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ON PERIODIC P. I. RINGS AND LOCALLY FINITE RINGS

YASUYUKI HIRANO

An element x of a ring R is called *periodic* if there exist distinct positive integers m, n for which $x^m = x^n$. Especially, x is called *potent* if $x^m = x$ for some positive integer $m > 1$. A ring R is called *periodic* if all elements of R are periodic. It is easily seen that a periodic ring R has the property that every element of R is expressible as a sum of a potent element and a nilpotent element. However it is not known whether a ring R with this property is periodic or not. On the other hand, by a result of the author and H. Tominaga [6], if R is a P. I. ring in which every element is the sum of two idempotents, then R is periodic. In this paper, we shall prove that a P. I. ring R in which every element is expressible as a sum of two periodic elements, is periodic.

We shall next consider the local finiteness of a periodic P. I. ring. A ring R is said to be *locally finite* if any finitely generated subring of R is a finite ring. Let R be a periodic P. I. ring, and S a finitely generated subring of R. We shall show that the additive group of S is finitely generated and that some power of S is a finite ring. Consequently a P. I. ring R is locally finite if and only if R is periodic and the additive group of R is a torsion group. Using this, we shall give a characterization of a locally finite ring.

We begin with the following lemma.

Lemma 1. *Let R be a ring. Then R is periodic if and only if all prime factor rings of R are periodic.*

Proof. Suppose that all prime factor rings of R are periodic. For each $x \in R$, let $S(x) = \{x^n - x^{n+1}f(x) \mid n > 0$ is an integer, $f(t) \in \mathbf{Z}[t]\}$, which is multiplicatively closed. By virtue of [3, Proposition 2], R is periodic if and only if $0 \in S(x)$ for all $x \in R$. Assume, to the contrary, that there exists $a \in R$ such that $0 \notin S(a)$. Then, by Zorn's lemma, we can find an ideal I of R which is maximal with respect to the property that $S(a) \cap I = \phi$. It is easy to check that I is a prime ideal of R. Hence R/I is periodic by hypothesis. But this contradicts the fact that $S(a) \cap I = \phi$.

A ring R is said to be *of bounded index (of nilpotence)* if there is a positive integer n such that $a^n = 0$ for any nilpotent element a in R. The least such integer is called the *index* of R. We shall show that a periodic ring of bounded index is a P. I. ring. Let G denote the symmetric group of degree n. The identity

$$s_n = \sum_{\sigma\in G}\mathrm{sgn}(\sigma)X_{1\sigma}X_{2\sigma}\cdots X_{n\sigma}$$

is called the *standard identity of degree n.*

Proposition 1. *Let n be a positive integer and let R be a periodic ring of index n. Then R satisfies the polynomial identity $(s_{2n})^n$.*

Proof. Let J denote the Jacobson radical of R, and x an element of J. Then there exist positive integers p, q such that $x^{p+q} = x^p$. By [4, Theorem 1.2.3] all elements of J are right-quasi-regular. Hence there exists $y \in R$ such that $(-x^q)+y+(-x^q)y = 0$. Then $x^p = x^p+x^p(-x^q+y-x^qy) = (x^p-x^{p+q})+(x^p-x^{p+q})y = 0$. This implies that J is a nil ideal. Let P be a primitive ideal of R. By [7, Theorem 2.3] $R/P = M_t(D)$ for some division ring D and some positive integer $t \leqq n$. Since D is a periodic division ring, D is commutative by [4, Lemma 3.1.3]. Hence R/P satisfies the standard identity s_{2n} of degree $2n$ by [8, Theorem 1.4.1]. Since R/J can be embedded in the direct product of all primitive factor rings of R, R/J also satisfies the identity s_{2n}, in other words, $s_{2n}(a_1, a_2, \cdots, a_{2n}) \in J$ for all elements $a_1, a_2, \cdots, a_{2n}$ in R. Since J is a nil ideal of index at most n, we have that $s_{2n}(a_1, a_2, \cdots, a_{2n})^n = 0$ for all $a_1, a_2, \cdots, a_{2n} \in R$. This completes the proof.

If R is a periodic ring, each element x in R can be expressed in the form $y+w$, where $y^n = y$ for some $n = n(y) > 1$ and w is nilpotent (e.g., see [2, Lemma 1]). However it is not known whether this property characterizes a periodic ring. On the other hand, by [6, Theorem 2], if R is a P. I. ring in which every element is the sum of two idempotents then, for any $x \in R$, x^3-x is nilpotent. Hence R is periodic by [3, Proposition 2]. We shall now prove the following

Theorem 1. *Let R be a P. I. ring. If every element of R is expressed as a sum of two periodic elements, then R is periodic.*

Proof. By virtue of Lemma 1, we may assume that R is a prime ring. Then, by [5, Theorem 1.4.2] the center C of R is nonzero. We claim that C is periodic. Let c be a nonzero element of C. Then, by hypothesis, there exist $x, y \in R$ such that $c = x+y$, $x^m = x^n$ for some $m > n > 0$, and $y^p = y^q$ for some $p > q > 0$. Then $(c-y)^m = (c-y)^n$, and so $c^m - c^n = zy$ for some $z \in C[y] (\subset R)$. If $c^m - c^n$ is nilpotent, then $c^m = c^n$, because C is an integral domain. Assume now that $c^m - c^n$ is not nilpotent. Then $e = y^{(p-q)q}$ is a nonzero idempotent and $y^q e = ey^q = y^q$. Therefore we have that $(c^m - c^n)^q (ae - a) = 0$ for all $a \in R$. Let us put $L = \{ae - a \mid a \in R\}$. Then L is a left ideal of R, and as seen above, $(c^m - c^n)^q L = 0$. Since $(c^m - c^n)^q \neq 0$ and since R is a prime ring, we obtain $L = 0$, that is, e is a right identity of R. We can similarly prove that e is a left identity of R. Hence e is the identity of R. We shall now prove that the characteristic of R is nonzero. Assume, to the contrary, that the characteristic of R is zero. Then we may assume that R contains the ring $\boldsymbol{Z}$ of integers as a subring. By hypothesis, there exist two periodic elements $v, w \in R$ such that $3 = v + w$. Obviously the subring $S = \boldsymbol{Z}[v, w]$ of R generated by v and w over $\boldsymbol{Z}$ is a commutative ring which is integral over $\boldsymbol{Z}$. By [1, Theorem 5.10] there exists a prime ideal P of S such that $P \cap \boldsymbol{Z} = 0$. Consider now the factor ring $\bar{S} = S/P$. Then $\bar{S}$ is an integral domain which is integral over $\boldsymbol{Z}$. So, without loss of generality, we may assume that $\bar{S}$ is a subring of the field $\boldsymbol{C}$ of complex numbers. In general, if a is a periodic element of $\boldsymbol{C}$ then the absolute value $|a|$ of a is either 0 or 1. Hence we have $3 = |\bar{v} + \bar{w}| \leqq |\bar{v}| + |\bar{w}| \leqq 2$, which is a contradiction. Therefore the characteristic of R is nonzero. Let F denote the prime field of C. Since x and y are integral over F, $c = x+y$ is integral over F. Hence c generates a finite subring of C, and so c is periodic. Therefore we proved that C is a periodic field. By [8, Corollary 1.6.28], R is a simple P. I. ring. Hence, by Kaplansky's theorem [8, Theorem 1.5.16], R can be identified with the matrix ring $M_t(D)$ over a division ring D which is finite dimensional over C. Then D is also periodic, and hence D is commutative. Thus we get $C = D$. Therefore $R = M_t(C)$ is periodic.

We shall next consider the finitely generated subrings of a periodic P. I. ring. Clearly a periodic P. I. ring need not be locally finite. For example, the subring

$$\begin{pmatrix} 0 & \boldsymbol{Z} \\ 0 & 0 \end{pmatrix} \text{ of } M_2(\boldsymbol{Z})$$

is a finitely generated periodic commutative ring, but this is not a finite ring. We shall prove the following:

Theorem 2. *Let R be a periodic P. I. ring and let S be a finitely gener-*

ated subring of R. Then the additive group S^+ of S is a finitely generated abelian group. Moreover there exists a positive integer n such that S^n is a finite ring. In particular, if S has an identity, then S is finite.

Proof. Let $t(S)$ denote the torsion submodule of the $\mathbf{Z}$-module S. Then $t(S)$ is an ideal of S and $S/t(S)$ is torsion-free. Let x be an element of $S/t(S)$. Then $x^{m+n} = x^m$ for some positive integers m, n. Then we can easily see that x^{mn} is an idempotent. Since $(2x^{mn})^{p+q} = (2x^{mn})^p$ for some positive integers p and q, we obtain a positive integer h such that $hx^{mn} = 0$. Since $S/t(S)$ is torsion-free, we conclude that $x^{mn} = 0$. Thus $S/t(S)$ is a nil ring. Since $S/t(S)$ is also a finitely generated P. I. ring, there exits a positive integer n such that $(S/t(S))^n = 0$ by [8, Proposition 1.6.34]. Hence we have $S^n \subset t(S)$. Let $c_1, c_2, \cdots, c_m$ generate the subring S. Then $A = \{c_{i_1}c_{i_2}\cdots c_{i_n} \mid 1 \leqq i_j \leqq m\}$ is a finite set, and hence there exists a positive integer k such that $kA = 0$. Hence we have $kS^n = 0$. Let B denote the set $\{c_{i_1}c_{i_2}\cdots c_{i_p} \mid 1 \leqq i_j \leqq m,\ 1 \leqq p \leqq n\}$. Then we can easily see that

$$kS = \sum_{b\in B} \mathbf{Z}kb.$$

Hence kS is a finitely generated $\mathbf{Z}$-module. Let S' denote the ring S/kS and let us write $k = \Pi_{i=1}^{t} p_i^{k_i}$ where the p_i are distinct primes and $k_i > 0$ for all i. Then, for each i, $S'_i = \{a \in S' \mid p_i^{k_i}a = 0\}$ is a subring of S' and S' is the direct sum of $S'_1, S'_2, \cdots, S'_t$. We shall show that S' is finite. To show it, it suffices to prove that S'_i is finite for each $i = 1, 2, \cdots, t$. Hence, without loss of generality, we may assume that $k = p^h$ for some prime p and some positive integer h. Let us set $I = pS'$. Then $I^h = 0$ and $p^{h-1}I = 0$. Then the ring S'/I is a finitely generated periodic algebra over $\mathbf{Z}/p\mathbf{Z}$ satisfying a polynomial identity. Hence S'/I is a finite dimensional algebra over $\mathbf{Z}/p\mathbf{Z}$ by [4, Theorem 6.4.3]. Let $S'/I = \{a_0+I, a_1+I, \cdots, a_d+I\}$ where $a_0 = 0$, $a_1, \cdots, a_d$ are elements of S'. Then we can choose elements $b_1, b_2, \cdots, b_f$ of I such that $a_1, a_2, \cdots, a_d, b_1, b_2, \cdots, b_f$ generate S'. For any i, j with $1 \leqq i, j \leqq d$, we have a unique integer $t(i, j)$ with $1 \leqq t(i, j) \leqq d$ such that $a_i a_j \equiv a_{t(i,j)}$ modulo I. Similarly we have a unique integer $s(i, j)$ such that $a_i + a_j \equiv a_{s(i,j)}$ modulo I. Let us now set $x_{ij} = a_i a_j - a_{t(i,j)}$ and $y_{ij} = a_i + a_j - a_{s(i,j)}$ for each $1 \leqq i, j \leqq d$. Let J denote the subring of S' generated by $x_{\alpha\beta}$, $y_{\mu\nu}$, b_λ, $a_\gamma x_{\alpha\beta}$, $a_\gamma y_{\mu\nu}$, $a_\gamma b_\lambda$, $x_{\alpha\beta}a_\gamma$, $y_{\mu\nu}a_\gamma$, $b_\lambda a_\gamma$ for $1 \leqq \alpha, \beta, \gamma \leqq d$, $1 \leqq \mu, \nu \leqq d$, and $1 \leqq \lambda \leqq f$. Then J is a finitely generated subring of I. Since $I^h = 0$ and $p^{h-1}I = 0$, J must be finite. We can now easily see that each element x of S' can be uniquely expressed in the form $a_i + z$, where $0 \leqq i \leqq d$ and $z \in J$. This implies that $I = J$. Therefore S' is a finite ring. Consequently S is a finitely generated $\mathbf{Z}$-module. Since the additive group of S^n is a torsion group, S^n is a finite ring. In particular, if S has an identity, then $S^n = S$, and hence S is finite.

As an immediate consequence of this theorem, we obtain the following:

Corollary 1. *Let R be a P. I. ring. Then R is locally finite if and only if R is periodic and the additive group of R is a torsion group.*

A ring R is said to be *of locally bounded index* if every finitely generated subring of R is of bounded index. Combining Corollary 1 with Proposition 1, we obtain the following characterization of a locally finite ring.

Corollary 2. *A ring R is locally finite if and only if R is a periodic ring of locally bounded index and the additive group of R is a torsion group.*

The following example due to Golod and Shafarevitch shows that a finitely generated periodic ring with torsion additive group need not be finite.

Example 1. Let p be a prime number. By [4, Theorem 8.1.3], there exists an infinite dimensional nil algebra A over $\mathbf{Z}/p\mathbf{Z}$ generated by three elements. Clearly A is generated by those three elements as a ring. Note that those elements generate infinite subsemigroup of the multiplicative semigroup of R.

As another corollary of Theorem 2, we obtain the following

Corollary 3. *Let R be a P. I. ring. Then the following statements are equivalent:*

(1) *R is periodic.*

(2) *For any finitely generated subring S of R, there exists a positive integer n such that S^n is a finite subring.*

(3) *For any finitely generated subring S of R, there exists a finite ideal I of S such that S/I is a nilpotent ring.*

(4) *The ideal $t(R) = \{a \in R \mid na = 0$ for some positive integer $n\}$ is locally finite and $R/t(R)$ is a nil ring.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 2 and (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let x be an element of R, and S denote the subring of R

generated by x. Then there exists a finite ideal I of S such that S/I is nilpotent. This implies that some power of x generates a finite subring. Hence there exist distinct positive integers m, n such that $x^m = x^n$.

(1)$\Leftrightarrow$(4). Assume that R is periodic. By Corollary 1 $t(R)$ is locally finite. We also know that $R/t(R)$ is a nil ring by the proof of Theorem 2.

Conversely, suppose that (4) holds, and let x be an element of R. Then some power of x generates a finite subring of R, and hence x is periodic.

A ring R is periodic if and only if each subsemigroup of R generated by a single element is finite. If R is a commutative periodic ring, then all finitely generated subsemigroups of R are finite. However Example 1 shows that this does not remain valid for noncommutative periodic rings. Thus we have the following

Conjecture. Let R be a periodic P. I. ring. Then all finitely generated subsemigroups of R are finite.

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Department of Mathematics
Okayama University
Okayama, 700 Japan

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