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COMMUTATIVITY THEOREMS FOR ALGEBRAS AND RINGS

Dedicated to Professor Masatoshi Ikeda on his 65th birthday

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0. Introduction. In his papers [3] and [4], M. Chacron studied R-algebras A satisfying the following condition:

(C) For each $x, y \in A$, there exist $f(X)$, $g(X) \in X^2R[X]$ such that $[x-f(x), y-g(y)] = 0$.

More generally, in [16], W. Streb introduced the following condition:

(S) For each $x, y \in A$, there exists $f(X, Y) \in \mathrm{K}_3$ such that $[x, y] = f(x, y)$.

Recently, in [17], he gave a classification of non-commutative rings. In our previous paper [14], the main results of [17] have been extended to algebras as follows:

Theorem 0.1 ([14, Theorem 1.1]). *Let A be a non-commutative R-algebra ($A \neq Z$). Then there exists a factorsubalgebra of A which is of type* a)$_l$, a)$_r$, b), c), d), e) *or* f)*:*

a)$_l$ $(R/\mathfrak{m})^{(2)} = \begin{pmatrix} R/\mathfrak{m} & R/\mathfrak{m} \\ 0 & 0 \end{pmatrix}$, *where $\mathfrak{m}$ is a maximal ideal of R.*

a)$_r$ ${}^{(2)}(R/\mathfrak{m}) = \begin{pmatrix} 0 & R/\mathfrak{m} \\ 0 & R/\mathfrak{m} \end{pmatrix}$, *where $\mathfrak{m}$ is a maximal ideal of R.*

b) *A non-commutative trivial extension $T \ltimes M$, where T is an R-algebra generated by one element without non-zero zero-divisors, and M is an irreducible bimodule over the R-algebra T and a faithful left and right T-module.*

c) *A non-commutative division R-algebra.*

d) *A simple radical R-algebra without non-zero zero-divisors.*

e) *An R-algebra B generated by two elements in $\mathrm{Ann}_B(D(B))$ such that $D(B)$ is the heart of B.*

f) *An R-algebra B generated by two elements such that $D(B)$ is the heart of B, $D(B) \subseteq Z(B)$ and $\mathrm{Ann}_B(D(B))$ is a commutative maximal ideal of B.*

Theorem 0.2 ([14, Theorem 1.2]). *Let A be a non-commutative R-*

algebra with 1. *Then there exists a factorsubalgebra of A which is of type* a)^1, b)^1, c), d)^1, e)^1 *or* f)^1*:*

a)^1 $\begin{pmatrix} R/\mathfrak{m} & R/\mathfrak{m} \\ 0 & R/\mathfrak{m} \end{pmatrix}$, *where* $\mathfrak{m}$ *is a maximal ideal of* R.

b)^1 *A non-commutative trivial extension* $T \ltimes M$, *where* T *is an integral domain which is an* R*-algebra generated by one element together with* 1, *and* M *is an irreducible bimodule over the* R*-algebra* T *and a faithful left and right* T*-module.*

c) *A non-commutative division* R*-algebra.*

d)^1 *A domain which is an* R*-algebra generated by* 1 *and a simple radical subalgebra.*

e)^1 *An* R*-algebra* B *with* 1 *generated by* 1 *and two elements in* $\mathrm{Ann}_B(D(B))$ *such that* $D(B)$ *is the heart of* B.

f)^1 *An* R*-algebra* B *with* 1 *generated by* 1 *and two elements such that* $D(B)$ *is the heart of* B, $D(B) \subseteq Z(B)$ *and* $\mathrm{Ann}_B(D(B))$ *is a commutative maximal ideal of* B.

A commutative ring R with 1 is called an *$\mathcal{A}$-ring* if R is either a finitely generated ring or a finitely generated S-algebra, where $S/\mathfrak{p}$ is an algebraically closed field for any prime ideal $\mathfrak{p}$ of S; R is called an *$\mathcal{P}$-ring* if R is a finitely generated S-algebra, where the quotient field of $S/\mathfrak{p}$ is a perfect field for any prime ideal $\mathfrak{p}$ of S. Needless to say, every $\mathcal{A}$-ring is an $\mathcal{P}$-ring.

Proposition 0.3 ([14, Proposition 1.6]). *Let* R *be an* $\mathcal{A}$*-ring.*

(1) *Suppose that* R *is a finitely generated ring. If an* R*-algebra* A *is of type* b) (*resp.* b)^1), *then* A *is isomorphic to some*

$$\mathrm{M}_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \alpha, \beta \in K \right\},$$

where K *is a finite field with a non-trivial automorphism* σ.

(2) *If* R *is not a finitely generated ring, then no* R*-algebra is of type* b) *or* b)^1.

Let $\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix}$ be an element of $\mathrm{M}_\sigma(K)$, and n a positive integer. Then

$$(*) \qquad \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix}^n = \begin{cases} \begin{pmatrix} \alpha^n & (\sigma(\alpha^n)-\alpha^n)(\sigma(\alpha)-\alpha)^{-1}\beta \\ 0 & \sigma(\alpha^n) \end{pmatrix} & \text{if } \alpha \notin K^\sigma \\ \begin{pmatrix} \alpha^n & n\alpha^{n-1}\beta \\ 0 & \alpha^n \end{pmatrix} & \text{if } \alpha \in K^\sigma. \end{cases}$$

This formula will be used repeatedly in the subsequent study.

Proposition 0.4 ([14, Proposition 1.7]). *There exists no algebra of type* f) *or* f)[1] *over an $\mathscr{P}$-ring.*

Theorem 0.5 ([14, Theorem 3.6]). *Let A be an algebra over an $\mathscr{P}$-ring R, and n a positive integer. Then the following conditions are equivalent:*

1) *A satisfies the identities $[X^n, Y^n] = 0$ and $[X-X^m, Y-Y^m] = 0$ for some $m > 1$.*

2) *A satisfies* (S) *and the identity $[X^n, Y^n] = 0$.*

3) *A is a subdirect sum of R-algebras each of which has one of the following types:*

i) *A commutative algebra.*

ii) $\mathrm{M}_\sigma(K)$, *where $(|K|-1)/(|K^\sigma|-1)$ divides n.*

In applications of the above results, we shall prove several commutativity theorems for algebras and rings, together with some related results.

Throughout the present paper, R will represent a commutative ring with 1, and A an R-algebra. As for notations and terminologies used without mention, we follow [14].

We shall consider also the following conditions for a non-empty subset M of A and a positive integer n:

(I$'-M$) For each $x \in A$, either $x \in Z$ or there exists $f(X) \in X^2R[X]$ such that $x-f(x) \in M$.

Q(n) If $x, y \in A$ and $n[x, y] = 0$ then $[x, y] = 0$.

1. Condition (C) and commutativity theorems. In this section, we study on the commutativity of algebras over an $\mathscr{N}$-ring satisfying (C).

In [3] and [4], M. Chacron considered the subset $\mathscr{C}' = \mathscr{C}'(A)$ of A defined to be the set of all elements c in A such that for each $x \in A$ there holds $[c, x-f(x)] = 0$ with some $f(X) \in X^2R[X]$. Patterning after the proof of [8, Corollary 1], we can easily see the following

Lemma 1.1. *Suppose that an R-algebra A satisfies* (C). *If A is either a division algebra or a radical algebra without non-zero zero-divisors, then $A = \mathscr{C}'$.*

Theorem 1.2. *Let R be a commutative ring with* 1 *such that every R-*

algebra B without non-zero zero-divisors satisfying the condition

(N) *for each $x \in B$, there exists $f(X) \in X^2R[X]$ such that $x-f(x) \in Z(B)$*

is commutative. If an R-algebra A satisfies (C), *then D is a nil ideal.*

Proof. We claim first that if A is either a division R-algebra or a radical R-algebra without non-zero zero-divisors (satisfying (C)), then A is commutative. In fact, let a and b be arbitrary elements of A, and $x \in \langle a, b\rangle_R$. Since $A = \mathscr{C}'$ by Lemma 1.1, there exist $f(X), g(X) \in X^2R[X]$ such that $[a, x-f(x)] = 0$ and $[b, x-f(x)-g(x-f(x))] = 0$. Then $h(X) = f(X)+g(X-f(X)) \in X^2R[X]$ and $[a, x-h(x)] = [b, x-h(x)] = 0$, and so $x-h(x) \in Z(\langle a, b\rangle_R)$. Hence, by hypothesis, $\langle a, b\rangle_R$ is commutative.

Now, to our end, it suffices to show that if A is a prime R-algebra satisfying (C) then A is commutative.

First, assume that A is semi-primitive. We may assume further that A is primitive. Obviously, every factorsubalgebra B of A inherits (C), and so $N^*(B)$ is commutative. Hence, by the structure theorem of primitive algebras (see, e.g. [14, Introduction]), A must be a division algebra, which is commutative by the above claim.

Next, assume that A is not semi-primitive. Now, let $a \in N^*(J)$, where $J \neq 0$ is the Jacobson radical of A. For each $x \in aA$, there exists $f(X) \in X^2R[X]$ such that $-xa+f(x)a = [a, x-f(x)] = 0$. Then we have $x^2 = f(x)x$, which implies evidently $x^2 = 0$. Combining this with $xa = f(x)a$, we obtain $xa = 0$, namely $aAa = 0$. We have thus seen that $N^*(J) = 0$. Therefore, as is well-known, J has no non-zero zero-divisors. Hence J is commutative by the above claim. Since $J^2 \subseteq Z$, we see that $[A, A]J^2 = [A, AJ^2] = 0$. Hence $[A, A] = 0$, namely A is commutative.

In view of [15, Proposition], we see that every $\mathscr{N}$-ring satisfies the hypothesis of Theorem 1.2. Therefore, combining Theorem 1.2 with [14, Lemma 2.1], we readily obtain the following

Corollary 1.3. *Suppose that an algebra A over an $\mathscr{N}$-ring satisfies* (C).

(1) *N is a commutative ideal of A containing D, and $[N, A]^2 = 0$.*

(2) *$C_A(N^*)$ is a maximal commutative subalgebra of A.*

(3) $\mathrm{Ann}([N^*, A])$ *is the largest commutative ideal of A and is contained in $C_A(N^*)$.*

(4) *For any non-empty subset M of N, $A/\mathrm{Ann}([M, A])$ has no non-zero nil ideals.*

(5) *Let $c \in N$, $x \in A$, k a positive integer, and p a prime number.*
(i) *If $x^k[c, x] = [c, x]x^k = 0$ then $[c, x] = 0$.*
(ii) *If $[c, x]_k = 0$ then $[c, x] = 0$.*
(iii) *If $[c, px] = [c, x^p] = 0$, then $[c, x] = 0$.*
(iv) *If the additive order of $[c, x]$ is finite, then it is square-free.*

The next proposition is the key result of this section.

Proposition 1.4. *Let A be a non-commutative algebra over an $\mathcal{N}$-ring R. Suppose that A satisfies* (C).

(1) *If R is a finitely generated ring, then there exists a factorsubalgebra of A which is of type* a)$_l$, a)$_r$ *or* $\mathrm{M}_\sigma(K)$.

(2) *If R is not a finitely generated ring, then there exists a factorsubalgebra of A which is of type* a)$_l$ *or* a)$_r$.

Proof. Since $D \subseteq N$ by Theorem 1.2, A has no factorsubalgebras of type c) or d). Further, by (C) and Proposition 0.4, A has no factorsubalgebras of type e) or f). Hence the assertions are immediate by Theorem 0.1 and Proposition 0.3.

In virtue of Corollary 1.3 and Proposition 1.4, many early commutativity theorems for rings satisfying (C) are still valid for algebras over an $\mathcal{N}$-ring.

Now, in the following theorem, we shall generalize the conditions [12, Theorem 2 1) and Theorem 3 4)] and the condition in [6, Lemma 3].

Theorem 1.5. *Let A be an algebra over an $\mathcal{N}$-ring R satisfying* (C). *Then the following conditions are equivalent:*

0) *A is commutative.*

1) *For each $c \in N^*$ and $x \in A$, there exist positive integers $m_1, \ldots, m_r$ and k such that $(m_1, \ldots, m_r) \mid 2$ and $[c, x^{m_i}]_k = 0$ $(i = 1, \ldots, r)$.*

2) *For each $c \in N^*$ and $x \in A$, either $[c, x] = 0$ or there exist positive integers m, n, m', n' and k such that $mn = m'n' > 1$, $(m+m', mn-1) \mid 2$, $x - x^{mn} \in N$ and $[(x^m(x+c)^m)^n, ((x+c)^{m'}x^{m'})^{n'}]_k = 0$.*

3) *For each $c \in N^*$ and $x \in A$, there exist positive integers m, n and k such that $mn \not\equiv 1 \pmod 4$, $[c, x - x^{mn}]_k = 0$ and $[(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k = 0$.*

4) *For each $c \in N^*$ and $x \in A$, there exist positive integers m, n and k such that $mn > 1$, $mn \not\equiv 2 \pmod 8$, $[c, x - x^{mn}]_k = 0$ and $[(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k = 0$.*

5) *For each* $c \in N^*$ *and* $x \in A$, *there exist positive integers* m, n *and* k *such that* $mn > 1$, $mn \not\equiv 6 \pmod 8$, $[c, x-x^{mn}]_k = 0$ *and* $[(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k = 0$.

Proof. Obviously 0) implies 1)−5).

1) ⇒ 0). In view of Proposition 1.4, it suffices to show that A has no factorsubalgebras of type $a)_l$, $a)_r$ or $M_\sigma(K)$.

(i) First, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $(R/\mathfrak{m})^{(2)}$, where $\mathfrak{m}$ is a maximal ideal of R. Now let

$$\phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \phi(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $c = [y, x]_2 \in N^*$ by Corollary 1.3 (1) and

$$\phi(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

But $[c, x^m]_k \neq 0$ for any positive integers m, k. Similarly, A has no factorsubalgebras isomorphic to ${}^{(2)}(R/\mathfrak{m})$.

(ii) Next, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $M_\sigma(K)$. Let γ be a generating element of the multiplicative group of K, and choose $x, y \in A$ such that

$$\phi(x) = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad \phi(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $c = [y, x]_2$ is in N^* by Corollary 1.3 (1), there exist positive integers $m_1, \ldots, m_r$ and k such that $(m_1, \ldots, m_r) | 2$ and

$$0 = \phi([c, x^{m_i}]_k) = (\sigma(\gamma^{m_i}) - \gamma^{m_i})^k (\sigma(\gamma) - \gamma)^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; (i = 1, 2, \ldots, r).$$

This means that $\gamma^{m_i} \in K^\sigma$, and hence $\gamma^2 \in K^\sigma$. But this forces a contradiction $(|K|-1)/(|K^\sigma|-1) | 2$.

2) ⇒ 0). In view of Proposition 1.4, it suffices to show that A has no factorsubalgebras of type $a)_l$, $a)_r$ or $M_\sigma(K)$.

(i) First, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $(R/\mathfrak{m})^{(2)}$, where $\mathfrak{m}$ is a maximal ideal of R. Choose $x, y \in A$ and $c \in N^*$ as in (i) of 1) ⇒ 0). Then

$$\phi(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad [c, x] \neq 0.$$

There exist positive integers m, n, m', n' and k such that $mn = m'n'$ and $\phi([(x^m(x+c)^m)^n, ((x+c)^{m'}x^{m'})^{n'}]_k) = 0$. But this is impossible. Similarly, A has no factorsubalgebras isomorphic to ${}^{(2)}(R/\mathfrak{m})$.

(ii) Next, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $\mathrm{M}_\sigma(K)$. Now, let γ be a generating element of the multiplicative group of K, and choose $x, y \in A$ and $c \in N^*$ as in (ii) of 1) $\Rightarrow$ 0). Since $[c, x] \neq 0$, there exist positive integers m, n, m', n', μ, ν and k such that $mn = m'n' > 1$, $2 = (m+m')\mu-(mn-1)\nu$, $x-x^{mn} \in N$ and $[(x^m(x+c)^m)^n, ((x+c)^{m'}x^{m'})^{n'}]_k = 0$. By a brief computation, we can easily see that

$$\phi(x-x^{mn}) = \begin{pmatrix} \gamma-\gamma^{mn} & 0 \\ 0 & \sigma(\gamma-\gamma^{mn}) \end{pmatrix}, \text{ and}$$

$$\begin{aligned} &\phi([(x^m(x+c)^m)^n, ((x+c)^{m'}x^{m'})^{n'}]_k) \\ &\quad = (\sigma(\gamma^{2mn})-\gamma^{2mn})^{k+1}(\sigma(\gamma^{m+m'})-\gamma^{m+m'})(\sigma(\gamma^m)+\gamma^m)^{-1} \\ &\qquad (\sigma(\gamma^{m'})+\gamma^{m'})^{-1}\ (\gamma-\sigma(\gamma))\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

provided $\gamma^{2m} \notin K^\sigma$ and $\gamma^{2m'} \notin K^\sigma$.

Hence $\gamma = \gamma^{mn}$, and $\gamma^{2mn} \in K^\sigma$ (if $\gamma^{m+m'} \in K^\sigma$ then $\gamma^2 = \gamma^{(m+m')\mu}(\gamma^{mn-1})^{-\nu} \in K^\sigma$). Consequently, $\gamma^2 = \gamma^{2mn} \in K^\sigma$, which means that $(|K|-1)/(|K^\sigma|-1)\,|\,2$. But this is impossible.

3) (4) or 5)) $\Rightarrow$ 0). Suppose, to the contrary, that A is not commutative. Then, in view of Proposition 1.4, there exists a factorsubalgebra of A which is of type a)$_l$, a)$_r$ or $\mathrm{M}_\sigma(K)$.

(i) First, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $(R/\mathfrak{m})^{(2)}$, where $\mathfrak{m}$ is a maximal ideal of R. Choose $x, y \in A$ and $c \in N^*$ as in (i) of 1) $\Rightarrow$ 0). Then $\phi([(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k) \neq 0$ for any positive integers m, n, k, which is a contradiction. Similarly, we can see that A has no factorsubalgebras isomorphic to ${}^{(2)}(R/\mathfrak{m})$.

(ii) Next, suppose that there exists a homomorphism ϕ of a subalgebra of A onto $\mathrm{M}_\sigma(K)$. Now, let γ be an arbitrary element of $K\backslash K^\sigma$, and choose $x, y \in A$ and $c \in N^*$ as in (ii) of 1) $\Rightarrow$ 0). Let m, n, k be positive integers such that $mn > 1$, $mn \not\equiv 1 \pmod 4$ ($mn \not\equiv 2 \pmod 8$, $mn \not\equiv 6 \pmod 8$, respectively), $[c, x-x^{mn}] = 0$ (Corollary 1.3 (5) (ii)), and $[(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k = 0$. Since

$$\phi([c, x-x^{mn}]) = (\sigma(\gamma)-\gamma)^2\{\sigma(\gamma-\gamma^{mn})-(\gamma-\gamma^{mn})\}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we get $\gamma-\gamma^{mn} \in K^\sigma$. Next, since

$$\phi([(x^m(x+c)^m)^n, ((x+c)^m x^m)^n]_k)$$
$$= (\sigma(\gamma^{2mn})-\gamma^{2mn})^{k+1}(\sigma(\gamma^m)-\gamma^m)(\sigma(\gamma^m)+\gamma^m)^{-1}(\gamma-\sigma(\gamma))\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

provided $\gamma^{2m} \notin K^{\sigma}$,

we get $\gamma^{2mn} \in K^{\sigma}$. Hence, by [12, Lemma 7], $K = \mathrm{GF}(9)$ and $K^{\sigma} = \mathrm{GF}(3)$. Noting that K is the splitting field of

$$X(X^2-1)(X^2+1)(X^2-X-1)(X^2+X-1),$$

we can choose γ such that $\gamma^2+1 = 0$. Since γ is of order 4 and $\gamma-\gamma^{mn} \in \mathrm{GF}(3)$, we can easily see that $mn \equiv 1 \pmod 4$. Next, choose γ such that $\gamma^2-\gamma-1 = 0$. Since γ^2 is of order 4 and $\gamma^{2mn} \in \mathrm{GF}(3)$, $mn = 2k$ with some k. Noting that $\gamma-(\gamma^2)^k = \gamma-\gamma^{mn} \in \mathrm{GF}(3)$, we can easily see that $k \equiv 1 \pmod 4$, so that $mn \equiv 2 \pmod 8$. Finally, choose γ such that $\gamma^2+\gamma-1 = 0$. Then we can easily see that $mn \equiv 6 \pmod 8$. But this is impossible, in either case.

Corollary 1.6. *Let A be an algebra over an $\mathcal{N}$-ring satisfying* (C), *and $n > 1$ an integer. Suppose that for each $c \in N^*$ and $x \in A$, there exists a positive integer k and a positive divisor m of n such that $[(x^m(x+c)^m)^{n/m}, ((x+c)^m x^m)^{n/m}]_k = 0$ and $[c, x-x^n]_k = 0$. Then A is commutative.*

Theorem 1.7. *Suppose that an algebra A over an $\mathcal{N}$-ring R satisfies* (C). *Then the following conditions are equivalent:*

0) *A is commutative.*

1) *For each $c \in N^*$ and $x \in A$, either $[c, x] = 0$ or there exist integers $n > 1$ and $k > 0$ such that*

i) $(n-1)[c, x] \neq 0$,

ii) $[c, x-x^n]_k = 0$,

iii) $[(x(1+c))^n-x^n(1+c)^n, x]_k = 0$, *and*

iv) $[((1+c)x)^n-(1+c)^n x^n, x]_k = 0$.

2) *For each $c \in N^*$ and $x \in A$, either $[c, x] = 0$ or there exist integers $n > 1$ and $k > 0$ such that*

i) $x-x^n \in N$, *and*

ii) *either* $[((1+c)x)^n-(x(1+c))^n, x]_k = 0$

or $[((1+c)x)^{n+1}-(x(1+c))^{n+1}, x]_k = 0$.

Proof. Obviously, 0) implies 1) and 2).

1) $\Rightarrow$ 0). Suppose that there exist $c \in N^*$ and $x \in A$ such that $[c, x]$

$\neq 0$. Then there exist integers $n > 1$ and $k > 0$ such that i)–iv) hold good. Note that, by Corollary 1.3 (5) (ii),

ii)' $[c, x-x^n] = 0$, i.e., $[c, x] = [c, x^n]$.

Since N is a commutative ideal by Corollary 1.3 (1), we see that $cAc = c^2A = 0$. Hence, by iii), iv) and ii)', $0 = [(x(1+c))^n - x^n(1+c)^n, x]_k - [((1+c)x)^n - (1+c)^n x^n, x]_k = (n-1)[[c, x^n], x]_k = (n-1)[c, x]_{k+1}$, and therefore $(n-1)[c, x] = 0$ again by Corollary 1.3 (5) (ii). But this contradicts i). Now, A is commutative by Corollary 1.3 (2).

2) $\Rightarrow$ 0). By Corollary 1.3 (1), N is a commutative ideal, and so $N^2 \subseteq Z$. Suppose that there exist $c \in N^*$ and $x \in A$ such that $[c, x] \neq 0$. Then there exist positive integers $n > 1$ and $k > 0$ such that $x - x^n \in N$ (and so $x^2 - x^{n+1} \in N$) and either

$$0 = [((1+c)x)^n - (x(1+c))^n, x]_k = [[c, x^n], x]_k \text{ or}$$
$$0 = [((1+c)x)^{n+1} - (x(1+c))^{n+1}, x]_k = [[c, x^{n+1}], x]_k.$$

Since $[c, x-x^n] = 0 = [c, x^2 - x^{n+1}]$, we get $[c, x]_{k+1} = 0$ or $[[c, x^2], x]_k = 0$. Thus we have seen that for each $c \in N^*$ and $x \in A$, there exists a positive integer k such that $[c, x^2]_{k+1} = 0$. Hence A is commutative, by Theorem 1.5 1).

Corollary 1.8 (cf. [12, Corollary 6]). *Let A be a ring with 1. Suppose that there exists a commutative subset M of A for which A satisfies the following condition*

$(3-M)^*$ *For each $x \in A$, either $x \in Z$ or there exist integers $n > 1$ and $k > 0$ such that*

1) $x - x^n \in M$,
2) $[x^n y^n - (xy)^n, x]_k = [y^n x^n - (yx)^n, x]_k = 0$ *for all* $y \in A$,
3) *for all* $c \in M$, $(n-1)[c, x] = 0$ *implies* $[c, x] = 0$.

Then A is commutative.

Corollary 1.9. *Let A be a ring satisfying $(\mathrm{I}' - C_A(N))$. Suppose that for each $c \in N^*$ and $x \in A$, there exist integers $n > 1$ and $k > 0$ such that*

1) *x is written in the form $x = b + a$ where $b^n = b$ and $a \in N$,*
2) $[((1+c)x)^n - (x(1+c))^n, x]_k = 0$ *or*
$[((1+c)x)^{n+1} - (x(1+c))^{n+1}, x]_k = 0$.

Then A is commutative.

Proof. By $(\mathrm{I}' - C_A(N))$, we can easily see that N is commutative. Hence, by [2, Theorem 2], N is a commutative ideal, and A satisfies (C).

Now, the commutativity of A is clear by Theorem 1.7 2).

Corollary 1.10 (cf. [6, Theorem 5 (1)]). *Let A be a ring with 1, and Q the intersection of the set of non-units of A with the set of quasi-regular elements of A. Suppose that there exists an integer $n>1$ such that*

1) $[x-x^n, y-y^n]=0$ *for all* $x, y \in A$,
2) $(xy)^n-x^ny^n \in Z$ *for all* $x, y \in A\backslash Q$, *and*
3) $(n-1)[c, x]=0$ *implies* $[c, x]=0$ *for all* $c \in N^*$ *and* $x \in A\backslash Q$.

Then A is commutative.

Proof. By the proof of 1) $\Rightarrow$ 0) in the proof of Theorem 1.7, we can easily see that $[c, x]=0$ for all $c \in N^*$ and $x \in A\backslash Q$. Hence $N^* \subseteq Z$, and A is commutative by Corollary 1.3 (2).

Theorem 1.11. *Let A be a ring with 1 satisfying* $(\mathrm{I}'-N)$, *and $n>1$ an integer. Suppose that for each $c \in N$ and $x \in A$, there exists a positive integer k such that*

1) $[c-c^n, x-x^n]=0$,
2) $(x(1+c))^n=x^n(1+c)^n$,
3) $[((1+c)x)^n-(1+c)^nx^n, x]_k=0$, *and*
4) $[(x(1+c))^{n+1}-x^{n+1}(1+c)^{n+1}, x]_k=0$.

Then A is commutative.

Proof. By 1), $[c, x]=[c, x^n]$ for all $c \in N$ and $x \in A$; in particular, $[c, c']=0$ for all $c, c' \in N$. Combining this with $(\mathrm{I}'-N)$, we see that A satisfies (C), and N is a commutative ideal (Corollary 1.3 (1)). Then, by 2) and 3), we can easily see that $(n-1)[c, x]=0$ for all $c \in N^*$ and $x \in A$. (See the proof of Theorem 1.7.) Now, let $c \in N^*$, and $x \in A$. Then $cAc=0$, and we see that

$$
\begin{aligned}
0 &= [(x(1+c))^{n+1}-x^{n+1}(1+c)^{n+1}, x]_k \\
&= [x^n(1+c)^nx(1+c)-x^{n+1}(1+c)^{n+1}, x]_k = [x^n[(1+c)^n, x](1+c), x]_k \\
&= n[x^n[c, x](1+c), x]_k = n[x^n[c, x], x]_k = nx^n[c, x]_{k+1}.
\end{aligned}
$$

Hence we obtain $n[c, x]_{k+1}=0$, and so $n[c, x]=0$ by Corollary 1.3 (5) (ii). This together with $(n-1)[c, x]=0$ implies that $[c, x]=0$. Therefore, Corollary 1.3 (2) shows that A is commutative.

Corollary 1.12 (cf. [5, Theorem 2.5]). *Let A be a ring with 1 satisfying* $(\mathrm{I}'-N)$, *and $n>1$ an integer. Suppose that*

1) $[c-c^n, x-x^n] = 0$ *for all* $c \in N$ *and* $x \in A$,
2) $(xy)^n = x^n y^n$ *for all* $x, y \in A \setminus Q$, *and*
3) $(xy)^{n+1} - x^{n+1}y^{n+1} \in Z$ *for all* $x, y \in A$.

Then A *is commutative.*

Proof. By the proof of Theorem 1.11, A satisfies (C), and for each $c \in N^*$ and $x \in A \setminus Q$, $(n-1)[c, x] = 0$ and $nx^n[c, x]_2 = 0$. Further, if $c \in N^*$ and $x \in Q$, then $(n-1)[c, x] = -(n-1)[c, 1-x] = 0$ and $n[c, x]_2 = -(1-x)^{-n}\{n(1-x)^n[c, 1-x]_2\} = 0$, by the above. Hence $(n-1)[c, x] = 0$ and $nx^n[c, x]_2 = 0$ for all $c \in N^*$ and $x \in A$. Then we can easily see that $[c, x]_2 = 0$ for all $c \in N^*$ and $x \in A$. Now, A is commutative, by Theorem 1.5.

Lemma 1.13 ([5, Theorem 3.1]). *Let* A *be a ring with* 1, *and* $n > 1$ *an integer. Suppose that*

1) N *is commutative*,
2) $(x-x^n)^2 = 0$ *for all* $x \in A \setminus N$, *and*
3) *any* $x \in A \setminus N$ *may be written in at most one way in the form* $x = b + c$ *where* $b^n = b$ *and* $c \in N$.

Then A *is commutative.*

Proof. As is easily seen, A is a normal ring (i.e., every idempotent is central) and satisfies (C). By 2), we see that A is of finite characteristic. Hence, without loss of generality, we may assume that A is of characteristic $q = p^k$, p a prime. Suppose, to the contrary, that A is not commutative. Then, in view of Proposition 1.4 and [12, Lemma 8], there exists a homomorphism ϕ of a subring of A onto $M_\sigma(K)$. Let γ be a generating element of the multiplicative group of K,

$$\phi(x) = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad \phi(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where we may assume that $c \in N$ (see 2)). Let $y \in A \setminus N$. By 2), $y^{2n} = 2y^{n+1} - y^2$. An easy induction shows that $y^{m(n-1)+2} = my^{n+1} - (m-1)y^2$, and so $y^{mn} = m(y^{n+m-1} - y^m) + y^m$ for any $m > 1$. In particular, $(y^q)^n = y^q$. By Corollary 1.3 (1), N is a commutative ideal of A and $[c, x] \in N^*$, and so $(x + [c, x])^q = x^q + \sum x^i[c, x]x^{q-i-1} = x^q + [c, x^q]$. Hence we have $[c, x^q] = 0$, by 3). Accordingly,

$$\begin{pmatrix} 0 & \sigma(\gamma^q) - \gamma^q \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix}^q \right] = 0,$$

namely $\gamma^q \in K^\sigma$. Noting that the map $\alpha \mapsto \alpha^q$ is an automorphism of K, we have a contradiction $K = K^\sigma$.

Theorem 1.14. *Let A be a ring, and $n > 1$ an integer. Suppose that*
1) *N is commutative,*
2) *$(x-x^n)^2 = 0$ for all $x \in A\backslash N$, and*
3) *any $x \in A\backslash N$ may be written in at most one way in the form $x = b+c$ where $b^n = b$ and $c \in N$.*

Then A is commutative.

Proof. As is easily seen, A is a normal ring satisfying (C). Suppose, to the contrary, that A is not commutative. Then, in view of Proposition 1.4 and [12, Lemma 8] there exists a homomorphism ϕ of a subring B of A onto some $\mathrm{M}_\sigma(K)$, and there exists a central idempotent $e \in B$ such that

$$\phi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

According to Lemma 1.13, eB is commutative, which forces a contradiction that $\mathrm{M}_\sigma(K) = \phi(B) = \phi(eB)$ is commutative.

Theorem 1.15. *Let A be a normal algebra over an $\mathcal{N}$-ring satisfying (C). Suppose that for each $x \in A\backslash N$, either $x \in Z$ or there exist positive integers $n_1, \dots, n_r$ such that $(n_1, \dots, n_r)$ has no factor of the form p^t-1, p any prime and $t > 1$, and that $x-x^{n_i+1} \in N$ $(i = 1, \dots, r)$. Then A is commutative.*

Proof. Suppose that A is not commutative. Then, in virtue of Proposition 1.4 and [12, Lemma 8], there exists a factorsubalgebra of A which is isomorphic to some $\mathrm{M}_\sigma(K)$. Now, let γ be a generating element of the multiplicative group of K. Then it is easy to see that there exist positive integers $n_1, \dots, n_r$ such that $(n_1, \dots, n_r)$ has no factor of the form p^t-1, p any prime and $t > 1$, and that $\gamma = \gamma^{n_i+1}$, namely $\gamma^{n_i} = 1$ $(i = 1, \dots, r)$. However, this forces a contradiction $|K|-1 \,|\, (n_1, \dots, n_r)$.

The next generalizes [5, Theorem 3.2].

Corollary 1.16. *Let A be a ring with N commutative. Suppose that for each $x \in A\backslash N$ there exist positive integers m, n such that $(m, n) | 2$, $x-x^{n+1} \in N$ and x is uniquely expressible in the form $x = b+c$, where $b^{m+1} = b$*

and $c \in N$. Then A is commutative.

Proof. Note that N is a commutative ideal by Corollary 1.3 (1), and that A is normal.

2. Further commutativity theorems. In this section, we shall prove further commutativity theorems for algebras over an $\mathcal{N}$-ring or an $\mathcal{S}$-ring, together with some structure theorems.

Theorem 2.1. *Let A be an algebra over an $\mathcal{N}$-ring R, and $m > 1$ an integer. If A satisfies* (S) *and the identity $(X+Y)^m - X^m - Y^m = 0$, then A is commutative.*

Proof. For any ring S with 1, neither $S^{(2)}$ nor ${}^{(2)}S$ satisfies the identity $(X+Y)^m - X^m - Y^m = 0$. Therefore, [7, Proposition 2] enables us to see that $D \subseteq N$. Hence, there exists no factorsubalgebra of A which is of type a)$_l$, a)$_r$, c) or d). Further, by (S) and Proposition 0.4, A has no factorsubalgebra of type e) or f). Now, suppose that $M_\sigma(K)$ satisfies the identity $(X+Y)^m - X^m - Y^m = 0$. Obviously, the map τ of K defined by $\tau(\alpha) = \alpha^m$ is an automorphism of K. Let $\gamma \in K \backslash K^\sigma$. Then, by $(*)$,

$$(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma) - \gamma)^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & 1 \\ 0 & \sigma(\gamma) \end{pmatrix}^m - \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix}^m - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^m = 0,$$

whence $\sigma(\gamma^m) = \gamma^m$ follows. We have thus seen that $K = \tau(K) \subseteq K^\sigma$, which is a contradiction. Hence, in virtue of Theorem 0.1 and Proposition 0.3, A is commutative.

As an application of Theorem 2.1, we shall prove the next

Corollary 2.2 (cf. [18, Theorem 1]). *Let $m > 1$ be an integer. Then the following statements are equivalent:*

1) *Every ring with 1 satisfying the identity $(X+Y)^m - X^m - Y^m = 0$ is commutative.*

2) *Either $m \equiv 2 \pmod 4$ or $p-1 \nmid m-1$ for each prime factor p of m.*

Proof. 1) $\Rightarrow$ 2). If $m = 2^l n$, $l > 1$ (resp. $m = p^l n$, $l > 0$, p an odd prime, and $p-1 \mid m-1$), then

$$\left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \ \alpha, \beta, \gamma, \delta \in \mathrm{GF}(2) \text{ (resp. } \mathrm{GF}(p)) \right\}$$

is a non-commutative ring with 1 and satisfying the identity $(X+Y)^m-X^m-Y^m=0$.

2) $\Rightarrow$ 1). Let A be a ring with 1 satisfying the identity $(X+Y)^m-X^m-Y^m=0$. Then, in view of [7, Proposition 2], D is contained in N, and so N is an ideal of A. Further, $k^m=k$ for all $k\in \boldsymbol{Z}\cdot 1$ ($\subseteq A$).

First, we consider the former case: $m=2n$, n is odd. Obviously, $2=(1)^m+(-1)^m=(1-1)^m=0$. Let $x, y\in A$. If $n=1$ then $(x+y)^2=x^2+y^2$ implies that $xy=yx$, and A is commutative. We assume henceforth that $n>1$. Then $1+x^m=(1+x)^m=(1+x^2)^n=1+x^2+x^3h(x)+x^m$ with some $h(X)\in X\boldsymbol{Z}[X]$, and so $x^2=x^3h(x)$. Hence $x^2=0$ for any $x\in N$. Now, noting that N is an ideal, we can easily see that N is commutative. Further, noting that $x-x^2h(x)\in N$ for all $x\in A$, we see that A satisfies (C). Hence A is commutative, by Theorem 2.1.

Next, we consider the latter case. Suppose, to the contrary, that A is not commutative. Then, without loss of generality, we may assume that A is subdirectly irreducible. As is easily seen, there exists a prime p such that $pA=0$. Now, for any $x\in N^*$, we have $(1+x)^m=1+x^m$, and so $mx=0$. Since $0\neq D\subseteq N$, this enables us to see that p is a factor of m. Recalling that $k^m=k$ for any $k\in \mathrm{GF}(p)$ ($\subseteq A$), we see that $p-1\mid m-1$, which is a contradiction.

Now, we denote by $\mathbf{W}$ the set of all words in X, Y, namely products of factors each of which is X or Y (together with 1). Further, by $\mathbf{K}_2^Y$ (resp. $\mathbf{L}_2^Y$) we denote the set of all $f\in R\langle X, Y\rangle[X, Y]R\langle X, Y\rangle$ (resp. $f\in R[Y][X, Y]R[Y]$) each of whose monomial terms has degree ≥ 2 in Y (together with 0).

Theorem 2.3. *Let A be an algebra over an $\mathscr{P}$-ring satisfying* (S) *and the identity $[X^n, Y^n]=0$. Suppose that for each $x, y\in A$, there exist non-negative integers $s\leq t\leq u$, $f_i\in \mathbf{K}_2^Y$ $(1\leq i\leq u)$, $w_i, w_i'\in \mathbf{W}$ $(1\leq i\leq u)$, and positive integers m_i, n_i $(1\leq i\leq u)$, l_i $(s+1\leq i\leq u)$ and k such that*

1) $(n, m_1n_1, \dots, m_sn_s, l_{s+1}m_{s+1}n_{s+1}, \dots, l_um_un_u)=1$,
2) $w_i(x, y)[y^{n_i}, x^{m_i}]_k w_i'(x, y)=f_i(x, y-x)$ $\quad (1\leq i\leq s)$,
3) $w_i(x, y)[(x^{m_i}y^{m_i})^{n_i}, x^{l_i}]_k w_i'(x, y)=f_i(x, y-x)$ $(s+1\leq i\leq t)$, *and*
4) $w_i(x, y)[(y^{m_i}x^{m_i})^{n_i}, x^{l_i}]_k w_i'(x, y)=f_i(x, y-x)$ $(t+1\leq i\leq u)$.

Then A is commutative.

Proof. Suppose, to the contrary, that A is not commutative. Then, in virtue of Theorem 0.5, we may assume that $A = \mathrm{M}_\sigma(K)$, where $\alpha^n \in K^\sigma$ for all $\alpha \in K$. Choose $\gamma \in K\backslash K^\sigma$, and put

$$x = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \gamma & 1 \\ 0 & \sigma(\gamma) \end{pmatrix}.$$

Then there exist non-negative integers $s \leq t \leq u$, $f_i \in \mathbf{K}_2^Y$ $(1 \leq i \leq u)$, w_i, $w_i' \in \mathbf{W}$ $(1 \leq i \leq u)$ and positive integers m_i, $n_i\,(1 \leq i \leq u)$, $l_i\,(s+1 \leq i \leq u)$ and k satisfying 1)–4). Since both x and y are units and

$$y - x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we obtain $[y^{n_i}, x^{m_i}]_k = 0$ $(1 \leq i \leq s)$, $[(x^{m_i}y^{m_i})^{n_i}, x^{l_i}]_k = 0$ $(s+1 \leq i \leq t)$, and $[(y^{m_i}x^{m_i})^{n_i}, x^{l_i}]_k = 0$ $(t+1 \leq i \leq u)$. By making use of $(*)$ in § 0, we can easily see that $\gamma^{2m_in_i} \in K^\sigma$ $(1 \leq i \leq s)$ and $\gamma^{2l_im_in_i} \in K^\sigma$ $(s+1 \leq i \leq u)$. Hence $\gamma^2 \in K^\sigma$ by $(n, m_1n_1, \ldots, m_sn_s, l_{s+1}m_{s+1}n_{s+1}, \ldots, l_um_un_u) = 1$, and so $\sigma(\gamma) = -\gamma$; similarly $\sigma(\gamma+1) = -(\gamma+1)$. But this forces a contradiction $2 = 0$.

Corollary 2.4. *Let A be an algebra over an $\mathscr{S}$-ring satisfying* (S) *and the identity $[X^n, Y^n] = 0$. Suppose that for each x, $y \in A$, there exist non-negative integers $r \leq s \leq t \leq u$ and positive integers $m_i > 1$ $(1 \leq i \leq r)$, $n_i\,(r+1 \leq i \leq u)$ and k such that*

$$\begin{aligned} &(n, m_1, \ldots, m_r, n_{r+1}, \ldots, n_u) = 1, \\ &[(x+y)^{m_i} - y^{m_i}, x]_k = 0 \ (1 \leq i \leq r), \\ &[y^{n_i}, x^{n_i}]_k = 0 \qquad (r+1 \leq i \leq s), \\ &[(xy)^{n_i}, x]_k = 0 \qquad (s+1 \leq i \leq t), \text{ and} \\ &[(yx)^{n_i}, x]_k = 0 \qquad (t+1 \leq i \leq u). \end{aligned}$$

Then A is commutative.

Proof. Replacing x by $-x$, we may assume that $[(y-x)^{m_i} - y^{m_i}, x]_k = 0$, and so $[y^{m_i}, x]_k = [(y-x)^{m_i}, x]_k$. Hence A is commutative by Theorem 2.3.

Theorem 2.5. *Let A be an algebra over an $\mathscr{S}$-ring satisfying* (S) *and the identity $[X^n, Y^n] = 0$. Suppose that for each x, $y \in A$, there exist positive integers m, k such that $(m, n) = 1$ and one of the following holds: $[(xy)^m, x]_k = 0$, $[(yx)^m, x]_k = 0$ and $[(x+y)^m, x]_k = 0$. Then A is com-*

mutative.

Proof. Suppose, to the contrary, that A is not commutative. Then, in view of Theorem 0.5, we may assume that $A = \mathrm{M}_\sigma(K)$, where $\alpha^n \in K^\sigma$ for all $\alpha \in K$. Choose $\gamma \in K \backslash K^\sigma$, and put

$$x = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then there exist positive integers m, k such that $(m, n) = 1$ and $[(xy)^m, x]_k = 0$, or $[(yx)^m, x]_k = 0$, or $[(x+y)^m, x]_k = 0$. As is easily seen,

$$[(xy)^m, x]_k = (\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma) - \gamma)^{k-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$[(yx)^m, x]_k = \sigma(\gamma)(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma) - \gamma)^{k-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and}$$

$$[(x+y)^m, x]_k = (\sigma(\gamma+1)^m - (\gamma+1)^m)(\sigma(\gamma) - \gamma)^{k-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence we see that $\gamma^m \in K^\sigma$ or $(\gamma+1)^m \in K^\sigma$. But this together with γ^n, $(\gamma+1)^n \in K^\sigma$ and $(m, n) = 1$ forces a contradiction $\gamma \in K^\sigma$.

By the above two results and [12, Corollary 2], we readily obtain

Corollary 2.6. *Let A be an s-unital ring satisfying* Q(n) *(see the introduction) and the identity $[X^n, Y^n] = 0$.*

(1) *Suppose that for each $x, y \in A$, there exist non-negative integers $r \le s \le t \le u$ and positive integers $m_i > 1$ $(1 \le i \le r)$, n_i $(r+1 \le i \le u)$ and k such that*

$$\begin{aligned} &(n, m_1, \dots, m_r, n_{r+1}, \dots, n_u) = 1, \\ &[(x+y)^{m_i} - y^{m_i}, x]_k = 0 \ (1 \le i \le r), \\ &[y^{n_i}, x^{n_i}]_k = 0 \qquad (r+1 \le i \le s), \\ &[(xy)^{n_i}, x]_k = 0 \qquad (s+1 \le i \le t), \text{ and} \\ &[(yx)^{n_i}, x]_k = 0 \qquad (t+1 \le i \le u). \end{aligned}$$

Then A is commutative.

(2) *Suppose that for each $x, y \in A$, there exist positive integers m, k such that $(m, n) = 1$ and one of the following holds: $[(xy)^m, x]_k = 0$, $[(yx)^m, x]_k = 0$ and $[(x+y)^m, x]_k = 0$. Then A is commutative.*

Obviously, Corollary 2.6 generalizes [9, Theorem 1 (1)]. All the results of [13] except Corollary 1 hold still for algebras over an $\mathscr{N}$-ring. ([13, Corollary 1] is stated incorrectly; "$h < t$" should be read as "$h < r$ (resp. $r < t$)".) Especially, the proof of [13, Theorem 5] enables us to see the following:

Theorem 2.7. *Let A be a left s-unital algebra over a commutative ring R. Let $f_i \in \mathrm{L}_2^Y$ $(1 \leq i \leq r)$, let m_i $(1 \leq i \leq r)$ be non-negative integers, and let n_i $(1 \leq i \leq r)$ be positive integers and $d = (n_1, \ldots, n_r)$. If A satisfies $\mathrm{Q}(d)$ and the identities $X^{m_i}[X^{n_i}, Y] - f_i(X, Y) = 0$ $(1 \leq i \leq r)$, then A is commutative.*

We insert here an easy lemma.

Lemma 2.8. *Let $x, y_1, \ldots, y_k$ be elements of a ring A such that $[y_i, y_j] = 0$ for all i, j.*

(1) *$[\cdots[x, y_1], \ldots, y_k] = [\cdots[x, y_{\omega(1)}], \ldots, y_{\omega(k)}]$ for any permutation ω on $\{1, \ldots, k\}$.*

(2) *If $[\cdots[x, y_1], \ldots, y_k] = 0$, then $[\cdots[x, y_1^{n_1}], \ldots, y_k^{n_k}] = 0$ for any positive integers $n_1, \ldots, n_k$.*

Proof. (1) It suffices to show that $[[a, y_i], y_j] = [[a, y_j], y_i]$ for any $a \in A$ and any i, j. But this is clear.

(2) Obviously, $[[\cdots[x, y_1], \ldots, y_{k-1}], y_k^{n_k}] = 0$. Therefore, in view of (1), we can get the assertion.

Proposition 2.9. *Let A be a non-commutative subdirectly irreducible algebra over an $\mathscr{S}$-ring R satisfying* (S).

(1) *Suppose that A satisfies the identity*

$$[\cdots[(X^m Y^m)^n - (Y^{m'} X^{m'})^{n'}, w_1(X, Y)], \ldots, w_k(X, Y)] = 0,$$

where m, n, m', n' are positive integers with $mn = m'n'$ and each $w_i \in \mathbf{W}$ is of length $n_i > 0$. Then A is isomorphic to some $\mathrm{M}_\sigma(K)$ and $(|K|-1)/(|K^\sigma|-1)$ divides one of the following numbers: $m+m'$, $2mn$, $n_1, \ldots, n_k$.

(2) *Suppose that A satisfies the identity*

$$[\cdots[(X^m Y^m)^n - X^{mn} Y^{mn}, X_1^{n_1}], \ldots, X_k^{n_k}] = 0,$$

where m, n, $n_1, \ldots, n_k$ are positive integers with $n > 1$ and each X_i is either X or Y. If A has 1, then A is isomorphic to some $\mathrm{M}_\sigma(K)$ and $(|K|-1)/(|K^\sigma|-1)$ divides one of the following numbers: $m(n-1)$, mn, $n_1, \ldots, n_k$.

Proof. (1) No $\mathrm{M}_2(\mathrm{GF}(p))$, p a prime, satisfies our identity, as a consideration of the following elements shows: $X = e_{11}$, $Y = e_{11}+e_{12}$. Therefore, by [7, Proposition 2], $D \subseteq N$. Hence, by [14, Lemma 2.1 (1)], N is a commutative ideal of A. Let $x \in A$ and $c \in N^*$, and put $a = x+[c, x]x^{m'}$, $b = x-x^{m'}[c, x]$ and $d = (a^m b^m)^n-(b^{m'}a^{m'})^{n'}$. Noting that $cAc = 0$, by induction, we can easily see that $a^l = x^l+[c, x^l]x^{m'}$ and $b^l = x^l-x^{m'}[c, x^l]$ for all positive integers l. It follows therefore that $a^m b^m = x^{2m}+[[c, x^m], x^{m+m'}]$ and $b^{m'}a^{m'} = x^{2m'}$, and so $d = \sum_{i=0}^{n-1} x^{2mi}[[c, x^m], x^{m+m'}]x^{2m(n-1-i)}$. Hence $[d, x^{2m}] = \sum_{i=0}^{n-1} x^{2mi}[[[c, x^m], x^{m+m'}], x^{2m}]x^{2m(n-1-i)} = [[[c, x^m], x^{m+m'}], x^{2mn}]$. On the other hand, noting that $d \in N$, we see that

$$0 = [\cdots[d, w_1(a, b)],\dots, w_k(a, b)] = [\cdots[d, x^{n_1}],\dots, x^{n_k}].$$

Therefore, by Lemma 2.8 (1), we obtain

$$\begin{aligned} 0 &= [\cdots[[d, x^{2m}], x^{n_1}],\dots, x^{n_k}] \\ &= [\cdots[[[[c, x^m], x^{m+m'}], x^{2mn}], x^{n_1}],\dots, x^{n_k}]. \end{aligned}$$

Putting $n'' = 2mn(m+m')n_1\cdots n_k$, we have $[c, x^{n''}]_{k+3} = 0$ by Lemma 2.8 (2), and hence $[c, x^{n''}] = 0$ by [14, Lemma 2.1 (5) (ii)]. We have thus seen that $x^{n''} \in C_A(N^*)$ for all $x \in A$. Since $C_A(N^*)$ is commutative by [14, Lemma 2.1 (2)], A satisfies the identity $[X^{n''}, Y^{n''}] = 0$. Hence, by Theorem 0.5, A is isomorphic to some $\mathrm{M}_\sigma(K)$. Now, let γ be a generating element of the multiplicative group of K, and put

$$x = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} 0 &= [\cdots[[[[c, x^m], x^{m+m'}], x^{2mn}], x^{n_1}],\dots, x^{n_k}] \\ &= (\sigma(\gamma^m)-\gamma^m)(\sigma(\gamma^{m+m'})-\gamma^{m+m'})(\sigma(\gamma^{2mn})-\gamma^{2mn})(\sigma(\gamma^{n_1})-\gamma^{n_1}) \\ &\quad \cdots(\sigma(\gamma^{n_k})-\gamma^{n_k})c \end{aligned}$$

implies that $(|K|-1)/(|K^\sigma|-1)$ divides $m+m'$, $2mn$, $n_1,\dots,$ or n_k.

(2) No $\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix}$, p a prime, satisfies our identity, as a consideration of the following elements shows: $X = e_{11}$, $Y = e_{12}+e_{22}$. Therefore, A has no factorsubalgebras of type a)^1, and hence A has no factorsubalgebras of type a)$_l$ or a)$_r$ by [14, Lemma 1.4 (1)]. As is easily seen, no algebra of type e) satisfies (S). Furthermore, by Proposition 0.4, there exists no R-algebra of type f). Hence, in view of [14, Proposition 1.3 (2)],

A is completely reflexive, i.e., $xy=0$ implies $yx=0$ for any $x, y \in A$. Also, in the same way as in the proof of (1), we see that $D \subseteq N$ and N is a commutative ideal of A. Let $x \in A$ and $c \in N^*$, and put $a = x+[c, x]x^m$, $b = x-x^m[c, x]$ and $d = (b^m a^m)^n - b^{mn}a^{mn}$. Then we can show that $b^m a^m = x^{2m}$ and $b^{mn}a^{mn} = x^{2mn} - x^m[[c, x^{mn}], x^{m(n-1)}]x^m$, and so $d = x^m[[c, x^{mn}], x^{m(n-1)}]x^m$. On the other hand, by our identity, we have $[\cdots[d, x^{n_1}], \dots, x^{n_k}] = 0$. Therefore,

$$\begin{aligned} &x^m[\cdots[[[c, x^{mn}], x^{m(n-1)}], x^{n_1}], \dots, x^{n_k}]x^m \\ &\quad = [\cdots[x^m[[c, x^{mn}], x^{m(n-1)}]x^m, x^{n_1}], \dots, x^{n_k}] = 0. \end{aligned}$$

Since A is completely reflexive, by [14, Lemma 2.1 (5) (i)], we get $[\cdots[[[c, x^{mn}], x^{m(n-1)}], x^{n_1}], \dots, x^{n_k}] = 0$. Now, the argument employed in the proof of (1) enables us to see the assertion.

Lemma 2.10. *Let l, m, n and k be positive integers. Let $A = \mathrm{M}_\sigma(K)$ and put $t = (|K|-1)/(|K^\sigma|-1)$.*

(1) *If A satisfies the identity $[(X^mY^m)^n-(Y^mX^m)^n, X^l]_k = 0$, then either* i) *$t$ divides l,* ii) *t divides m,* iii) *t divides $2m$ and $mK = 0 = nK$, or* iv) *t divides mn and $nK = 0$.*

(2) *If A satisfies the identity $[(X^mY^m)^{n+1}-X^{m(n+1)}Y^{m(n+1)}, X^l]_k = 0$, then either* i) *$t$ divides l,* ii) *t divides m,* iv) *t divides mn and $nK = 0$, or* v) *t divides $m(n+1)$ and $(n+1)K = 0$.*

(3) *If A satisfies the identity $[(X^mY^m)^n - Y^{mn}X^{mn}, X^l]_k = 0$, then A satisfies the identity $[(X^mY^m)^{n+1}-X^{m(n+1)}Y^{m(n+1)}, X^l]_k = 0$.*

(4) *If* i) *t divides l, A satisfies the identity $[XY-YX, W^l] = 0$; if* ii) *t divides m, A satisfies the identity $[X^m, Y^m] = 0$; if* iii) *t divides $2m$ and $mK = 0 = nK$, A satisfies the identity $(X^mY^m)^n-(Y^mX^m)^n = 0$; if* iv) *$t$ divides mn and $nK = 0$, A satisfies the identities $(X^mY^m)^n-(Y^mX^m)^n = 0$, $(X^mY^m)^n - X^{mn}Y^{mn} = 0$ and $(X^mY^m)^{n-1} - Y^{m(n-1)}X^{m(n-1)} = 0$.*

Proof. Let γ be a generating element of the multiplicative group of K. Obviously, t divides an integer r if and only if $\gamma^r \in K^\sigma$. Put

$$x = \begin{pmatrix} \gamma & 1 \\ 0 & \sigma(\gamma) \end{pmatrix},\ y = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix},\ b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(1) Assume that A satisfies our identity and that $\gamma^l \notin K^\sigma$ and $\gamma^m \notin K^\sigma$. Then we can see that $0 = [(x^my^{-m})^n-(y^{-m}x^m)^n, x^l]_k = n(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma^{-m}) - \gamma^{-m})(\sigma(\gamma^l) - \gamma^l)^k(\sigma(\gamma) - \gamma)^{-1}c$, which implies that $nK = 0$.

Now, we assume that $\gamma^{mn} \notin K^{\sigma}$, and suppose, to the contrary, that $\gamma^{2m} \notin K^{\sigma}$. Then, by Proposition 2.9 (1), we have $\gamma^{2mn} \in K^{\sigma}$. Since $\gamma^{mn} \notin K^{\sigma}$, we have $\gamma^{3mn} \notin K^{\sigma}$, and so $\gamma^{3m} \notin K^{\sigma}$. This forces a contradiction $[(x^m y^{2m})^n - (y^{2m}x^m)^n, x^l]_k = (\sigma(\gamma^{3mn}) - \gamma^{3mn})(\sigma(\gamma^{2m}) - \gamma^{2m})(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma^l) - \gamma^l)^k(\sigma(\gamma^{3m}) - \gamma^{3m})^{-1}(\sigma(\gamma) - \gamma)^{-1}c \neq 0$. Hence $\gamma^{2m} \in K^{\sigma}$. Further, we see that $0 = [(y^m b^m)^n - (b^m y^m)^n, y^l]_k = -m(\sigma(\gamma^{mn}) - \gamma^{mn})(\sigma(\gamma^l) - \gamma^l)^k c$, which shows that $mK = 0$.

(2) Assume that A satisfies our identity and that $\gamma^l \notin K^{\sigma}$ and $\gamma^m \notin K^{\sigma}$. Then, by Proposition 2.9 (2), we have $\gamma^{mn} \in K^{\sigma}$ or $\gamma^{m(n+1)} \in K^{\sigma}$. Furthermore, we see that $0 = [(y^{-m}x^m)^{n+1} - y^{-m(n+1)}x^{m(n+1)}, y^{-l}]_k = \alpha(\sigma(\gamma^{-l}) - \gamma^{-l})^k(\sigma(\gamma) - \gamma)^{-1}c$, where $\alpha = (n+1)(\sigma(\gamma^m) - \gamma^m)\gamma^{-m} - (\sigma(\gamma^{m(n+1)}) - \gamma^{m(n+1)})\gamma^{-m(n+1)}$. Therefore $\alpha = 0$. If $\gamma^{m(n+1)} \in K^{\sigma}$ then $\alpha = 0$ implies that $(n+1)K = 0$. If $\gamma^{mn} \in K^{\sigma}$ then $\sigma(\gamma^{m(n+1)}) - \gamma^{m(n+1)} = \gamma^{mn}(\sigma(\gamma^m) - \gamma^m)$ and $\alpha = 0$ imply that $nK = 0$.

(3) Noting that $D^2 = 0$, for any $d \in D$ and $u, v, w \in A$, we have $[udv, w] = ud[v, w] + u[d, w]v + [u, w]dv = u[d, w]v$. Since $(v^m u^m)^n - u^{mn}v^{mn} \in D$, we see that $[(u^m v^m)^{n+1} - u^{m(n+1)}v^{m(n+1)}, u^l]_k = [u^m((v^m u^m)^n - u^{mn}v^{mn})v^m, u^l]_k = u^m[(v^m u^m)^n - u^{mn}v^{mn}, u^l]_k v^m = 0$.

(4) Let $B = \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \middle| \alpha \in K^{\sigma}, \beta \in K\right\}$, which is a commutative subring of A. If i) t divides l, then, for any $u, v, w \in A$, $w^l \in B$ and $[u, v] \in B$ imply that $[[u, v], w^l] = 0$. If ii) t divides m, then $u^m \in B$ for all $u \in A$, and so $[u^m, v^m] = 0$ for all $u, v \in A$.

Next, we assume that iii) t divides $2m$ and $mK = 0 = nK$. Let

$$u = \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \alpha' & \beta' \\ 0 & \sigma(\alpha') \end{pmatrix}$$

be arbitrary elements in A. Since $mK = 0$, if $u^m \in B$ then $u^m \in Z$, and hence $u^m v^m = v^m u^m$. Suppose now that $u^m \notin B$ and $v^m \notin B$. Since $\alpha^{2m} \in K^{\sigma}$ and $\alpha'^{2m} \in K^{\sigma}$, we can easily see that $\alpha^m \equiv \alpha'^m \pmod{K^{\sigma}\setminus\{0\}}$, and so $\alpha^m \alpha'^m \in K^{\sigma}$. Hence both $u^m v^m$ and $v^m u^m$ belong to B. Noting here that $nK = 0$, we can easily see that $(u^m v^m)^n = (v^m u^m)^n$. We have thus seen that $(u^m v^m)^n = (v^m u^m)^n$ for all $u, v \in A$.

Finally, we assume that iv) t divides mn and $nK = 0$. Noting that $\alpha^{mn} \in K^{\sigma}$ for all $\alpha \in K$ and $nK = 0$, we can easily see that $(u^m v^m)^n = (v^m u^m)^n = v^{mn}u^{mn} = u^{mn}v^{mn}$ for all $u, v \in A$. Furthermore, since t divides mn, we have $mn > 2$, and hence $m(n-1) > 1$. Therefore, if either $u \in N$ or $v \in N$, then $(v^m u^m)^{n-1} = 0 = u^{m(n-1)}v^{m(n-1)}$. If $u \notin N$ and $v \notin N$, then both u

and v are invertible, and so $(u^m v^m)^n = u^{mn} v^{mn}$ implies that $(v^m u^m)^{n-1} = u^{m(n-1)} v^{m(n-1)}$, completing the proof.

Following [10], we denote by Φ the additive mapping of $\boldsymbol{Z}\langle X, Y\rangle$ to $\boldsymbol{Z}$ defined as follows: For each monic monomial $X_1 \cdots X_r$ (X_i is either X or Y), $\Phi(X_1 \cdots X_r)$ is the number of pairs (i, j) such that $1 \leq i < j \leq r$ and $X_i = X$, $X_j = Y$. It is easy to see that, for any $f(X, Y) \in \boldsymbol{Z}\langle X, Y\rangle$, $\Phi(f(X, Y))$ equals the coefficient of XY occurring in $f(1+X, 1+Y)$. Now, let $f(X, Y) \in \boldsymbol{Z}\langle X, Y\rangle [X, Y] \boldsymbol{Z}\langle X, Y\rangle$. Then $f(1+X, 1+Y) \in \boldsymbol{Z}\langle X, Y\rangle [X, Y] \boldsymbol{Z}\langle X, Y\rangle$, and so there exists $g(X, Y) \in \mathrm{K}_3(\boldsymbol{Z})$ such that $f(1+X, 1+Y) = \Phi(f(X, Y))[X, Y] + g(X, Y)$.

Theorem 2.11. *Let A be a ring with 1, and let m, n be positive integers. Then the following conditions are equivalent:*

1) *A satisfies* Q(mn) *and the identity $[X^m, Y^m] = 0$.*

2) *A satisfies* Q(mn) *and the identity $(X^m Y^m)^n - (Y^m X^m)^n = 0$.*

3) *A is a subdirect sum of rings each of which has one of the following types:*

i) *A commutative ring.*

ii) $\mathrm{M}_\sigma(K)$, *where $(|K|-1)/(|K^\sigma|-1)$ divides m and $mnK \neq 0$.*

Proof. Obviously, 1) implies 2).

2) $\Rightarrow$ 3). Since $\Phi((X^m Y^m)^n - (Y^m X^m)^n) = m^2 n$, there exists $g(X, Y) \in \mathrm{K}_3(\boldsymbol{Z})$ such that $((1+X)^m (1+Y)^m)^n - ((1+Y)^m (1+X)^m)^n = m^2 n[X, Y] + g(X, Y)$. Furthermore, by [10, Theorem] and Q(mn), there exists an integer k such that $(mn, k) = 1$ and $kD = 0$. Hence, there exists an integer j such that $[x, y] + jg(x, y) = 0$ for all $x, y \in A$, namely A satisfies (S) as $\boldsymbol{Z}$-algebra. Now, suppose that A has a subdirectly irreducible homomorphic image A' which is non-commutative. Then, by Proposition 2.9 (1), A' is isomorphic to some $\mathrm{M}_\sigma(K)$. Since $kD(A') = 0$, we have $mnK \neq 0$. Hence, by Lemma 2.10 (1), $(|K|-1)/(|K^\sigma|-1)$ divides m.

3) $\Rightarrow$ 1). It is easy to see that A satisfies Q(mn). By Lemma 2.10 (4), A satisfies the identity $[X^m, Y^m] = 0$.

In [11], Y. Kobayashi investigated the following problem: Given an integer $n > 1$, determine the structure of $n(n-1)/2$-torsion-free rings with 1 satisfying the identity $(XY)^n - X^n Y^n = 0$. He solved this problem, when n is even ([11, Theorems 1 and 2]). The next theorem includes an answer to this problem. Given a positive integer n, we put $e(n) = n$ or $n-1$ according as n is even or odd.

Theorem 2.12. *Let A be a ring with 1, and let m, n be positive integers. Put $l = mn(n+1)/2$.*

(I) *If l is even, then the following conditions are equivalent:*

1) *A satisfies $Q(l)$ and the identity $[X^m, Y^m] = 0$.*

2) *A satisfies $Q(l)$ and the identity $(X^m Y^m)^n - Y^{mn} X^{mn} = 0$.*

3) *A satisfies $Q(l)$ and the identity $(X^m Y^m)^{n+1} - X^{m(n+1)} Y^{m(n+1)} = 0$.*

4) *A is a subdirect sum of rings each of which has one of the following types:*

i) *A commutative ring.*

ii) *$M_\sigma(K)$, where $(|K|-1)/(|K^\sigma|-1)$ divides m and $lK \neq 0$.*

(II) *If l is odd, then the following conditions are equivalent:*

2) *A satisfies $Q(l)$ and the identity $(X^m Y^m)^n - Y^{mn} X^{mn} = 0$.*

3) *A satisfies $Q(l)$ and the identity $(X^m Y^m)^{n+1} - X^{m(n+1)} Y^{m(n+1)} = 0$.*

4)' *A is a subdirect sum of rings each of which has one of the following types:*

i) *A commutative ring.*

ii) *$M_\sigma(K)$, where $(|K|-1)/(|K^\sigma|-1)$ divides m and $lK \neq 0$.*

iii) *$M_\sigma(K)$, where $(|K|-1)/(|K^\sigma|-1)$ divides $me(n+1)$ and $2K = 0$.*

Proof. (I) It suffices to show that 3) $\Rightarrow$ 4) $\Rightarrow$ 1).

3) $\Rightarrow$ 4). Suppose that A has a subdirectly irreducible homomorphic image A' which is non-commutative. Since $\Phi((X^m Y^m)^{n+1} - X^{m(n+1)} Y^{m(n+1)}) = -m^2 n(n+1)/2$, the proof of 2) $\Rightarrow$ 3) in Theorem 2.11 enables us to see that A' is isomorphic to some $M_\sigma(K)$ and $lK \neq 0$. (Apply Proposition 2.9 (2) instead of Proposition 2.9 (1).) Now, suppose that $nK = 0$ or $(n+1)K = 0$. Then $2lK = mn(n+1)K = 0$, and so $2K = 0$. But this forces a contradiction $lK = 0$. Hence, by Lemma 2.10 (2), $(|K|-1)/(|K^\sigma|-1)$ divides m.

4) $\Rightarrow$ 1). It is easy to see that A satisfies $Q(l)$. By Lemma 2.10 (4), A satisfies the identity $[X^m, Y^m] = 0$.

(II) By the proof of (I), this is almost clear.

The next includes a generalization of [1, Theorems 2 and 3].

Theorem 2.13. *Let A be a ring with 1. Let m, n, m', n', m'' and n'' be positive integers with $m'n' = m''n''$, and $w_i \neq 1$ $(i = 1, \ldots, k)$ in* **W**.

(1) *Suppose that A satisfies $Q(mn(n+1))$ and the identities*

$$(X^mY^m)^n - Y^{mn}X^{mn} = (Y^mX^m)^n - X^{mn}Y^{mn}, \text{ and}$$
$$[\cdots[(X^{m'}Y^{m'})^{n'} - (Y^{m''}X^{m''})^{n''}, w_1(X, Y)], \ldots, w_k(X, Y)] = 0.$$

Then A satisfies the identity $[X^m, Y^m] = 0$.

(2) *Suppose that A satisfies* $Q(mn(n+1))$ *and the identities*

$$(X^mY^m)^{n+1} - X^{m(n+1)}Y^{m(n+1)} = (Y^mX^m)^{n+1} - Y^{m(n+1)}X^{m(n+1)}, \text{ and}$$
$$[\cdots[(X^{m'}Y^{m'})^{n'} - (X^{m''}Y^{m''})^{n''}, w_1(X, Y)], \ldots, w_k(X, Y)] = 0,$$
where $m' < m''$.

Then A satisfies the identity $[X^m, Y^m] = 0$.

Proof. (1) Suppose that A has a subdirectly irreducible homomorphic image A' which is non-commutative. Since $\Phi(\{(X^mY^m)^n - Y^{mn}X^{mn}\} - \{(Y^mX^m)^n - X^{mn}Y^{mn}\}) = m^2n(n+1)$, by the same argument as in the proof of Theorem 2.11, we can see that A' is isomorphic to some $M_\sigma(K)$ and $mn(n+1)K \neq 0$. Now, let γ be a generating element of the multiplicative group of K, and put

$$x = \begin{pmatrix} \gamma & 1 \\ 0 & \sigma(\gamma) \end{pmatrix},\ y = \begin{pmatrix} \gamma & 0 \\ 0 & \sigma(\gamma) \end{pmatrix},\ b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Suppose that $\gamma^m \notin K^\sigma$. Then we see that $0 = \{(y^mb^m)^n - b^{mn}y^{mn}\} - \{(b^my^m)^n - y^{mn}b^{mn}\} = m(n+1)(\gamma^{mn} - \sigma(\gamma^{mn}))c$; hence $\gamma^{mn} \in K^\sigma$. Noting this fact, we see that $\{(x^my^{-m})^n - y^{-mn}x^{mn}\} - \{(y^{-mn}x^m)^n - x^{mn}y^{-mn}\} = n(\sigma(\gamma^m) - \gamma^m)(\sigma(\gamma^{-m}) - \gamma^{-m})(\sigma(\gamma) - \gamma)^{-1}c \neq 0$. This contradiction shows that $\gamma^m \in K^\sigma$. Hence, by Lemma 2.10 (4), A' satisfies the identity $[X^m, Y^m] = 0$.

(2) Suppose that A has a subdirectly irreducible homomorphic image A' which is non-commutative. Since $\Phi(\{(X^mY^m)^{n+1} - X^{m(n+1)}Y^{m(n+1)}\} - \{(Y^mX^m)^{n+1} - Y^{m(n+1)}X^{m(n+1)}\}) = -m^2n(n+1)$, the proof of 2) $\Rightarrow$ 3) in Theorem 2.11 enables us to see that A' satisfies (S) as $\boldsymbol{Z}$-algebra. Furthermore, no

$$\begin{pmatrix} \mathrm{GF}(p) & \mathrm{GF}(p) \\ 0 & \mathrm{GF}(p) \end{pmatrix},\ p \text{ a prime},$$

satisfies the identity $(X^mY^m)^{n+1} - X^{m(n+1)}Y^{m(n+1)} = (Y^mX^m)^{n+1} - Y^{m(n+1)}X^{m(n+1)}$, as a consideration of the following elements shows: $X = e_{11}$, $Y = e_{12} + e_{22}$. Therefore, by the proof of Proposition 2.9 (2), A' is completely reflexive, $D(A') \subseteq N(A')$ and $N(A')$ is a commutative ideal of A'. Now, let $x \in A'$ and $c \in N^*(A')$, and put $a = x + [c, x]x^{m'}$, $b = x - x^{m'}[c, x]$ and $d = (b^{m'}a^{m'})^{n'} - (b^{m''}a^{m''})^{n''}$. Then, we can see that $b^{m'}a^{m'} = x^{2m'}$ and $b^{m''}a^{m''} = x^{2m''} -$

$x^{m'}[[c, x^{m''}], x^{m''-m'}]x^{m'}$, and so

$$d = x^{m'}\left(\textstyle\sum_{i=0}^{n''-1} x^{2m''i}[[c, x^{m''}], x^{m''-m'}]x^{2m''(n''-1-i)}\right)x^{m'}.$$

Hence $[d, x^{2m''}] = x^{m'}[[[c, x^{m''}], x^{m''-m'}], x^{2m''n''}]x^{m'}$. On the other hand, we see that

$$0 = [\cdots[d, w_1(a, b)], \ldots, w_k(a, b)] = [\cdots[d, x^{n_1}], \ldots, x^{n_k}],$$

where n_i is the length of w_i. Therefore,

$$\begin{aligned} &x^{m'}[\cdots[[[[c, x^{m''}], x^{m''-m'}], x^{2m''n''}], x^{n_1}], \ldots, x^{n_k}]x^{m'} \\ &\quad = [\cdots[[d, x^{2m''}], x^{n_1}], \ldots, x^{n_k}] = 0. \end{aligned}$$

Since A' is completely reflexive, by [14, Lemma 2.1 (5) (i)], we have

$$[\cdots[[[[c, x^{m''}], x^{m''-m'}], x^{2m''n''}], x^{n_1}], \ldots, x^{n_k}] = 0.$$

Then, the argument employed in the proof of Proposition 2.9 (1) enables us to see that A' is isomorphic to some $M_\sigma(K)$. Moreover, by the same argument as in the proof of Theorem 2.11, we have $mn(n+1)K \neq 0$. Hence, we can get the assertion by the same procedure as in (1).

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