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GLOBAL DIMENSION AND A QUESTION OF ARMENDARIZ

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M. Auslander has shown that the global dimension of a ring Λ is bounded by the projective dimension of Λ/I for left ideals I of Λ[AU]. For noetherian rings satisfying a polynomial identity, Rainwater [R] restricted I to being a two sided maximal ideal. In this note we consider a somewhat dual statement. More precisely :

The aim is to give a positive answer to the following question of Armendariz in case of semiprimary rings and classical orders:

(1) Let Λ be a noetherian ring with a polynomial identity. If the injective dimension of all maximal two sided ideals is bounded by n, does n then also bound the global dimension of Λ?

We shall prove the

(2) **Proposition.** (i) *If Λ is semiprimary, then the question* (1) *has a positive answer.*

(ii) *Let R be a Dedekind domain with the field of fractions K and Λ an R-order in a finite dimensional semisimple K-algebra A ; i.e., Λ is finitely generated over R as module. Then the question* (1) *has a positive answer.*

The **proof** is done in several steps :

Step 1 (for(ii)): Reduction to the case where R is complete. Since Λ is an R-order, we note that a two sided Λ-ideal $\mathfrak{M}$ is maximal if and only if all its completions $\mathfrak{M}_p$ at the maximal ideals p of R coincide with Λ_p, except for one, p_0, where $\mathfrak{M}_{p_0}$ is a maximal two sided ideal of Λ_{p_0}. Moreover, each such set of local data determines a unique maximal two sided ideal of Λ. In addition, if M and N are Λ-lattices ; i. e., left Λ-modules, which are finitely generated and projective over R, then

$$\mathrm{Ext}^n_\Lambda(M, N) = \bigoplus_{p \in \max(R)} \mathrm{Ext}^n_{\Lambda_p}(M_p, N_p)$$

where the subscript denotes the completion. Since every finitely generated module has a resolution by Λ-lattices, this formula also holds for finitely generated Λ-modules. Thus it is enough to prove the proposition in case R is complete. The importance of this is that in the complete situation we have the Krull-Schmidt theorem available.

Hence we assume from now on that R is complete and Λ is basic. In the semiprimary case the Krull-Schmidt theorem always holds, and projective covers exist.

Step 2: Assume that Λ is not local. Let $\Lambda = \oplus_{i=1}^{m} P_i$, where $\{P_i\}_{1\le i\le m}$ are the indecomposable projective Λ-modules with $J_i = \mathrm{rad}(P_i)$; note that $m > 1$. The maximal two sided ideals of Λ are then $\mathfrak{M}_i = \bigoplus_{j\neq i} P_j \oplus J_i$. According to the hypothesis,

$$0 = \mathrm{Ext}_{\Lambda}^{n+1}(-, \mathfrak{M}_i) = \bigoplus_{j\neq i} \mathrm{Ext}_{\Lambda}^{n+1}(-, P_j) \oplus \mathrm{Ext}_{\Lambda}^{n+1}(-, J_i).$$

Since $m > 1$, we conclude

$$\mathrm{Ext}_{\Lambda}^{n+1}(-, P_j) = 0 = \mathrm{Ext}_{\Lambda}^{n+1}(-, J_i),\ 1 \le i \le m.$$

Recall that given a short exact sequence of Λ-modules

$$0 \to X' \to X \to X'' \to 0, \tag{3}$$

we get functorially exact sequences

$$\begin{aligned} &\mathrm{Ext}_{\Lambda}^{n}(-, X') \to \mathrm{Ext}_{\Lambda}^{n}(-, X) \to \mathrm{Ext}_{\Lambda}^{n}(-, X'') \to \\ &\mathrm{Ext}_{\Lambda}^{n+1}(-, X') \to \mathrm{Ext}_{\Lambda}^{n+1}(-, X) \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\mathrm{Ext}_{\Lambda}^{n}(X'', -) \to \mathrm{Ext}_{\Lambda}^{n}(X, -) \to \mathrm{Ext}_{\Lambda}^{n}(X', -) \to \\ &\mathrm{Ext}_{\Lambda}^{n+1}(X'', -) \to \mathrm{Ext}_{\Lambda}^{n+1}(X, -). \end{aligned} \tag{5}$$

Applying this to the exact sequence

$$0 \to J_i \to P_i \to S_i \to 0,$$

where S_i is the associated simple module, we conclude

$$\mathrm{Ext}_{\Lambda}^{n+1}(-, S_i) = 0$$

and by induction — using (4) — we get

$$\mathrm{Ext}_{\Lambda}^{n+1}(-, L) = 0$$

for every finitely generated artinian Λ-module L. (Since $\Lambda/\mathfrak{M}$ is finitely generated artinian for every maximal two sided ideal $\mathfrak{M}$, we could quote a result of Rainwater [R] to conclude that the global dimension of Λ is bounded by n; however, the arguments below give a very short direct proof.) Now let M be a Λ-lattice, and let π be a parameter of R. Then the exact sequence

$$0 \to M \xrightarrow{\cdot\pi} M \to M/\pi\cdot M \to 0,$$

where $\cdot\pi$ is multiplication by π, gives rise to the exact sequence (cf. (4))

$$\mathrm{Ext}_\Lambda^n(-, M) \overset{\mu}{\to} \mathrm{Ext}_\Lambda^n(-, M) \to \mathrm{Ext}_\Lambda^{n+1}(-, M/\pi\cdot M);$$

however, $M/\pi\cdot M$ is artinian and finitely generated, and so the map μ, which is induced from $\cdot\pi$ is surjective. Since π is a central element, μ is still multiplication by π, which generates the radical of R. But for each finitely generated Λ-module X, $\mathrm{Ext}_\Lambda^n(X, M)$ is finitely generated over R. Thus Nakayama's lemma implies $\mathrm{Ext}_\Lambda^n(X, M) = 0$, and so $\mathrm{Ext}_\Lambda^n(-, M) = 0$. If now Y is an arbitrary finitely generated left Λ-module, then we have an exact sequence

$$0 \to t(Y) \to Y \to Y/t(Y) \to 0,$$

where $t(Y)$ is the R-torsion submodule of Y and $Y/t(Y)$ is a Λ-lattice. Again the sequence (4) implies that for every finitely generated left Λ-module Y,

$$\mathrm{Ext}_\Lambda^{n+1}(-, Y) = 0$$

on finitely generated modules. This implies that Λ has global dimension bounded by n. In fact, the global dimension of any ring is bounded by the projective dimension of the finitely generated modules, and for a noetherian ring the syzygies of finitely generated modules are finitely generated, thus the above formula guarantees that the global dimension is bounded by n.

Assume now that Λ is semiprimary. In that case $\Lambda/\mathrm{rad}(\Lambda)$ is semisimple artinian and $\mathrm{rad}(\Lambda)$ is nilpotent; consequently every finitely generated left Λ-module has finite Loewy length. The above argument has shown that $\mathrm{Ext}_\Lambda^{n+1}(-, S_i) = 0$ for every simple module S_i. But then $\mathrm{Ext}_\Lambda^{n+1}(-, \Lambda/\mathrm{rad}(\Lambda)) = 0$, and quoting a result of Eilenberg [E, Theorem 12] we conclude $\mathrm{gl.dim.}(\Lambda) \leq n$.

Step 3: Λ is local semiprimary. Let E be the injective envelope of the unique simple Λ-module. The radical of Λ now is the unique maximal two sided ideal, which has injective dimension n. So we get a minimal injective resolution

$$0 \to \mathrm{rad}(\Lambda) \to E_1 \to \cdots \to E_{n-1} \overset{\alpha}{\to} E_n \overset{\beta}{\to} E_{n+1} \to 0,$$

where $\{E_i\}_{1\leq i\leq n+1}$ are injective Λ-modules. The natural map $\mathrm{Im}(\alpha) \to E_n$ is an essential monomorphism and hence $\mathrm{Soc}(E_n)$, the socle of E_n, is contained in $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. Thus we obtain a factorization of β as

$$E_n \to E_n/\mathrm{Soc}(E_n) \to E_{n-1}.$$

An argument with the Loewy lengths now shows that this can not happen. This also proves that for a local semiprimary ring the only modules of finite injective dimension are the injective ones.

Step 4: An R-order Λ has also injective lattices; i.e., Λ-lattices, which are injective with respect to the category of left Λ-lattices. They are the modules $Q_i^* = \mathrm{Hom}_R(Q_i, R)$, where Q_i are the indecomposable projective right Λ-modules. For a Λ-lattice M we write $\mathrm{LExt}_\Lambda^n(-, M)$ for the functor $\mathrm{Ext}_\Lambda^n(-, M)$ restricted to the category of Λ-lattices. Let now Λ be a local R-order, where R is a complete Dedekind domain. Then arguments similar to the ones above show

$$\mathrm{Ext}_\Lambda^{n+1}(-, M) = 0 \text{ iff } \mathrm{LExt}_\Lambda^n(-, M) = 0.$$

Since Λ is a local order, it has a unique indecomposable injective left Λ-lattice $E = \mathrm{Hom}_R(\Lambda, R)$, and if $\mathrm{rad}(\Lambda)$ has injective dimension bounded by n, then

$$0 = \mathrm{Ext}_\Lambda^{n+1}(-, \mathrm{rad}(\Lambda)) = \mathrm{LExt}_\Lambda^n(-, \mathrm{rad}(\Lambda)),$$

and so we have a minimal injective resolution in the category of left Λ-lattices

(6) $$0 \to \mathrm{rad}(\Lambda) \to E^{(s_1)} \to \cdots \to E^{(s_{n-1})} \to E^{(s_n)} \to 0, \ s_i \in \mathbf{N}.$$

Applying the exact functor $\mathrm{Hom}_R(-, R)$, we get a minimal projective resolution for $\mathrm{Hom}_R(\mathrm{rad}(\Lambda), R)$, which ends at the left hand side as

$$0 \to \Lambda^{(s_n)} \xrightarrow{\beta} \Lambda^{(s_{n-1})} \to \cdots.$$

Since this is part of a minimal projective resolution, the map β factorizes via $\mathrm{rad}(\Lambda)^{(s_{n-1})}$. Since $\mathrm{rad}(\Lambda/\pi\cdot\Lambda) = \mathrm{rad}(\Lambda)/\pi\cdot\Lambda$, and since reduction modulo π is exact, we get a monomorphism

$$\beta^\wedge : (\Lambda/\pi\cdot\Lambda)^{(s_n)} \to (\Lambda/\pi\cdot\Lambda)^{(s_{n-1})},$$

which factorizes via $\mathrm{rad}(\Lambda/\pi\cdot\Lambda)^{(s_{n-1})}$. Now an argument as above with the Loewy lengths shows that this is impossible. Hence there can not be any Λ-lattice of finite injective dimension. *This completes the proof of the proposition.*

Remarks. 1) The fact that Λ is an R-order in a semisimple K-algebra is only used to pass from the global to the local situation: The ext-formula linking global and local extensions of lattices.

2) The arguments in Step 2 are totally general for rings, where the

Krull-Schmidt theorem is valid for finitely generated modules, thanks to Rainwater's argument [R].

3) That for a local perfect ring finitely generated modules of finite injective dimension must be injective should be a general fact; however, we were not able to prove this.

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