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A GENERALIZATION OF A THEOREM OF POSNER

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Throughout, R will represent a ring with center C, σ and τ ring endomorphisms of R, and A a non-zero ideal of R. A mapping $d: x \mapsto x'$ of R into itself is called a *generalized* (σ, τ)*-derivation* of R if $(x+y)'-x'-y' \in C$ and $(xy)'-x'\sigma(y)-\tau(x)y' \in C$ for all $x, y \in R$. A generalized $(1,1)$-derivation of R is called a *generalized derivation*. Needless to say, any (usual) derivation of R is a generalized derivation; in case R is commutative, any mapping of R into itself is a generalized derivation. Given a subset S of R, we put $V_R(S) = \{x \in R \mid xs = sx \text{ for all } s \in S\}$. For any $x, y \in R$, we write $[x, y] = xy-yx$.

Our present objective is to prove the following theorem which generalizes a theorem of Posner [1, Theorem 2].

Theorem 1. *Let R be a prime ring, A a non-zero ideal of R, and σ and τ surjective ring endomorphisms of R such that $\sigma(A) \neq 0$ and $\tau(A) \neq 0$.*

(1) *If* $\operatorname{char} R \neq 2$, *then the following are equivalent*:

1) *R is commutative.*

2) *There exists a generalized (σ, τ)-derivation $d: x \mapsto x'$ of R into itself such that $A' \neq 0$, $[a', \sigma(a)] \in C$ and $[a', \tau(a)] \in C$ for all $a \in A$.*

(2) *If* $\operatorname{char} R = 2$, *then the following are equivalent*:

1) *R is commutative.*

2) *There exists a generalized (σ, σ)-derivation $d: x \mapsto x'$ of R into itself such that $A' \neq 0$ and $[a', \sigma(a)] \in C$ for all $a \in A$.*

In preparation for proving our theorem, we state two lemmas.

Lemma 1. *Let R be a prime ring, A a non-zero ideal and K a non-zero right (or left) ideal of R. Then*

(1) $V_R(K) = C$.

(2) *If $xy = 0$ and $x \in C$, then $x = 0$ or $y = 0$.*

(3) *If $xy \in C$ and $x \in C$, then $x = 0$ or $y \in C$.*

(4) *If $xAy = 0$ for $x, y \in R$, then $x = 0$ or $y = 0$.*

(5) *If K is commutative, then R is commutative.*

Proof. (1) Let K be a right ideal of R. Then, for any $x \in V_R(K)$,

$k \in K$ and $y \in R$, it follows that $0 = [x, ky] = k[x, y]$, and so $K[x, R] = 0$. Since R is prime and $K \neq 0$, we get $[x, R] = 0$, that is, $x \in C$.

(2) This is almost evident.

(3) Let $xy \in C$ and $x \in C$, then it follows that $0 = [xy, r] = x[y, r]$ for all $r \in R$. Hence we have $xs[y, r] = x[y, sr] = 0$ for all $s \in R$, and so we get $xR[y, R] = 0$. Since R is prime, we have either $x = 0$ or $y \in C$.

(4) This is almost evident.

(5) Since $K \subseteq V_R(K) = C$ by (1), we obtain $R = V_R(K) = C$.

Lemma 2. *Let R be a prime ring, A a non-zero ideal of R, and σ and τ surjective ring endomorphisms of R such that $\sigma(A) \neq 0$ and $\tau(A) \neq 0$. Then the following are equivalent:*

1) *R is commutative.*

2) *There exists a generalized (σ, τ)-derivation $d: x \mapsto x'$ such that $A' \neq 0$ and $[a', \tau(a)] = 0$ for all $a \in A$.*

3) *There exists a generalized (σ, τ)-derivation $d: x \mapsto x'$ such that $A' \neq 0$ and $[a', \sigma(a)] = 0$ for all $a \in A$.*

Proof. The implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) are evident, and the proof of 3) $\Rightarrow$ 1) is quite similar to that of 2) $\Rightarrow$ 1). So we shall only prove 2) $\Rightarrow$ 1).

For any $a, b \in A$, it follows that $0 = [(a+b)', \tau(a+b)] = [a', \tau(b)] + [b', \tau(a)]$, so that $[a', \tau(b)] = [\tau(a), b']$. Since $a'[\sigma(b), \tau(a)] = [(ab)', \tau(a)] - \tau(a)[b', \tau(a)] = [\tau(ab), a'] - \tau(a)[\tau(b), a'] = 0$, it follows that $a'\sigma(s)[\sigma(b), \tau(a)] = a'[\sigma(sb), \tau(a)] - a'[\sigma(s), \tau(a)]\sigma(b) = 0$ for all $s \in A$, and so $a'\sigma(A)[\sigma(A), \tau(a)] = 0$. By Lemma 1(2) and (1), we have either $a' = 0$ or $\tau(a) \in V_R(\sigma(A)) = C \subseteq V_R(A')$. If $a' = 0$, then we have $[\tau(a), b'] = [a', \tau(b)] = 0$ for all $b \in A$, that is, $\tau(a) \in V_R(A')$. Hence, in either case, we have $\tau(A) \subseteq V_R(A')$, whence $A' \subseteq V_R(\tau(A)) = C$. We can easily see that $0 = [\sigma(b), (ab)'] = [\sigma(b), \tau(a)]b' = b'[\sigma(b), \tau(a)]$ for all $a, b \in A$. Also we have $0 = [(b^2)', \tau(a)] = b'\tau[b, a]$. Since $0 = [(bs)', \tau(a)] = b'[\sigma(s), \tau(a)] + \tau[b, a]s'$ for all $s \in A$, we get $0 = (b')^2[\sigma(s), \tau(a)] + b'\tau[b, a]s' = (b')^2[\sigma(s), \tau(a)]$. But, since $b' \in C$, we get either $b' = 0$ for all $b \in A$ or $[\tau(a), \sigma(A)] = 0$ for all $a \in A$. Since $A' \neq 0$, we conclude $\tau(A) \subseteq C$ by Lemma 1(1). Hence R is commutative by Lemma 1(5).

Corollary 1. *Let R be a prime ring, A a non-zero ideal of R, and σ and*

τ surjective ring endomorphisms of R such that $\sigma(A) \neq 0$ and $\tau(A) \neq 0$. If R has a generalized (σ, τ)-derivation such that $A' \neq 0$ and $[a', \tau[a, b]] = 0$ for all $a, b \in A$, then R is commutative.

Proof. Let a, b, s be arbitrary elements of A. Then we have $0 = [a', \tau[a, ba]] = \tau[a, b][a', \tau(a)]$. So we have $0 = \tau[a, bs][a', \tau(a)] = \tau[a, b]\tau(s)[a', \tau(a)]$, and so $\tau[a, b]\tau(A)[a', \tau(a)] = 0$. Hence we have either $\tau[a, b] = 0$ for all $b \in A$ or $[a', \tau(a)] = 0$ by Lemma 1(3). In case $\tau[a, b] = 0$ for all $b \in A$, we get $[\tau(a), \tau(A)] = 0$, and so $\tau(a) \in C$ by Lemma 1(1). Hence, in either case, we have $[a', \tau(a)] = 0$. Therefore R is commutative by Lemma 2.

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. (1) It suffices to prove that 2) implies 1). Let a, b be arbitrary elements of A. Since $[a', \tau(b)] + [b', \tau(a)] \in C$, we get

$$(*) \qquad \{3\tau(a) + \sigma(a)\}[a', \tau(a)] + a'[\sigma(a), \tau(a)] = [a', \tau(a^2)] + [(a^2)', \tau(a)] \in C.$$

Hence we have $0 = [a', 3\tau(a) + \sigma(a)][a', \tau(a)] + a'[a', [\sigma(a), \tau(a)]] = \{3[a', \tau(a)] + [a', \sigma(a)]\}[a', \tau(a)]$. By Lemma 1(3), we have either $[a', \tau(a)] = 0$ or $3[a', \tau(a)] + [a', \sigma(a)] = 0$. Similarly we have

$$(**) \qquad \{3\sigma(a) + \tau(a)\}[a', \sigma(a)] + a'[\tau(a), \sigma(a)] \in C,$$

since $[a', \sigma(a)] \in C$. So we have either $[a', \sigma(a)] = 0$ or $3[a', \sigma(a)] + [a', \tau(a)] = 0$. Now we claim that $[a', \tau(a)] = 0$. By the argument above, the following four cases occur.

(i) $[a', \tau(a)] = 0$ and $[a', \sigma(a)] = 0$.
(ii) $[a', \tau(a)] = 0$ and $3[a', \sigma(a)] + [a', \tau(a)] = 0$.
(iii) $3[a', \tau(a)] + [a', \sigma(a)] = 0$ and $3[a', \sigma(a)] + [a', \tau(a)] = 0$.
(iv) $3[a', \tau(a)] + [a', \sigma(a)] = 0$ and $[a', \sigma(a)] = 0$.

To prove our claim, we must only consider cases (iii) and (iv).
Case(iii) : Since $\operatorname{char} R \neq 2$, we can easily see that $[a', \sigma(a)] = [a', \tau(a)]$, and so we have $[a', \tau(a)] = 0$.
Case(iv) : Obviously, $3[a', \tau(a)] = 0$. If $\operatorname{char} R \neq 3$, obviously we have $[a', \tau(a)] = 0$. So assume $\operatorname{char} R = 3$. Then $a'[\sigma(a), \tau(a)] + \sigma(a)[a', \tau(a)] \in C$ by $(*)$, and $a'[\tau(a), \sigma(a)] \in C$ by $(**)$. Hence $\sigma(a)[a', \tau(a)] \in C$, and so we get either $\sigma(a) \in C$ or $[a', \tau(a)] = 0$ by Lemma 1(3). If $\sigma(a) \in C$, then $C \ni [a', \sigma(b)] + [b', \sigma(a)] = [a', \sigma(b)]$ for all $b \in A$. Since $\operatorname{char} R$

$= 3$, we have $C \ni [a', \sigma(b^2)] = 2\sigma(b)[a', \sigma(b)] = -\sigma(b)[a', \sigma(b)]$. Then we can easily see that $[a', \sigma(b)]^2 = 0$, and so $[a', \sigma(b)] = 0$ for all $b \in A$. Hence we have $a' \in V_R(\sigma(A)) = C$ by Lemma 1(1), so that $[a', \tau(a)] = 0$. Therefore, in any case, we have $[a', \tau(a)] = 0$ for all $a \in A$. Thus R is commutative by Lemma 2.

(2) Assume $\operatorname{char} R = 2$. It suffices to prove that 2) implies 1). Let a, b, c, e, s be arbitrary elements of A. Since $[a', \sigma(b)] + [b', \sigma(a)] \in C$, $(b^2)' - [b', \sigma(b)] = (b^2)' - \{b'\sigma(b) + \tau(b)b'\} \in C$ and $[b', \sigma(b)] \in C$, we have $(b^2)' \in C$, and so $C \ni [a', \sigma(b^2)] + [(b^2)', \sigma(a)] = [a', \sigma(b^2)]$. Hence we have $[a', \sigma(b^4)] = 2[a', \sigma(b^2)]\sigma(b^2) = 0$. Since $[(a')^2, \sigma(b^2)] = 2a'[a', \sigma(b^2)] = 0$, it follows that $[a', \sigma(b^2)]^2 a' = \{a'\sigma(b^2)a'\sigma(b^2) + 2(a')^2\sigma(b^4) + \sigma(b^2)a'\sigma(b^2)a'\}a' = [\{\sigma(b^2)a'\}^2, a'] = [\{\sigma(b^2x)\}^2, a'] \in C$, where x is an element of R with $a' = \sigma(x)$. Thus, we get $[a', \sigma(b^2)]^3 = 2\sigma(b^2)[a', \sigma(b^2)]^2 a' = 0$. Since $0 = [a', \sigma(b+c)^2] = a'\sigma[b, c] + \sigma[b, c]a'$, we get $\sigma[b, c]a' = a'\sigma[b, c]$. Hence we have $\sigma[b, c][a', \sigma(c)] = a'\sigma[bc, c] - \sigma[bc, c]a' = 0$. So, linearlizing this on c, we obtain $\sigma[b, c][a', \sigma(e)] + \sigma[b, e][a', \sigma(c)] = 0$ for all $a, b, c, e \in A$. Since $[a', \sigma(c^2)] = 0$, we have $\sigma[b, c^2][a', \sigma(e)] = 0$. Hence we can easily see that $\sigma[b, c^2]\sigma(s)[a', \sigma(e)] = 0$, that is, $\sigma[b, c^2]\sigma(A)[a', \sigma(e)] = 0$. By Lemma 1(3), we have either $\sigma(c^2) \in V_R(\sigma(A)) = C$ for all $c \in A$ or $a' \in V_R(\sigma(A)) = C$ for all $a \in A$. If $A' \subseteq C$, then R is commutative by Lemma 2. Now we suppose that $\sigma(c^2) \in C$ for all $c \in A$. Since $\operatorname{char} R = 2$, we can easily see that $\sigma[a, b] = \sigma((a+b)^2) + \sigma(a^2) + \sigma(b^2) \in C$ and $\sigma[a, b]^3 = 2\sigma[a, ab]\sigma[ab, b] = 0$. Hence $\sigma[a, b] = 0$ and $\sigma(A)$ is commutative. Therefore R is commutative by Lemma 1(5).

Remark 1. Let R be a prime ring, σ, τ surjective ring endomorphisms of R and $d: x \mapsto x'$ a generalized (σ, τ)-derivation of R. As is easily seen, $0' \in C$, $0'\sigma(x) \in C$ and $\tau(x)0' \in C$ for all $x \in R$, so that $0'\sigma[x, y] = 0$ and $\tau[x, y]0' = 0$ for all $x, y \in R$. Thus, if $0' \neq 0$, then $R (= \sigma(R) = \tau(R))$ is commutative.

Similarly, we can easily see that if $R \ni 1$ and $1' \neq 0$, then R is commutative.

Remark 2. Let R be a prime ring, and A a nonzero ideal of R. As is well known, if d is a derivation of R such that $d(A) = 0$ then $d = 0$. However even if d is a generalized derivation of R such that $d(A) = 0$, d is not necessarily zero.

For example, consider $R = \boldsymbol{Z}$ and $A = 2\boldsymbol{Z}$. Then the mapping $d: \boldsymbol{Z} \to \boldsymbol{Z}$ defined by $d(n) = 1-(-1)^n$ for all $n \in \boldsymbol{Z}$, is a non-zero generalized derivation of $\boldsymbol{Z}$ and $d(A) = 0$.

Now, we give an example of a generalized $(\sigma, 1)$-derivation which is not a $(\sigma, 1)$-derivation.

Example 1. Let $R = \begin{pmatrix} \boldsymbol{Z}[x] & \boldsymbol{Z}[x]/2\boldsymbol{Z}[x] \\ 0 & \boldsymbol{Z}[x] \end{pmatrix}$. Let σ and δ be the mappings of R into itself defined as follows :

$$\sigma: \begin{pmatrix} f(x) & g(x) \\ 0 & h(x) \end{pmatrix} \mapsto \begin{pmatrix} f(-x) & g(-x) \\ 0 & h(-x) \end{pmatrix}, \text{ and}$$

$$\delta: \begin{pmatrix} f(x) & g(x) \\ 0 & h(x) \end{pmatrix} \mapsto \begin{pmatrix} f(x)-f(-x) & 0 \\ 0 & h(x)-h(-x) \end{pmatrix}.$$

Then δ is a $(\sigma, 1)$-derivation. Now, let ϕ and d be the mappings of R into itself defined as follows :

$$\phi: \begin{pmatrix} f(x) & g(x) \\ 0 & h(x) \end{pmatrix} \mapsto \begin{pmatrix} 2f(x) & 0 \\ 0 & 2h(x) \end{pmatrix}, \ d(\alpha) = \delta(\alpha)+\phi(\alpha) \text{ for all } \alpha \in R.$$

Then d is not a $(\sigma, 1)$-derivation, but is a generalized $(\sigma, 1)$-derivation with $[\alpha', \sigma(\alpha)] = 0$ for all $\alpha \in R$.

Finally, we state two questions.

Question 1. Does Theorem 1(1) remain valid without the assumption that $[a', \tau(a)] \in C$ for all $a \in A$?

Question 2. Does Theorem 1(1) remain valid even if $\operatorname{char} R = 2$?

References

[1] E. C. Posner : Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

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