlis [7, Theorem 37].

Many of our results are closely related to reflexive modules. The key of our proofs is to alter a characterization of reflexive modules in Masaike [6, Corollary 3.2] into our setting.

We shall prove Theorem B in § 1, and Theorem C in § 2. Right 1-Gotrenstein rings are characterized in terms of reflexive modules. In the final section § 3, we shall investigate reflexive modules and right 1-Gorenstein rings from the point of view of the properties in Theorem A, and we shall obtain another characterization of right 1-Gorensteon rings.

Throughout the present paper, R is a noether ring and we denote a fixed minimal injective resolution of ${}_RR$ by $0 \to R \to E_0 \to E_1 \to \ldots$. As for torsion theory, we adopt Lambek torsion theory and denote the corresponding torsion radical by t.

1. Proof of Theorem B.

Proposition 1. *Let R be a noether ring and M a dense submodule of a finitely generated free left module F. If M is reflexive, then F/M is embedded into a direct sum of copies of E_0/R. If every dense maximal left ideal is reflexive, then $E_0 \oplus E_1$ is a cogenerator.*

Proof. In the sequel we denote the R-dual $Hom_R(B, R)$ of left or right R-module B by B^*. Let M be a finitely generated reflexive left R-module. Since R is a noether ring, M^* is a finitely generated right R-module. Consider an exact sequence of right R-modules : $0 \to L \to G \to M^* \to 0$ with G finitely generated free. Then we have the following exact sequence of left R-modules : $0 \to M^{**} \to G^* \to L^*$. Let $P = G^*$ and $N = Coker(M^{**} \to P)$. Then P is finitely generated free and N is torsionless because it is a submodule of L^*. Since M is reflexive, we have an exact sequence of left R-modules :

$$0 \to M \xrightarrow{\lambda} P \to N \to 0$$

such that P is finitely generated free and N is torsionless. Take the push out of $\lambda : M \to P$ and the inclusion map : $M \to F$. Then we have the following commutative diagram with exact rows and exact columns :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M & \to & P & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & F & \to & X & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F/M & = & F/M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since F and N are torsionfree, so is X. By injectivity, we have the following commutative diagram with exact rows :

$$\begin{array}{ccccccccc} 0 & \to & P & \to & X & \to & F/M & \to & 0 \\ & & \| & & \alpha\downarrow & & \beta\downarrow & & \\ 0 & \to & P & \to & E(P) & \to & E(P)/P & \to & 0 \end{array}$$

Since F/M is torsion by assumption, the map : $P \to X$ in the above is an essential monomorphism. Thus α is a monomorphism and so is β. This shows the first part of our statement. For, $P \cong R^{(m)}$ for some $m > 0$ and $E(P)/P \cong (E_0/R)^{(m)}$.

For the latter part, let S be a simple module. Then S is torsionfree or torsion. If S is torsionfree, then S is embedded into E_0. If S is torsion, then there exists a dense maximal left ideal I such that

$$0 \to I \to R \to S \to 0$$

is exact. Since I is reflexive by our assumption, it follows from the first half of our statement that $S \cong R/I$ is embedded into E_0/R and so into E_1. This shows that $E_0 \oplus E_1$ is a cogenerator.

Proposition 2. *Let R be a noether ring with dom.dim $R \geqq 1$. Assume that M is a finitely generated left R-module for which there exists an exact sequence : $0 \to M \to F \to (E_0/R)^{(m)}$, with F finitely generated free. Then M is reflexive.*

Proof. Let $U = Coker(M \to F)$. Take a pull back of the inclusion : $U \to (E_0/R)^{(m)}$ and the map : $E_0^{(m)} \to (E_0/R)^{(m)}$ induced by the canonical projection : $E_0 \to (E_0/R)$. Then we have the following commutative diagram with exact rows and exact columns :

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & R^{(m)} & \to & X & \to & U & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & R^{(m)} & \to & E_0^{(m)} & \to & (E_0/R)^{(m)} & \to & 0 \end{array}$$

The condition *dom.dim* $R \geqq 1$ implies that $E_0^{(m)}$ is flat. Since X is finitely generated, it follows from Lazard [5, Theoreme 1.2] that X is torsionless. By taking a pull back of two maps $X \to U$ and $F \to U$, we have an exact sequence : $0 \to M \to R^{(m)} \oplus F \to X \to 0$. Then M is reflexive by Masaike [6, Corollary 3.2].

Proof of Theorem B. The implication $(2) \Rightarrow (1)$ is completed in Proposition 1. It remains to show the implication $(1) \Rightarrow (2)$. Let I be a maximal left ideal of R and let $S = R/I$. By the assumption (1), S is embedded into E_0 or into E_1. Assume that S is embedded into E_0. Then S is torsionless and hence I is reflexive by Masaike [6, Corollary 3.2]. On the other hand, assume that S is embedded into E_1. Then S is embedded into E_0/R because S is simple and E_0/R is essential in E_1. Our statement follows from Proposition 2.

2. Proof of Theorem C.

The implication (1) ⇒ (2) : Since U is uniform, U appears as a direct summand of either E_0 or E_1. Assume that U is embedded into E_0. As is well-known, there exists an irreducible left ideal I such that $U \cong E(R/I)$. Since E_0 is flat, it follows from Lazard [5, Theoreme 1.2] that R/I is torsionless. Thus I is reflexive by Masaike [6, Corollary 3.2]. Next assume that U is embedded into E_1. Then $U \cap (E_0/R)$ is nonzero. Take a nonzero element u in $U \cap (E_0/R)$. Then $Ru \cong R/I$ for some left ideal I of R. By Proposition 2, I is reflexive and obviously we see $U \cong E(R/I)$.

The implication (2) ⇒ (1) : Let I be an irreducible left ideal which is reflexive. Let $V = R/I$. If V is torsionfree, then V can be embedded into a product Q of copies of E_0. Since E_0 is flat and R is a noether ring, Q is also flat. So V is torsionless by Lazard [5]. In other words, V is embedded into a finite direct sum of copies of ${}_RR$. Since V is uniform, V is emdedded into ${}_RR$. This implies $U \cong E(V) \subsetneq E_0$.

Thus we can assume that V is not torsionfree. Let $W = t(V) \neq 0$. Since I is reflexive, we see similarly as in the proof of Proposition 1 that there exists an exact sequence : $0 \to I \xrightarrow{\lambda} P \to N \to 0$ with P finitely generated free and N torsionless. Take a push out of $\lambda\colon I \to P$ and the inclusion : $I \to R$. Then we have the following commutative diagram with exact rows and exact columns :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & \lambda & \downarrow & & & & \\
0 & \to & I & \to & P & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & R & \to & X & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow \mu & & & & \\
 & & V & = & V & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since N is finitely generated torsionfree, so is X. Take a pull back of μ and the inclusion map : $W \to V$. Then we have the following commutative diagram with exact rows and exact columns :

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & P & \to & Y & \to & W & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & P & \to & X & \to & V & \to & 0 \end{array}$$

Since Y is torsionfree and W is torsion, P is essential in Y. Thus we have the following commutative diagram with exact rows and exact columns :

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & P & \to & Y & \to & W & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & P & \to & E(P) & \to & E(P)/P & \to & 0 \end{array}$$

Therefore we have the following chain of monomorphisms :

$$W \subsetneq E(P)/P \subsetneq (E_0/R)^{(r)} \subsetneq E_1^{(r)}$$

where $P \cong R^{(r)}$ for some $r > 0$. Since W is uniform, we have a monomorphism of W into E_1 and an extended monomorphism of $U = E(W)$ into E_1. This completes the proof.

3. Reflexive modules and right 1-Gorenstein rings.

Let R be a right n-Gorenstein ring. Then Iwanaga showed in [2] that $E_0 \oplus \ldots \oplus E_n$ is a cogenerator without the assumption *dom.dim* $R \geqq 1$. But it is not known yet whether $E_0 \oplus \ldots \oplus E_n$ is a finitely embedding cogenerator or not even if $n = 1$. If R is a right 1-Gorenstein ring with *dom.dim* $R \geqq 1$, then $E_0 \oplus E_1$ is a finitely embedding cogenerator by Theorem A.

On the other hand, the condition (2) in Theorem A is satisfied by a right 1-Gorenstein ring R without the assumption *dom.dim* $R \geqq 1$. Furthermore there exists an example of a noether ring with *dom.dim* $R \geqq 1$, which satisfies the conditions in Theorem A, but is not right 1-Gorenstein (See [3]). Concerning these problems, we show the following. (Compare the condition (2) below with that of Theorem A).

Proposition 3. *The following conditions are equivalent for a noether ring R.*

(1) *R is right 1-Gorenstein.*

(2) *For any finitely generated left R-module X, there exists an exact sequence*

$$0 \to L \to F \to X \to 0$$

with F finitely generated free and L reflexive.

Proof. As is well-known, Jans [4] showed that the condition (1) is equivalent to that any finitely generated torsionless left R-module is reflexive. Thus the implication (1) $\Rightarrow$ (2) is obvious. In order to prove the implication (2) $\Rightarrow$ (1), it is sufficient to prove that every submodule K of a finitely generated free left R-module G is reflexive. Let $X = G/K$. By assumption there exists an exact sequence $0 \to L \to F \to X \to 0$ with F finitely generated free and L reflexive. By Schanuel's lemma, we have $L \oplus G \cong K \oplus F$. Since L, G and F are all reflexive, so is K.

In the sequel we let R a noether ring with *dom.dim* $R \geqq 1$. In view of our proof of Theorem B, the module E_0/R plays an important role. More precisely, we consider the property that $E_0 \oplus (E_0/R)$ cogenerates a certain module B finitely. The property then makes a module related to B reflexive. Also it is connected with 1-Gorenstein rings.

We mention here the following, as an immediate consequence of Propositions 1 and 2.

Proposition 4. *Let R be a noether ring with dom.dim $R \geqq 1$ and M a dense submodule of a finitely generated free left R-module F. Then M is reflexive if and only if F/M is embedded into a direct sum of copies of E_0/R.*

Proposition 5. *Let R be a non-singular noether ring with dom.dim $R \geqq$ 1. Then the following conditions are equivalent.*

(1) *R is right 1-Gorenstein.*

(2) *Every finitely generated torsion left R-module is embedded into a direct sum of copies of E_0/R.*

Proof. (1) $\Rightarrow$ (2) : Let X be a finitely generated torsion left R-module. Then we have an exact sequence $0 \to M \to F \to X \to 0$ such that F is finitely generated free. Since R is right 1-Gorenstein, it follows from Jans [4] that

M is reflexive. Since X is torsion, M is dense in F. By Proposition 4, we see that $X \cong F/M$ is embedded into a direct sum of copies of E_0/R.

$(2) \Rightarrow (1)$: It is sufficient to show that every finitely generated torsionless left R-module L is reflexive. L is embedded into a finitely generated free module F, and thus let K be a complement of L in F and $M = L \oplus K$. Since R is a non-singular ring, M is dense in F. Now it follows from the assumption (2) and Proposition 4 that M is reflexive. Therefore L is also reflexive.

Next we mention the reflexivity of irreducible left ideals.

Proposition 6. *Let R be a noether ring with dom.dim $R \geqq 1$ and I an irreducible left ideal of R. If R/I is embedded into a direct sum of copies of $E_0 \oplus (E_0/R)$, then I is reflexive.*

Proof. We have essentially shown in Proof of Theorem C. Let $U = R/I$. Then U is uniform. By the assumption, U is embedded into either E_0 or E_0/R. We can follow the proof of Theorem C to complete the proof of Proposition 6.

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