

RINGS FOR WHICH THE CONVERSE OF SCHUR'S LEMMA HOLDS

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By Schur's lemma, if M is an irreducible right module over a ring R , then the endomorphism ring $End_R(M)$ is a division ring. However the converse assertion is not true in general. In this paper, we consider when the converse assertion is true.

In § 1 we give some results concerning modules. Let R be a semiprime ring, and e a nonzero idempotent of R . Then it is well known that eR is irreducible if and only if $End_R(eR) (\cong eRe)$ is a division ring. This result was generalized by R. Ware [10]. He proved that a projective right module P over a semiprime ring R is irreducible if and only if $End_R(P)$ is a division ring. In this case, P is isomorphic to a (non-nilpotent) minimal right ideal of R . As the main result of this section, we prove that a right module M over a ring R is isomorphic to a non-nilpotent minimal right ideal of R if and only if $End_R(M)$ is a division ring and the annihilator of the trace ideal of M in R is zero. As a corollary of this result, we extend the above result of Ware. We prove that a torsionless module M over a semiprime ring R is irreducible if and only if $End_R(M)$ is a division ring.

In § 2 we study rings over which a given right module is irreducible whenever its endomorphism ring is a division ring. For simplifying our notations, we denote this property by (CS) which may be regarded as a converse of Schur's lemma. We show that this property is Morita invariant. Obviously a semisimple Artinian ring has the property (CS). More generally we show that a von Neumann regular ring with primitive factor rings Artinian has (CS). Hence a von Neumann regular P.I. ring has (CS). However, in general, a von Neumann regular ring need not have this property. In fact, we show that the endomorphism ring $End_D(V)$ of a vector space V of a division ring D has the property (CS) if and only if V is finite dimensional. It seems to be very difficult to determine the class of rings with (CS), even the class of P.I. rings with (CS). However we show that a P.I. ring with (CS) is necessarily π -regular. Using this result, we give a characterization of an Azumaya algebra with (CS). Actually for an Azumaya algebra, having

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the property (CS) is equivalent to the π -regularity.

§ 1. Some results on modules. Let R be a ring, M a right R -module and $M^* = \text{Hom}_R(M, R)$. Then the subset $t(M)$ of R consisting of the elements of the form $\sum_i f_i(m_i)$ where the f_i are from M^* and the m_i are from M , forms a two-sided ideal of R , called the *trace ideal* of M . And $\text{Ann}_M(t(M))$ denotes the set $\{m \in M \mid mt(M) = 0\}$.

We begin this section with the following :

Theorem 1. *Let R be a ring, and M a right R -module. Then the following statements are equivalent :*

- (1) M is isomorphic to a non-nilpotent minimal right ideal of R .
- (2) $\text{End}_R(M)$ is a division ring and $\text{Ann}_M(t(M)) = 0$.

Proof. (1) \Rightarrow (2). By Schur's lemma $\text{End}_R(M)$ is a division ring. Since M is isomorphic to a minimal right ideal of R , $t(M)$ is the sum of all minimal right ideals of R which are isomorphic to M . By hypothesis $t(M)^2 \neq 0$, and hence $Mt(M) \neq 0$. Since $t(M)$ is an ideal of R , $\text{Ann}_M(t(M))$ is an R -submodule of M . Since M is irreducible and $Mt(M) \neq 0$, we conclude that $\text{Ann}_M(t(M)) = 0$.

(2) \Rightarrow (1). Let m be a nonzero element of M . Then $mt(M) \neq 0$ by hypothesis, and so there exist $f \in M^*$ and $x \in M$ such that $mf(x) \neq 0$. Then mf is a nonzero element of the division ring $\text{End}_R(M)$, and hence mf is an automorphism of M . Therefore $mR \supset mf(M) = M$. This implies that M is irreducible.

Let I be a minimal right ideal of R which is isomorphic to M . Clearly, if $I^2 = 0$, then $MI = 0$. Since $t(M)$ is the sum of all minimal right ideals which are isomorphic to M and since $Mt(M) \neq 0$ by hypothesis, we conclude that there exists a non-nilpotent minimal right ideal which is isomorphic to M .

Let R be a ring. A right R -module M is called *semiprime* if, for each nonzero element m of M , there exists $f \in M^*$ such that $mf(m) \neq 0$.

As an immediate consequence of Theorem 1, we obtain the following :

Corollary 2. *Let R be a ring, and M a semiprime right R -module. Then M is irreducible if and only if $\text{End}_R(M)$ is a division ring.*

A module M over a ring R is called *torsionless* if M is embedded in a direct product of copies of R . As was mentioned in [11, p.555], torsionless

modules over semiprime rings are semiprime. Hence we can improve [8, Corollary 4.4] as follows.

Corollary 3. *Let R be a semiprime ring, and M a torsionless right R -module. Then M is irreducible if and only if $\text{End}_R(M)$ is a division ring.*

A Frobenius algebra with a symmetric associative nondegenerate bilinear form is called a *symmetric algebra*. Let G be a finite group and let K be an arbitrary field. Then the group algebra KG is a symmetric K -algebra.

Proposition 4. *Let R be a symmetric algebra over a field K , and M an injective (or equivalently projective) right R -module. Then M is irreducible if and only if $\text{End}_R(M)$ is a division ring.*

Proof. By [10, Proposition 4.3] we may assume that $M = eR$ for some primitive idempotent e of R . Let J denote the Jacobson radical of R , and $r(J)$ the right annihilator of J . By [4, Proposition 9.12], eR/eJ is isomorphic to the unique minimal submodule $r(J)e$ of eR . Let φ denote the composite map of the natural epimorphism $eR \rightarrow eR/eJ$ with an isomorphism $eR/eJ \rightarrow r(J)e$. Then φ is a nonzero endomorphism of eR . By hypothesis φ is an automorphism of eR , and hence we conclude that eR equals to its socle. Therefore eR is irreducible.

In case R is a commutative ring, we have the following:

Proposition 5. *Let R be a commutative ring and let M be either a finitely generated module or a projective module. Then M is irreducible if and only if $\text{End}_R(M)$ is a division ring.*

Proof. Suppose that $\text{End}_R(M)$ is a division ring. If M is projective, then M is cyclic by [10, Proposition 4.3]. So, in either case, M is finitely generated. Then we can easily see that every element of $D = \text{End}_R(M)$ is integral over R . Since D is a division ring, $\bar{R} = R/\text{Ann}_R(M)$ must be a field, where $\text{Ann}_R(M)$ denotes the annihilator of M in R . Since $\text{End}_R(M)$ is a division ring, M must be indecomposable, and hence M is a one-dimensional vector space over \bar{R} . Therefore M is an irreducible R -module.

The following example shows that Proposition 5 does not remain true for P.I. rings.

Example 6. Let F be a field and consider the subring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

of the ring $M_2(F)$ of all 2×2 matrices over F . Then

$$M = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$$

is a right ideal of R generated by the idempotent

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and so M is a cyclic projective right R -module. Clearly M is not irreducible, but $\text{End}_R(M)$ is isomorphic to the field F .

§ 2. Rings for which the converse of Schur's lemma holds. A ring R is said to have *the property (CS)* if a given right R -module M is irreducible whenever $\text{End}_R(M)$ is a division ring.

We start this section with the following :

Proposition 7. *The property (CS) is Morita invariant.*

Proof. Assume that a ring R has the property (CS) and a ring S is Morita equivalent to R . Then we have two functors $F : \text{Mod-}R \rightarrow \text{Mod-}S$ and $G : \text{Mod-}S \rightarrow \text{Mod-}R$ with $FG \cong 1$ and $GF \cong 1$.

Now to prove that S also has the property (CS), suppose that M is a right S -module with $\text{End}_S(M)$ a division ring. Then $\text{End}_S(M)$ is isomorphic to $\text{End}_R(G(M))$ by [1, Proposition 21.2]. Since R has the property (CS), $G(M)$ is irreducible and hence M , which is isomorphic to $FG(M)$, is also an irreducible right S -module.

In relation with property (CS), we consider the following condition :

(*) *Every nonsingular uniform right R -module is irreducible.*

Lemma 8. *If R has the property (CS), then every factor ring of R satisfies (*).*

Proof. Since every factor ring of R has (CS), it suffices to prove that R satisfies the condition (*). Let M be a nonsingular uniform right R -mod-

ule, and \widehat{M} denote the injective envelope of M . Then \widehat{M} is a nonsingular injective indecomposable right R -module. By [1, Lemma 25.4], $S = \text{End}_R(\widehat{M})$ is a local ring. By [1, Proposition 18.20], an element f of S is in the Jacobson radical $J(S)$ of S if and only if $\text{Ker}(f)$ is essential in \widehat{M} . So, if $f \in J(S)$, then the singular module $\widehat{M}/\text{Ker}(f)$ is isomorphic to a submodule of \widehat{M} . But this implies $f = 0$, because \widehat{M} is nonsingular. This implies $J(S) = 0$, and hence the local ring S is a division ring. By virtue of the property (CS), we conclude that \widehat{M} and hence M is irreducible.

We show that quasi-Frobenius rings satisfy (*).

Example 9. Let R be a quasi-Frobenius ring, and M a nonsingular uniform right R -module. Then the socle $\text{Soc}(M)$ of M is nonsingular and hence projective. Since R is quasi-Frobenius, every projective module is injective. Thus $\text{Soc}(M)$ is injective, and so $\text{Soc}(M)$ is a direct summand of M . But, since M is uniform, M must coincide with its socle $\text{Soc}(M)$, so that M is irreducible.

The ring R in Example 6 is a right nonsingular finite uniform dimensional ring, but R does not satisfy (*). We show that a right nonsingular finite uniform dimensional ring with (*) must be a semisimple Artinian ring. To show this, we need the following:

Lemma 10. *Let R be a ring. Then the following statements are equivalent:*

- (1) R satisfies the condition (*).
- (2) Every nonsingular uniform right R -module is injective.

Proof. (1) \Leftrightarrow (2). Let M be a nonsingular uniform right R -module and let \widehat{M} denote the injective envelope of M . Then \widehat{M} is also nonsingular and uniform, and hence \widehat{M} is irreducible by (*). Thus we obtain $M = \widehat{M}$.

(2) \Leftrightarrow (1). Let M be a nonsingular uniform right R -module, and N a nonzero submodule of M . Then N is also nonsingular and uniform, and hence N is injective by (2). Then N is a direct summand of M , and so $N = M$ because M is uniform. This proves that M is irreducible.

Proposition 11. *The following statements are equivalent:*

- (1) R is a semisimple Artinian ring.
- (2) R is a semiprime right Goldie ring with (*).

(3) R is a right nonsingular finite uniform dimensional ring with (*).

Proof. It suffices to prove only (3) \Leftrightarrow (1). So suppose that R is a right nonsingular finite uniform dimensional ring with (*). Then R contains an essential right ideal I which is a finite direct sum of uniform right ideals, say $I_\lambda (\lambda \in \Lambda)$. Since R is right nonsingular, each I_λ is also nonsingular (and uniform). Then by (*) and Lemma 10, each I_λ is irreducible and injective. Then $I = \bigoplus_{\lambda \in \Lambda} I_\lambda$ is also injective. Since I is essential in R , we conclude that $I = R$. Therefore R is the direct sum of the minimal right ideals I_λ .

Of course, there are non-semisimple rings with (CS).

Proposition 12. *Suppose that the Jacobson radical $J(R)$ of a ring R is a nil ideal generated by central elements and that $R/J(R)$ is semisimple Artinian. Then R has the property (CS).*

Proof. Let M be a right R -module with $End_R(M)$ a division ring. Consider the factor ring \bar{R} of R by the annihilator $Ann_R(M)$ of M in R . It is easy to see that \bar{R} also satisfies the hypotheses on R . The center $Z(\bar{R})$ of \bar{R} can be naturally considered as a subring of $End_R(M)$, and so $Z(\bar{R})$ is a domain. Then we can easily see that \bar{R} is a simple Artinian ring. Clearly this implies that M is an irreducible right R -module.

A ring R is called π -regular if for each a in R , there exists a positive integer n depending on a and an element x in R such that $a^n = a^n x a^n$.

By virtue of Proposition 11, we get a necessary condition for a P.I. ring to have the property (CS).

Proposition 13. *A P.I. ring with property (CS) is π -regular.*

Proof. Let R be a P.I. ring with (CS). Then, for each prime ideal P of R , R/P is a prime Goldie ring by Posner's theorem [9, Theorem 7.3.2]. Since R/P also has the property (CS), R/P is a simple Artinian ring by Proposition 11. Therefore R is π -regular by Fisher and Snider [7, Theorem 2.3].

The ring R in Example 6 is a π -regular P.I. ring, but R does not have the property (CS). Moreover the ring R in [8, Example 1.19] is a π -regular semiprime P.I. ring, however R does not have the property (CS).

With help of Proposition 13, we can completely characterize Azumaya

algebras with property (CS).

Theorem 14. *Let R be an Azumaya algebra. Then the following statements are equivalent :*

- (1) R has the property (CS).
- (2) R is π -regular.
- (3) The center $Z(R)$ of R is π -regular.

Proof. (1) \Leftrightarrow (2). Since an Azumaya algebra is a P.I. ring, R is π -regular by Proposition 13.

(2) \Leftrightarrow (3). Let a be an element of $Z(R)$. Then there exists a positive integer n and an element x in R such that $a^n = a^n x a^n$. Since a is in $Z(R)$, a^n is strongly regular in the sense of [3]. Then by [3, Lemma 1], there exists an element z in $Z(R)$ such that $a^{2n} z = a^n$. This proves that $Z(R)$ is π -regular.

(3) \Leftrightarrow (1). Let M be a right R -module with $End_R(M)$ a division ring. Consider the factor ring \bar{R} of R by the annihilator ideal of M in R . By [5, Proposition 2.1.11], \bar{R} is an Azumaya algebra and the center $Z(\bar{R})$ of \bar{R} is the natural homomorphic image of $Z(R)$ in \bar{R} . Hence $Z(\bar{R})$ is also π -regular. By the way, $Z(\bar{R})$ can be naturally considered as a subring of the division ring $End_R(M)$. Hence $Z(\bar{R})$ must be a field. Hence \bar{R} is indecomposable as a ring. By [5, Theorem 2.2.5], we conclude that \bar{R} is a simple Artinian ring. This implies that M is irreducible.

Corollary 15. *Let R be a semiprime ring which is finitely generated as a module over its center. Then the following statements are equivalent :*

- (1) R has the property (CS).
- (2) R is a von Neumann regular ring.
- (3) The center $Z(R)$ of R is a von Neumann regular ring.

Proof. (1) \Leftrightarrow (3). By Proposition 13 R is π -regular. Hence, by the proof of (2) \Leftrightarrow (3) of Theorem 14, we know that the center $Z(R)$ of R is also π -regular. Since $Z(R)$ has no nonzero nilpotent elements, $Z(R)$ is von Neumann regular.

(3) \Leftrightarrow (2). If $Z(R)$ is von Neumann regular, then R is von Neumann regular by Armendariz [2, Theorem 1].

(2) \Leftrightarrow (1). If R is von Neumann regular, then R is an Azumaya algebra by Armendariz [2, Theorem 2]. Then R has the property (CS) by Theorem 14.

By Corollary 15, it may be suspected that von Neumann regular rings have the property (CS). Unfortunately, the following example shows that there is a von Neumann regular ring R with a non-irreducible cyclic right R -module M whose endomorphism ring $End_R(M)$ is a field.

Example 16. Let F be a field and let A be the set of countable matrices over F of the form

$$\begin{bmatrix} C_n & & 0 & & & \\ & a & & & & \\ & & \ddots & & & \\ 0 & & & \ddots & & \\ & & & & a & \\ & & & & & \ddots \end{bmatrix}$$

where $a \in F$ and C_n is an arbitrary $n \times n$ matrix over F and n is allowed to be any integer. Obviously the center $Z(A)$ of A is (isomorphic to) the field F . Now consider the enveloping algebra $R = A \otimes_F A^{op}$ of A over F . For each x in R , there is a positive integer N such that x is in $B \otimes_F B^{op}$, where

$$B = \left\{ \left[\begin{bmatrix} C_N & & 0 & & & \\ & a & & & & \\ & & \ddots & & & \\ 0 & & & \ddots & & \\ & & & & a & \\ & & & & & \ddots \end{bmatrix} \right] \mid C_N \in M_N(F) \text{ and } a \in F \right\}$$

Now we may observe that B is isomorphic to the ring $M_N(F) \oplus F$. Hence the F -subalgebra $B \otimes_F B^{op}$ of R is a unit von Neumann regular ring, and so there exists an invertible element y in $B \otimes_F B^{op}$, hence in R such that $x = xyx$. So R is a unit von Neumann regular ring. In this situation, we can say more about R . Indeed, as a right R -module, A is a cyclic module generated by the identity element 1 of A , but A is not irreducible, because A is not a simple ring. However $End_R(A) = Z(A) = F$, so that R does not have the property (CS).

Another typical example of a von Neumann regular ring is the endomorphism ring $End_D(V)$ of a vector space V over a division ring D . Now we shall prove

Proposition 17. *Let V be a left vector space over a division ring D . Then the following statements are equivalent :*

- (1) $End_D(V)$ has the property (CS).
- (2) $End_D(V)$ satisfies the condition (*).

(3) V is finite dimensional over D .

Proof. The implication (1) \Leftrightarrow (2) follows from Lemma 8, and (3) \Leftrightarrow (1) is clear.

(2) \Leftrightarrow (3). Let us set $R = \text{End}_D(V)$ and observe that V is irreducible as a right R -module. Also we have a primitive idempotent e in R such that eR is isomorphic to V . It is well known that R is von Neumann regular, and hence R is right nonsingular. Thus V is a nonsingular uniform right R -module. By Lemma 10, V is an injective right R -module. Then V must be finite dimensional over D by [6, Proposition 19.46].

Deposite of Example 16 and Proposition 17, some class of von Neumann regular rings still enjoy the property (CS).

Theorem 18. *A von Neumann regular ring with primitive factor rings Artinian has the property (CS).*

Proof. Let R be a von Neumann regular ring whose primitive factor rings are Artinian. Assume that M is a right R -module with $\text{End}_R(M)$ a division ring. To prove that M is irreducible, without loss of generality we may assume that M is a faithful right R -module. By observing that the center $Z(R)$ of R can be embedded as a subring in the ring $\text{End}_R(M)$ in the natural way, we know that $Z(R)$ is an integral domain. Now for R , if R is not a simple ring, then R would have a non-trivial central idempotent by [8, Theorem 6.6]. But this is contradictory to the fact that $Z(R)$ is an integral domain. So R is a simple Artinian ring by assumption. By this fact, M is an irreducible right R -module.

By Kaplansky's theorem [9, Theorem 6.3.1], every primitive factor ring of a P.I. ring is Artinian. Hence we have the following:

Corollary 19. *A von Neumann regular P.I. ring has the property (CS).*

Using Corollary 19 together with Proposition 13, we obtain another corollary.

Corollary 20. *For a reduced P.I. ring R , the following statements are equivalent:*

(1) R has the property (CS).

(2) R is a von Neuman regular ring.

Proof. By Corollary 19, (2) implies (1). Now assume that the reduced P.I. ring R has the property (CS). Then by Proposition 13 and its proof, every prime factor ring of R is a simple Artinian ring. Since R is reduced, R is von Neumann regular by Fisher and Snider [7, Corollary 1.4].

As mentioned in the proof of Corollary 20, every prime P.I. ring with property (CS) is a simple Artinian ring. Comparing this fact with results in Corollaries 19 and 20, the following question might be interesting.

Question 21. Is a semiprime P.I. ring with property (CS) von Neumann regular ?

In the light of Proposition 17 and Theorem 18, we also have the following question.

Question 22. Let R be a von Neumann regular ring with (CS). Is every primitive factor ring of R Artinian ?

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