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# LOCALLY INTEGRAL DERIVATIONS AND ENDOMORPHISMS

Dedicated to Professor Kazuo Kishimoto on his 60th birthday

MIGUEL FERRERO* and ANDRZEJ NOWICKI**

**Introduction.** Let $R$ be a ring and $d$ a derivation of $R$. The derivation $d$ is said to be integral if it satisfies a monic polynomial with coefficients in $R$ [13]. A weaker condition was considered in [3] and [4], the so called condition $(F)$. A derivation $d$ is said to satisfy the condition $(F)$ on $R$ if for every $a \in R$ there exists a positive integer number $m = m(a)$ such that $d^m(a)$ is contained in the ideal of $R$ generated by $a, d(a), \dots, d^{m-1}(a)$. In this paper a derivation of this type is said to be locally integral. A locally integral endomorphism of $R$ is defined similarly. The purpose of this paper is to study locally integral derivations and endomorphisms of commutative rings.

Throughout this paper $K$ is a commutative ring with an identity and $R$ is a commutative $K$-algebra. In § 1 we give some basic facts concerning a locally integral $K$-derivation or $K$-endomorphism ($K$-mapping, for short) $\sigma$ of $R$. We collect here some properties of prime, $\sigma$-prime and quasi-prime ideals. This properties are easy and some of them are known for derivations, but we include it here for the sake of completeness and to justify the convenience of studying locally integral mappings.

In § 2 we consider $K$-mappings of finitely generated algebras and power series rings, and we include several examples and contraexamples.

In § 3 we prove that the $\sigma$-prime spectrum of $R$ is a spectral space, when $\sigma$ is locally integral (see [6]). It follows that in this case there exists a ring $S$ such that the topological spaces $Spec_\sigma(R)$ and $Spec(S)$ are homeomorphic.

**1. Definitions and basic facts.** Let $R$ be a commutative $K$-algebra and let $\sigma$ be either a $K$-derivation or a $K$-endomorphism of $R$.

An ideal $I$ of $R$ is said to be a $\sigma$-ideal if $\sigma(I) \subseteq I$. If $T$ is a subset of $R$, then we denote by $(T)$ the ideal of $R$ generated by $T$ and by $[T]$ the smallest $\sigma$-ideal of $R$ containing $T$. When $T = \{a_1, \dots, a_n\}$ is a finite set we write simply $(a_1, \dots, a_n)$ and $[a_1, \dots, a_n]$ instead of $(\{a_1, \dots, a_n\})$ and $[\{a_1, \dots, a_n\}]$. It is

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clear that for $a \in R$ we have $[a] = (a, \sigma(a), \sigma^2(a), \ldots)$.

The following is easy to prove

**Lemma 1.1.** *Let $a \in R$. The ideal $[a]$ is finitely generated if and only if $[a] = (a, \sigma(a), \ldots, \sigma^n(a))$, for some natural number $n$.*

The mapping $\sigma$ is said to be *integral* if there exists a finite set $\{a_0, a_1, \ldots, a_{n-1}\}$ of elements of $R$ such that

$$\sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_1\sigma + a_0\sigma^0 = 0.$$

Denote by $\mathrm{I}(\sigma)$ the set of all the elements $a \in R$ such that the ideal $[a]$ is a finitely generated ideal of $R$. By Lemma 1.1 we see that $a \in \mathrm{I}(\sigma)$ if and only if there exists a natural number $n$ and elements $b_0, \ldots, b_{n-1}$ in $R$ such that

$$\sigma^n(a) + b_{n-1}\sigma^{n-1}(a) + \cdots + b_1\sigma(a) + b_0 a = 0.$$

We say that $\sigma$ is *locally integral* if $\mathrm{I}(\sigma) = R$. Thus, if $\sigma$ is integral, then it is locally integral. Also if $R$ is a Noetherian ring, then every $\sigma$ is locally integral.

Locally integral derivations have already appeared in [4] and [3]. In these papers a locally integal derivation is a derivation which satisfies the so called condition $(F)$ on $R$. This condition is very useful in those papers.

If $I$ is an ideal of $R$, we denote by $I_\#$ the biggest $\sigma$-ideal contained in $I$. Note that

$$I_\# = \{a \in I : \sigma^n(a) \in I, \text{ for all } n \geq 1\}.$$

A proper $\sigma$-ideal $P$ of $R$ is said to be *$\sigma$-prime* if $AB \subseteq P$ for any $\sigma$-ideals $A$ and $B$ implies either $A \subseteq P$ or $B \subseteq P$. This definition is standard when $\sigma$ is a derivation (see [3], [4], [5], [7]). But if $\sigma$ is an automorphism of $R$ there are some other definitions which are used (see [1], [2], [14], [15]). We have chosen here the above definition mainly to unify the exposition. But evident modifications can be introduced in order to get results in the other cases.

For example, if $\sigma$ is an automorphism of $R$ an ideal $I$ is said to be $\sigma$-invariant if $\sigma(I) = I$. A definition of $\sigma$-prime ideals using $\sigma$-invariant ideals instead of $\sigma$-ideals is used in [1] and [2]. Then we may say that $\sigma$ is locally integral if the smallest $\sigma$-invariant ideal of $R$ containing $a$ is finitely generated, for every $a \in R$. With this new approaching one can get similar properties to those we will prove in this paper.

A proper $\sigma$-ideal $P$ of $R$ is said to be *quasi-prime* if there exists a prime ideal $Q$ of $R$ such that $P = Q_\#$. Quasi-prime ideals of differential rings may be characterized in several useful ways and they play an important role in differential algebra (see [8], [9], [10], [11], [12]).

It is not difficult to see that every quasi-prime ideal is $\sigma$-prime and that if $R$ is a Noetherian ring these two notions coincide [11]. Also, it is well-known that if $\sigma$ is a derivation, then there are $\sigma$-prime ideals which are not quasi-prime ([10], [5], [3]). It is easy to give an example when $\sigma$ is an automorphism too.

Now we prove the following

**Proposition 1.2.** *Let $P$ be a $\sigma$-ideal of $R$. If $\sigma$ is locally integral, then $P$ is $\sigma$-prime if and only if $P$ is quasi-prime.*

*Proof.* Assume that $P$ is $\sigma$-prime and consider the family of ideals $A$ of $R$ with $A_\# = P$. If $(A_\iota)_{\iota\in\Omega}$ is a chain of ideals of this type and we put $A = \bigcup_{\iota\in\Omega} A_\iota$, then $P = A_\#$ follows since $\sigma$ is locally integral. Therefore there exists an ideal $Q$ of $R$ which is maximal with respect to $Q_\# = P$. It is easy to see that $Q$ is prime. The proof is complete because the converse is always true.

If $A$ is an ideal of $R$ we denote by $r(A)$ the radical of $A$, i.e., the intersection of all prime ideals of $R$ containing $A$. The ideal $A$ is said to be a radical ideal (or semiprime) if $r(A) = A$. Similarly, a $\sigma$-ideal $I$ is said to be $\sigma$-semiprime if $I$ equals the intersection of all the $\sigma$-prime ideals of $R$ containing $I$.

As an immediate consequence of Proposition 1.2 we obtain

**Corollary 1.3.** *If $\sigma$ is locally integral and $I$ is a $\sigma$-semiprime (in particular, $\sigma$-prime) ideal, then $r(I)_\# = I$.*

For the next Corollary we need the following

**Lemma 1.4.** *Let $d$ be a derivation of $R$. Then every radical $d$-prime ideal is prime.*

*Proof.* First observe that if $A$ is a radical $d$-ideal of $R$ and $x \in R$, then the ideal $(A : x) = \{a \in R : ax \in A\}$ is again a $d$-ideal. Assume now that $P$ is a radical $d$-prime ideal and for $x, y \in R$ we have $xy \in P$. Therefore $y \in (P : x)$ and so $[y] \subseteq (P : x)$. Consequently $x \in (P : [y])$, hence $[x] \subseteq (P : [y])$ and it follows that $[x][y] \subseteq P$. Thus either $x \in P$ or $y \in P$ and we are done.

**Corollary 1.5.** *Assume that $R$ contains the field $\mathbf{Q}$ of rational numbers* *and $d$ is a locally integral derivation of $R$. Then every $d$-prime ideal of $R$ is prime.*

*Proof.* Suppose that $P$ is a $d$-prime ideal of $R$. It is well-known that the radical $r(P)$ is a $d$-ideal and, by Corollary 1.3, $P = r(P)_\# = r(P)$. Then $P$ is radical and so prime, by Lemma 1.4.

**2. Examples and Contraexamples.** We begin this section with the following extension of ([3], Theorem 4.2). Note that in [3] the result was proved for a differential ring and $n = 1$ (see Theorem 4.1).

**Theorem 2.1.** *Let $R = K[a_1,\dots,a_n]$ a finitely generated algebra over $K$ and $\sigma : R \to R$ either a $K$-derivation or a $K$-endomorphism of $R$. Then $\sigma$ is locally integral.*

*Proof.* Take any $b \in R$ and choose polynomials $g, f_1,\dots,f_n$ in $K[X_1,\dots, X_n]$ such that $b = g(a_1,\dots,a_n)$ and $\sigma(a_i) = f_i(a_1,\dots,a_n)$, for $i = 1,\dots,n$. Denote by $k$ the smallest subring of $K$ containing all the coefficients of the polynomials $g, f_1,\dots,f_n$, and by $S$ the ring $k[a_1,\dots,a_n]$. Then $b \in S$ and $\sigma(S) \subseteq S$. Put $r = \sigma/S$. So $r$ is locally integral since $S$ is Noetherian. Thus there exist an integer $m$ and a finite set $\{b_0,\dots, b_{m-1}\}$ of elements of $S$ such that

$$r^m(b) + b_{m-1}r^{m-1}(b) + \cdots + b_0 b = 0.$$

Now it is clear that $b \in \mathbb{I}(\sigma)$.

The following example shows that a similar result does not hold for an algebra which is not finitely generated.

**Example 2.2.** *Let $R = K[X_1, X_2,\dots]$ a polynomial ring in infinitely many indeterminates and let $\sigma$ be the $K$-derivation (or $K$-endomorphism) of $R$ defined by $\sigma(X_n) = X_{n+1}$, for $n \geq 1$. Then $[X_1] = (X_1, X_2,\dots)$ is not finitely generated and so $\sigma$ is not locally finite.*

Now, let $R$ be the power series ring $K[[X_1,\dots,X_n]]$ and $d$ a $K$-derivation of $R$. If $K$ is a Noetherian ring, then $d$ is locally integral. Also, if $K$ is a ring of characteristic a prime integer $p$, then $d$ is integral ([13], Theorem 4.1) and so $d$ is locally integral. It is natural to ask whether $d$ is always locally integral. The following example shows this is not true even for $n = 1$.

**Example 2.3.** *Let $K = \mathbf{Z}[Y_1, Y_2, \ldots]$ a polynomial ring in infinitely many indeterminates over the ring of integer numbers $\mathbf{Z}$ and put $R = K[[X]]$. Then the derivation $d = \frac{\partial}{\partial X}$ is not locally integral.*

*Proof.* Put $f = \sum_{i=1}^{\infty} Y_i X^i \in R$. We show that $f \notin \mathbf{I}(d)$.

Observe that

$$d^n(f) = n!\,Y_n + a_{1n} Y_{n+1} X + a_{2n} Y_{n+2} X^2 + \cdots,$$

for every $n \geq 1$, where $a_{in} \in \mathbf{Z}$.

Suppose that $f \in \mathbf{I}(d)$. Then

$$d^{n+1}(f) = b_0 f + b_1 d(f) + \cdots + b_n d^n(f),$$

for some $b_0, b_1, \ldots, b_n$ in $R$. Comparing the constant terms in the above equality we get

$$(n+1)!\,Y_{n+1} = a_1 Y_1 + \cdots + a_n Y_n,$$

for some $a_1, \ldots, a_n$ in $K$. This is clearly a contradiction.

Next we will give a similar example as above for automorphisms. But for this we need the following results which are also interesting by their own.

Let $R$ be an algebra over the field of rational numbers $\mathbf{Q}$ and $d : R \to R$ a derivation of $R$. We denote by $S = R[[X]]$ the power series ring in one indeterminate $X$. The derivation $d$ can be extended to a derivation $d^* : S \to S$ by $d^*(\sum_{i=0}^{\infty} a_i X^i) = \sum_{i=0}^{\infty} d(a_i) X^i$. Hereafter we denote $d^*$ simply by $d$.

It is well-known that the derivation $d$ induces an automorphism $exp(d) : S \to S$ defined by $exp(d)(f) = \sum_{i=0}^{\infty} \frac{X^i}{i!} d^i(f)$, for every $f \in S$. We put $exp(d) = \sigma$. We will keep these notations until the end of the proof of the following

**Theorem 2.4.** *The derivation $d$ is locally integral if and only if $R \subseteq \mathbf{I}(\sigma)$.*

To prove the theorem we need a preparation. To avoid writing too much we will give a sketch of the proof omitting several computations.

First, suppose that $r$ is an automorphism of the ring $R$ and denote by $R[t ; r]$ the skew polynomial ring of automorphism type. It is clear that for any monic polynomial $h(t) \in R[t ; r]$ there exists a monic polynomial $g(t) \in R[t ; r]$ such that $h(t) = g(t-1)$. Therefore, for any $a \in R$ we have $h(r)(a) = 0$ if and only if $g(r-1)(a) = 0$ (we will denote by 1 the identity mapping).

So the following is clear.

**Lemma 2.5.** *$a \in \mathbb{I}(r)$ if and only if there exists a monic polynomial $g(t) \in R[t\,;r]$ such that $g(r-1)(a)=0$.*

Let $f \in S = R[[X]]$ and $\sigma = exp(d)$. We have $(\sigma-1)(f) = \sum_{j=1}^{\infty} \frac{X^j}{j!} d^j(f)$. Denote by $\lambda_{s,r}$ the following rational number

$$\lambda_{s,r} = \sum_{\substack{j_1+\cdots+j_s=r \\ j_i \geq 1}} \frac{1}{j_1!\cdots j_s!}$$

By induction we easily get

**Lemma 2.6.** *For any $s \geq 1$, $(\sigma-1)^s(f) = \sum_{r=s}^{\infty} \lambda_{s,r} X^r d^r(f)$, where $\lambda_{s,r} \in \mathbb{Q}$ and $\lambda_{s,s}=1$.*

We will need also the following

**Lemma 2.7.** *Assume that $a \in R$ and $d^n(a)+\sum_{j=0}^{n-1} a_i d^i(a)=0$, for some given elements $a_0,\dots,a_{n-1}$ in $R$. Then for any integer $m \geq n$ and elements $c_0,\dots,c_m \in R$, there exist $y_0,\dots,y_{n-1}$ in $R$ with $\sum_{i=0}^{n-1} y_i d^i(a)+\sum_{i=0}^{m} c_j d^j(a)=0$.*

*Proof.* If $m=n$ it is enough to choose $y_i = c_n a_i - c_i$, for $i=0,\dots,n-1$.

Since

$$d^{n+1}(a) = -\sum_{i=0}^{n-1} (d(a_i)d^i(a)+a_i d^{i+1}(a)),$$

we see that for $s>n$ there exist $b_0,\dots,b_{s-1}$ in $R$ with

$$d^s(a)+\sum_{i=0}^{s-1} b_i d^i(a)=0.$$

Now the proof can easily be completed by induction.

*Proof of Theorem* 2.4. Assume that $R \subseteq \mathbb{I}(\sigma)$ and take any $a \in R$. By Lemma 2.5 there exists a monic polynomial $g(t) \in S[t\,;\sigma]$ with $g(\sigma-1)(a)=0$. So, from Lemma 2.6 we get a relation of the following type

$$\sum_{i=n}^{\infty} \lambda_{n,i} d^i(a)X^i + f_{n-1}\left(\sum_{i=n-1}^{\infty} \lambda_{n-1,i} d^i(a)X^i\right)+ \cdots + f_1\left(\sum_{i=1}^{\infty} \lambda_{1,i} d^i(a)X^i\right)+f_0 a = 0, \tag{1}$$

where $f_i = \sum_{j=0}^{\infty} r_{ij}X^j$, $i = 0,\dots,n-1$ ($r_{ij} \in R$). We get $r_{00}a = 0$ and $r_{10}d(a) = -r_{01}a$. Hence it follows by induction that $r_{10}d^s(a) = \sum_{i=0}^{s-1} c_{1si}d^i(a)$, for every $s \geq 1$, where $c_{1si} \in R$.

Take any $u < n$. Considering the coefficient of $X^u$ in (1) and using an induction argument we prove that for all $s \geq u$ we have

$$r_{u0}d^S(a) = \sum_{i=0}^{S-1} c_{usi}d^i(a), \text{ where } c_{usi} \in R. \tag{2}$$

Now, consider the coefficient of $X^n$ in the relation (1) and use the relations (2) to substitute the terms of the form $\lambda_{j,n}r_{j0}d^n(a)$. We get a relation of the type $d^n(a)+b_{n-1}d^{n-1}(a)+\cdots+b_1d(a)+b_0a = 0$, where $b_i \in R$. This gives $a \in \mathbb{I}(d)$ and so $d$ is locally integral.

Conversely, assume that $d$ is locally integral and take again any $a \in R$. Then there exist $a_0,\dots,a_{n-1}$ in $R$ such that

$$d^n(a)+\sum_{i=0}^{n-1} a_id^i(a) = 0. \tag{3}$$

We have to find a solution for the following equation

$$\begin{aligned}(\sigma-1)^n(a) + f_{n-1}(\sigma-1)^{n-1}(a)+ \\ \cdots+f_1(\sigma-1)(a)+f_0a = 0, \\ \text{where } f_i = \sum_{j=0}^{\infty} r_{ij}X^j,\ i = 0,\dots,n-1\ (r_{ij} \in R).\end{aligned} \tag{4}$$

We put $r_{ij} = 0$ for $i+j < n$, $r_{n-11} = a_{n-1},\dots,r_{1n-1} = a_1$ and $r_{0n} = a_0$. So all the coefficients of $X^j$ in the first member of (4) are zero, for $0 \leq j \leq n$. Finally, using (3), Lemma 2.7 and an induction argument we obtain all the coefficients $r_{ij}$ in order to get a solution for (4). The proof is complete.

Now we are in position to give the following

**Example 2.8.** *Let $R$ and $d$ be as in Example 2.3 and let $\sigma$ be the associated automorphism $\exp(d) : R[[Y]] \to R[[Y]]$. Then $\sigma$ is an $S$-automorphism of the ring $S[[X]]$, where $S = K[[Y]]$, since $\sigma(Y) = Y$. By Theorem 2.4, $\sigma$ is not locally integral.*

Suppose $d$ is a derivation of a ring $R$. Then we put $Nil(d) = \{a \in R :$ there exists $n \geq 0$ with $d^n(a) = 0\}$. It is easy to see that $Nil(d)$ is a subring of $R$. The derivation $d$ is said to be locally nilpotent if $Nil(d) = R$. Clearly a locally nilpotent derivation is locally integral.

The fact that $Nil(d)$ is a subring of $R$ is a very useful fact concerning locally nilpotent derivations. There is a natural question to be risen : is $\mathbb{I}(d)$ a subring of $R$ when $d$ is a derivation? We finish this section with two examples showing that the elements $a+b$ and $ab$ are not, in general, in $\mathbb{I}(d)$ when $a, b \in \mathbb{I}(d)$.

**Example 2.9.** *Let $K$ be a ring and $R = K[X_1, X_2, \ldots]$ a polynomial ring in infinitely many indeterminates. Define the $K$-derivation $d$ of $R$ by $d(X_n) = X_n X_{n+1}$, for $n = 1,2,\ldots.$ Then $X_1, X_2 \in \mathbb{I}(d)$ and $X_1+X_2 \notin \mathbb{I}(d)$.*

*Proof.* It is clear that $X_1, X_2 \in \mathbb{I}(d)$. Consider the sequence of elements of $R$ defined by $a_n = X_1^n(X_1+X_{n+2})$, $n = 0,1,2,\ldots.$ First we prove

$$d(a_n) = X_1^{n-1} a_0 + (nX_2+X_{n+3})a_n - a_{n+1}, \text{ for any } n \geq 0. \tag{5}$$

In fact

$$\begin{aligned} d(a_n) &= nX_1^{n-1} X_1X_2(X_1+X_{n+2}) + X_1^n(X_1X_2+X_{n+2}X_{n+3}) \\ &= nX_2a_n + X_1^{n+1} X_2 + X_1^n X_{n+2} X_{n+3} \\ &= nX_2a_n + X_1^{n+1} a_0 + X_{n+3}a_n - X_1^{n+1}(X_1+X_{n+3}) \\ &= X_1^{n+1}a_0 + (nX_2+X_{n+3})a_n - a_{n+1}. \end{aligned}$$

Let $A$ be the ideal of $R$ generated by $(a_0, a_1, a_2, \ldots)$. By (5), $d(A) \subseteq A$ and so $[a_0] \subseteq A$. Also $a_1 = (X_1+X_3)a_0 - d(a_0) \in [a_0]$. Therefore we easily get $A = [a_0]$ using (5) and an induction argument.

Suppose $[a_0]$ is finitely generated. Then there exists $n \geq 2$ with $[a_0] = (a_0, \ldots, a_n)$. Thus $a_{n+1} = b_0a_0 + \cdots + b_na_n$ for some $b_i \in R$, $i = 0, \ldots, n$. Applying to this relation the $K$-homomorphism of rings $\varphi: R \to K$ defined by $\varphi(X_1) = 1$, $\varphi(X_2) = \cdots = \varphi(X_{n+2}) = -1$ and $\varphi(X_j) = 0$ for $j \geq n+3$, we get the contradiction $1 = 0$. Therefore $X_1+X_2 = a_0 \notin \mathbb{I}(d)$.

**Example 2.10.** *Let $\mathbb{Z}_2$ be the field of two elements and put $R = \mathbb{Z}_2[X_1, X_2, \ldots]$ the polynomial ring in infinitely many indeterminates. Let $d$ be the derivation of $R$ defined by $d(X_1) = X_1X_2+1$, $d(X_2) = X_2X_3+1$ and $d(X_n) = X_{n+1}$, for $n \geq 3$. Then $X_1, X_2 \in \mathbb{I}(d)$ and $X_1X_2 \notin \mathbb{I}(d)$.*

*Proof.* The elements $X_1$ and $X_2$ belong to $\mathbb{I}(d)$ because for $i = 1,2$ we have

$$d^2(X_i) = d(X_{i-1})X_i + X_{i+1}\, d(X_i).$$

We define a sequence as follows: $a_1 = X_1X_2$, $a_2 = X_1+X_2$, $a_n = X_2X_n + \varepsilon_n X_{n-1}$, for $n \geq 3$, where $\varepsilon_n = 0$ if $2 \nmid n$ and $\varepsilon_n = 1$ if $2 \mid n$.

It is not difficult to check that for every $n \geq 1$ there exist $b_n \in R$ and $s \leq n$ such that $d(a_n) = b_n a_s + a_{s-1}$. Therefore $[a_1] = (a_1, a_2, \ldots)$.

Suppose that $[a_1]$ is a finitely generated ideal of $R$. Then there exists $n \geq 2$ such that $[a_1] = (a_1, a_2, \ldots, a_n)$. We may assume $2 \nmid n$ and $n = 2l+1 \geq 3$. Therefore

$$a_{2l+2} = c_1 a_1 + \cdots + c_{2l+1} a_{2l+1},$$

for some $c_1, \ldots, c_{2l+1}$ in $R$. Applying to this relation the ring homomorphism $\varphi : R \to \mathbb{Z}_2$ defined by $\varphi(X_{2l+1}) = 1$ and $\varphi(X_j) = 0$ for $j \neq 2l+1$, we get the contradiction $1 = 0$. So $X_1 X_2 = a_1 \notin \mathbb{I}(d)$.

**3. Spectral spaces.** For any ring $R$ we denote by $Spec(R)$ the set of all prime ideals of $R$. It is well-known that $Spec(R)$ is a topological space (with the Zariski topology) in which closed sets are of the form $V(E) = \{P \in Spec(R) : P \supseteq E\}$, and open sets are of the form $D(E) = \{P \in Spec(R) : P \not\supseteq E\}$, where $E$ is an arbitrary subset of $R$.

A topological space is said to be spectral [6] if it is $T_0$ and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every non-empty irreducible closed subset has a generic point. For any ring $R$, $Spec(R)$ is spectral. Moreover, if $X$ is an spectral topological space there exists a ring $S$ such that $X$ is homeomorphic to $Spec(S)$([6], Theorem 6).

For a differential ring $(R, d)$, a corresponding question has been considered in [9]. It is proved that $Quas(R)$, the set of all quasi-prime ideals of $R$ with the evident topology, is spectral if $R$ satisfies the ascending chain condition on the so called $q$-radical ideals ([9], Theorem 2.9).

The purpose of this section is to prove a result for $(R, \sigma)$, where $\sigma$ is either a derivation or an endomorphism which is locally integral, without any further assumption.

Assume that $R$ is a $K$-algebra $\sigma : R \to R$ is either a $K$-derivation or a $K$-endomorphism and consider $Spec_\sigma(R)$, the set of all $\sigma$-prime ideals of $R$. The set $Spec_\sigma(R)$ is always non-empty because every maximal $\sigma$-ideal is $\sigma$-prime. If $E$ is a subset of $R$ denote

$$V_\sigma(E) = \{P \in Spec_\sigma(R) : P \supseteq E\} \text{ and}$$
$$D_\sigma(E) = \{P \in Spec_\sigma(R) : P \not\supseteq E\}.$$

It is clear that $Spec_\sigma(R)$ is a topological space in which closed and open sets are of the form $V_\sigma(E)$ and $D_\sigma(E)$, respectively. We will prove the following

**Theorem 3.1.** *If $\sigma$ is locally integral, then $Spec_\sigma(R)$ is a spectral space.*

Combining this with ([6], Theorem 6) we have

**Corollary 3.2.** *If $\sigma$ is locally integral, then there exists a ring $S$ such that the topological spaces $Spec_\sigma(R)$ and $Spec(S)$ are homeomorphic.*

To prove Theorem 3.1 we start with the following

**Lemma 3.3.** *If $\sigma$ is locally integral and $x \in R$, then $D_\sigma(x)$ is a quasi-compact open set, where $D_\sigma(x) = D_\sigma(\{x\})$.*

*Proof.* Define $\varphi: Spec(R) \to Spec_\sigma(R)$ by $\varphi(P) = P_{\neq}$, for any $P \in Spec(R)$. By Proposition 1.2 we know that $\varphi$ is a surjective mapping. It is easy to check that if $E$ is a subset of $R$ then $\varphi^{-1}(D_\sigma(E)) = D([E])$. So $\varphi$ is a continuons mapping. Since $\sigma$ is locally integral the ideal $[x]$ is finitely generated and so the set $D([x])$ is quasi-compact. Hence $D_\sigma(x) = \varphi(D[x])$ is also quasi-compact.

**Corollary 3.4.** *Let $E$ be a subset of $R$. If $\sigma$ is locally integral, then the set $D_\sigma(E)$ is quasi-compact if and only if $D_\sigma(E) = D_\sigma(E_0)$, for some finite subset $E_0$ of $R$.*

*Proof.* If $E_0 = \{x_1, \dots, x_n\}$ is a finite subset of $R$, then $D_\sigma(E_0) = \bigcup_{i=1}^{n} D_\sigma(x_i)$ is quasi-compact by Lemma 3.3. The converse follows easily from the equality $D_\sigma(E) = \bigcup_{x \in E} D_\sigma(x)$.

**Lemma 3.5.** *Assume that $\sigma$ is locally integral. Then*

(i) *The space $Spec_\sigma(R)$ is quasi-compact.*

(ii) *Quasi-compact open sets form an open basis of $Spec_\sigma(R)$.*

(iii) *Quasi-compact open sets are closed under finite intersection.*

*Proof.* (i) and (ii) follow from Lemma 3.3 since $Spec_\sigma(R) = D_\sigma(1)$ and $D_\sigma(E) = \bigcup_{x \in E} D_\sigma(x)$.

(iii) First note that every ideal of the form $[a_1, \dots, a_n]$ is finitely generated since $[a_1, \dots, a_n] = [a_1] + \cdots + [a_n]$ and $\sigma$ is locally integral. Suppose that $D_i = D_\sigma(E_i)$ are quasi-compact open sets, $i = 1, \dots, n$. By Corollary 3.4 we may assume that $E_1, \dots, E_n$ are finite. So the ideal $I = [E_1] \cdots [E_n]$ is finitely generated and $D_1 \cap \cdots \cap D_n = D_\sigma(I)$ is quasi-compact.

**Lemma 3.6.** *If $F$ is a non-empty irreducible closed subset of $Spec_\sigma(R)$, then there exists a $\sigma$-prime ideal $P$ of $R$ such that $F = V_\sigma(P)$.*

*Proof.* Since $V_\sigma(E) = V_\sigma([E])$ we may assume there exists a $\sigma$-ideal $A$ of $R$ with $F = V_\sigma(A)$. Using the Zorn's Lemma it is easy to see that there exists a $\sigma$-ideal $P$ such that $F = V_\sigma(P)$ and $P$ is maximal among the $\sigma$-ideals $A$ with $F = V_\sigma(A)$. Then $P \neq R$ because $F$ is non-empty. We show that $P$ is $\sigma$-prime. In fact, let $A$ and $B$ be $\sigma$-ideals of $R$ such that $A \supseteq P$, $B \supseteq P$ and $AB \subseteq P$. So $F = V_\sigma(P) = V_\sigma(A) \cup V_\sigma(B)$. Since $F$ is irreducible either $V_\sigma(A) = F$ or $V_\sigma(B) = F$ and it follows that either $A = P$ or $B = P$.

*Proof of Theorem* 3.1. It is clear that $Spec_\sigma(R)$ is $T_0$. By Lemma 3.6 every non-empty irreducible closed subset has a generic point. The other three properties hold by Lemma 3.5.

Finally, assume that $R_1$ and $R_2$ are $K$-algebras, $\sigma_1 : R_1 \to R_1$ and $\sigma_2 : R_2 \to R_2$ are both $K$-derivations or $K$-endomorphisms and $f : R_1 \to R_2$ is a homomorphism of $K$-algebras with $f \circ \sigma_1 = \sigma_2 \circ f$. Denote by $\bar{f} : Spec_{\sigma_2}(R_2) \to Spec_{\sigma_1}(R_1)$ the mapping defined by $\bar{f}(P) = f^{-1}(P)$, for all $P \in Spec_{\sigma_2}(R_2)$. It is easy to check that $\bar{f}$ is a (well-defined) continuous mapping and that the inverse images of quasi-compact open sets are quasi-compact. Therefore, using again ([6], Theorem 6) we have

**Corollary 3.7.** *Under the above assumptions, if $\sigma_1$ and $\sigma_2$ are locally integral then there exist rings $S_1$ and $S_2$ and a ring homomorphism $h : S_1 \to S_2$ such that the following diagram is commutative:*

$$\begin{array}{ccc} Spec_{\sigma_2}(R_2) & \xrightarrow{\bar{f}} & Spec_{\sigma_1}(R_1) \\ \wr\| & & \wr\| \\ Spec(S_2) & \xrightarrow{\bar{h}} & Spec(S_1), \end{array}$$

*where the vertical arrows are canonical and $\bar{h}(Q) = h^{-1}(Q)$, for any $Q \in Spec(S_1)$.*

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M. FERRERO
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO
RIO GRANDE DO SUL, 90049-PORTO ALEGRE, BRAZIL

A. NOWICKI
INSTITUTE OF MATHEMATICS
N. COPERNICUS UNIVERSITY
UL. CHOPINA 12/18, 87-100-TORUŃ, POLAND

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